Classification and Support Vector Machine

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Outline

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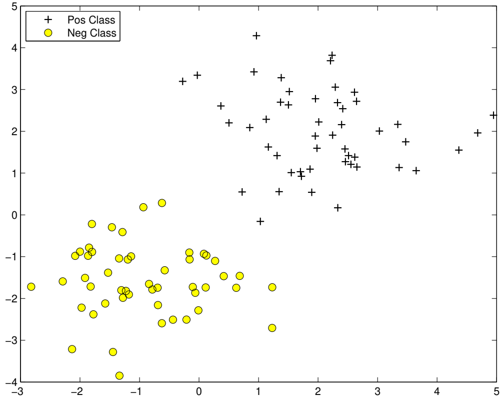
Classification Problem

- A set of input points with binary labels: $\mathbf{x}_i \in \mathbb{R}^n \to y_i \in \{-1, 1\}$, $i = 1, \ldots, N$
- How to classify the $\mathbf{x}_i$'s?



Classification Problem

- Separate the two classes using a linear model:

$$\hat{y} = \boldsymbol{\beta}^T \mathbf{x} + \beta_0$$

- We can classify as follows:

$$\begin{cases} \text{predict “Pos Class”}, & \text{if } \operatorname{sign}(\hat{y}) = +1 \\ \text{predict “Neg Class”}, & \text{if } \operatorname{sign}(\hat{y}) = -1 \end{cases}$$

- Thus, misclassification happens if $\operatorname{sign}(\hat{y}) \neq y$!
- Note that then the decision boundary is the hyperplane $\boldsymbol{\beta}^T \mathbf{x} + \beta_0 = 0$

Classification Problem

- Then our goal is to minimize the number of misclassifications:

$$\begin{array}{ll} \underset{\beta_0, \boldsymbol{\beta}, \{\hat{y}_i\}}{\text{minimize}} & \sum_{i=1}^{N} \mathbf{1}_{\{\text{sign}(\hat{y}_i) \neq y_i\}} \\ \text{subject to} & \hat{y}_i = \boldsymbol{\beta}^T \mathbf{x}_i + \beta_0, \; \forall i \end{array}$$

- However, the objective loss function is nonconvex and nondifferentiable

- What could we do?

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Classification as Linear Regression

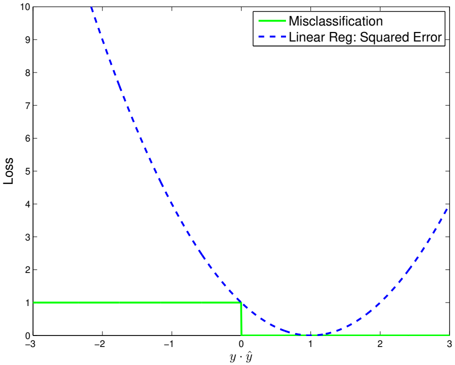
- A straightforward idea is to do linear regression, i.e., replacing the misclassification loss with the residual sum of squares loss:

$$\begin{array}{ll} \underset{\beta_0, \boldsymbol{\beta}, \{\hat{y}_i\}}{\text{minimize}} & \sum_{i=1}^{N} \left(\hat{y}_i - y_i\right)^2 \\ \text{subject to} & \hat{y}_i = \boldsymbol{\beta}^T \mathbf{x}_i + \beta_0, \ \forall i \end{array}$$

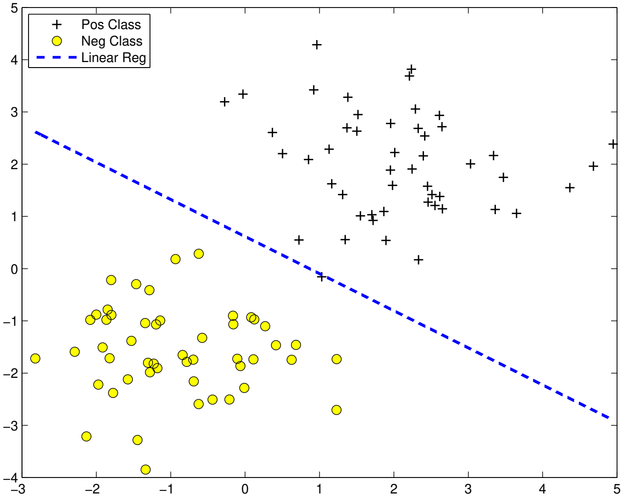
- Note that $\left(\hat{y}_i - y_i\right)^2 = \left(\hat{y}_i - y_i\right)^2 \cdot y_i^2 = \left(1 - y_i \hat{y}_i\right)^2$.

From Loss Function Point of View

- Misclassification loss $\mathbf{1}_{\{\text{sign}(\hat{y})\neq y\}}$; squared error loss $(\hat{y}-y)^2 = (1-y_i\hat{y}_i)^2$



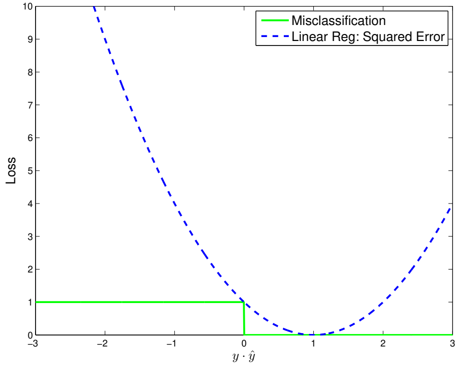
Decision Boundary



Outline

- Can we do better?

- Some idea?



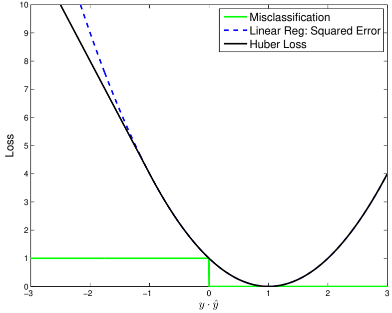
- Can we find some better loss function approximation?

Famous Huber Loss

- Huber Loss (with parameter M):

$$\phi_{hub}(x) = \begin{cases} |x|^2 & |x| < M \\ M(2|x| - M) & |x| \geq M \end{cases}$$

- Select $M = 2$, define loss as $\phi_{hub}(1 - y\hat{y})$:

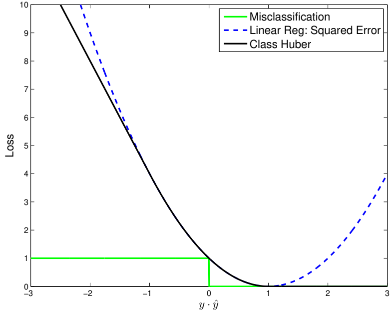


Famous Huber Loss

- Actually, we don't need to penalize when $y\hat{y} > 0$!
- Define

$$\phi_{hub\_pos}(x) = \begin{cases} \phi_{hub}(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- Select $M = 2$, define loss as $\phi_{hub\_pos}(1 - y\hat{y})$:

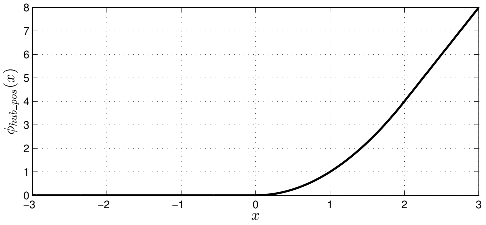


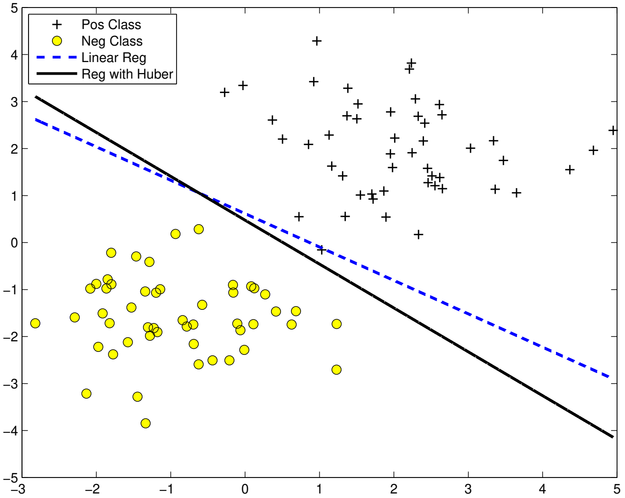
Minimize the "Huberized" Loss

- Then, we take the "Huberized" loss as the approximation and intend to minimize it:

$$\begin{array}{ll} \underset{\beta_0, \boldsymbol{\beta}, \{\hat{y}_i\}}{\text{minimize}} & \sum_{i=1}^{N} \phi_{hub\_pos}\left(1 - y_i \hat{y}_i\right) \\ \text{subject to} & \hat{y}_i = \boldsymbol{\beta}^T \mathbf{x}_i + \beta_0, \ \forall i \end{array}$$

where $\phi_{hub\_pos}(x)$ with $M = 2$ looks like:




Pos Class
Neg Class
Linear Reg
Reg with Huber
5
4
3
2
1
0
−1
−2
−3
−4
−5
−3
−2
−1
0
1
2
3
4
5

Outline

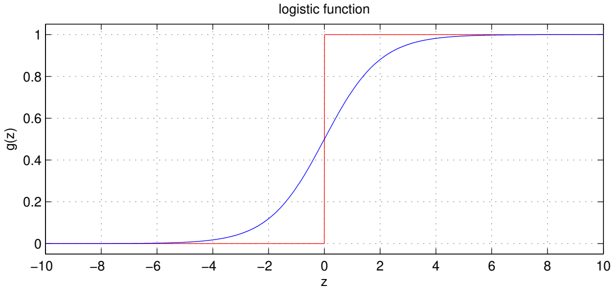
- Can we find more approximations?

Logistic Model

- Given a data point $\mathbf{x} \in \mathbb{R}^n$, model the probability of label $y \in \{-1, +1\}$ as (recall that $\hat{y} = \boldsymbol{\beta}^T \mathbf{x} + \beta_0$):

$$P(Y = y | \mathbf{x}) = \frac{1}{1 + e^{-y \cdot \hat{y}}}$$

- Here, the function $g(z) = \frac{1}{1+e^{-z}}$ is called logistic function:



Logistic Model

- The above probability formula amounts to modeling the log-odds ratio as linear model $\hat{y}$:

$$\log \frac{P(Y=1|\mathbf{x})}{P(Y=-1|\mathbf{x})} = \log \frac{1+e^{\hat{y}}}{1+e^{-\hat{y}}} = \hat{y} = \boldsymbol{\beta}^T \mathbf{x} + \beta_0$$

- Classification rule:

$$\begin{cases} \text{predict “Pos Class”}, & \text{if } \operatorname{sign}(\hat{y}) = +1 \\ \text{predict “Neg Class”}, & \text{if } \operatorname{sign}(\hat{y}) = -1 \end{cases}$$

- The above classification rule is exactly the same as we wanted before!

Negative Log-likelihood as Loss

- Armed with logistic model, we can have the likelihood function:

$$LH\left(\boldsymbol{\beta}, \beta_0\right) = \prod_{i=1}^{N} P\left(Y = y_i | \mathbf{x}_i\right) = \prod_{i=1}^{N} \frac{1}{1 + e^{-y_i \cdot \hat{y}_i}}$$

- The loss function can be defined as negative log-likelihood:

$$Loss\left(\boldsymbol{\beta}, \beta_0\right) = -\log LH\left(\boldsymbol{\beta}, \beta_0\right) = \sum_{i=1}^{N} \log\left(1 + e^{-y_i \cdot \hat{y}_i}\right)$$

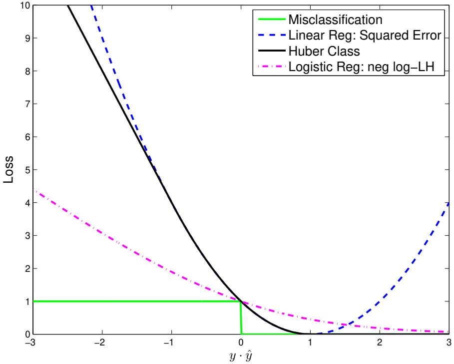
Logistic Regression

- Now, minimize the loss function (i.e., maximize the likelihood):

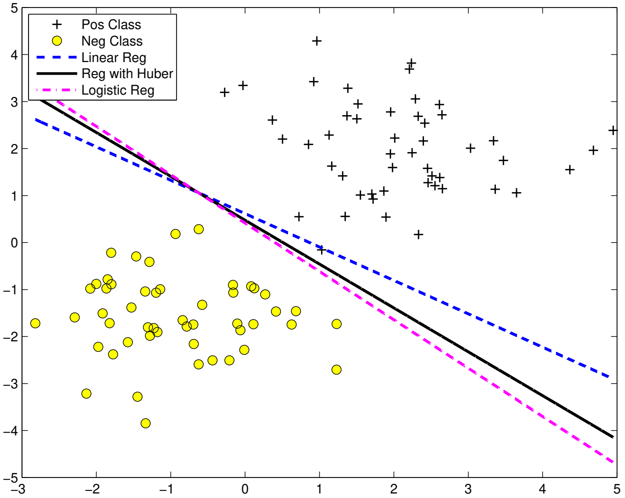
$$\begin{array}{ll}\underset{\beta_0,\boldsymbol{\beta},\{\hat{y}_i\}}{\text{minimize}} & \sum_{i=1}^{N}\log\left(1+e^{-y_i\cdot\hat{y}_i}\right)\\ \text{subject to} & \hat{y}_i=\boldsymbol{\beta}^T\mathbf{x}_i+\beta_0,\ \forall i\end{array}$$

From Loss Function Point of View

- Misclassification loss $\mathbf{1}_{\{\text{sign}(\hat{y}) \neq y\}}$; squared error loss $(\hat{y} - y)^2$; Class Huber $\phi_{hub\_pos}(1 - y\hat{y})$; Negative log-likelihood $\log\left(1 + e^{-y \cdot \hat{y}}\right)$



Decision Boundary



Outline

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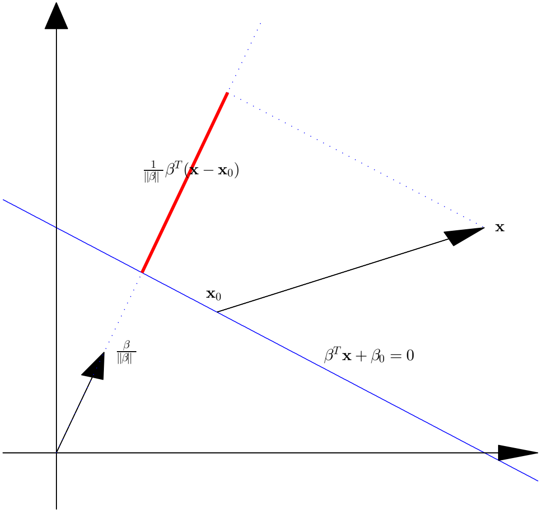
Optimal Separating Hyperplane

- So many hyperplanes can separate the two classes
- BUT, what is the optimal separating hyperplane?
- Optimal separating hyperplane should:
    - separate the two classes, and
    - maximize the distance to the closest point from either class.

Signed Distance



- decision boundary:

$$\boldsymbol{\beta}^T\mathbf{x}+\beta_0=0$$

- signed distance

$$\begin{aligned}&\frac{1}{\|\boldsymbol{\beta}\|}\boldsymbol{\beta}^T(\mathbf{x}-\mathbf{x}_0)\\=&\frac{1}{\|\boldsymbol{\beta}\|}(\boldsymbol{\beta}^T\mathbf{x}+\beta_0)\end{aligned}$$

Maximize the Margin

- Then the "margin" is defined as

$$\frac{1}{\|\boldsymbol{\beta}\|} y \left(\boldsymbol{\beta}^T \mathbf{x} + \beta_0 \right)$$

- Now, maximize the distance to the closest point from either class

$$\begin{array}{ll} \underset{\beta_0, \boldsymbol{\beta}, M}{\text{maximize}} & M \\ \text{subject to} & \frac{1}{\|\boldsymbol{\beta}\|} y_i \left(\boldsymbol{\beta}^T \mathbf{x}_i + \beta_0 \right) \geq M, \ \forall i \end{array}$$

Support Vector Machine

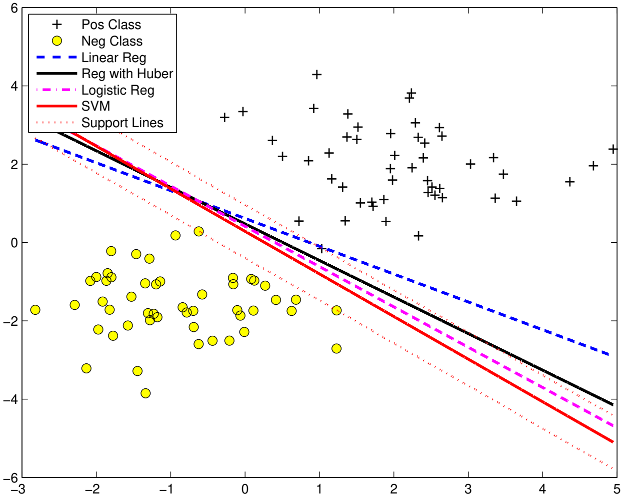
- For any $\boldsymbol{\beta}$ and β_0 satisfying these inequalities, any positively scaled multiple satisfies them too, then arbitrarily set

$$\|\boldsymbol{\beta}\| = \frac{1}{M}$$

- The above problem amounts to

$$\begin{array}{ll} \underset{\beta_0, \boldsymbol{\beta}}{\text{minimize}} & \|\boldsymbol{\beta}\| \\ \text{subject to} & y_i \left(\boldsymbol{\beta}^T \mathbf{x}_i + \beta_0 \right) \geq 1, \ \forall i \end{array}$$

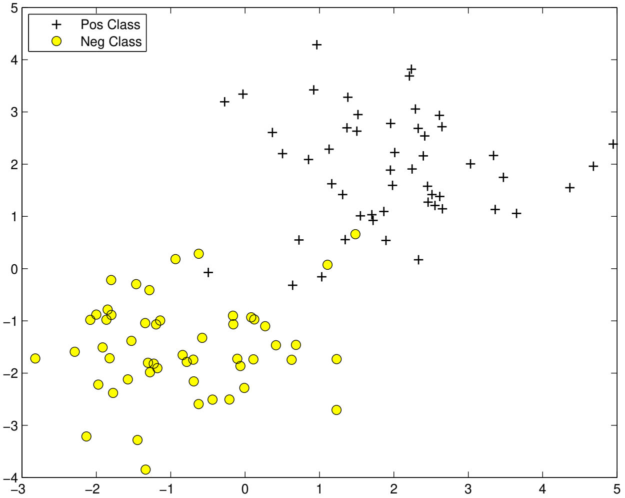
Boundary and Support Vectors



Outline

- The case we considered before is linearly separable
- What if it's linearly nonseparable?

Linearly Nonseparable Case



Linearly Nonseparable Case

- Relax the constraints by introducing positive slack variables ξ_i's:

$$y_i\left(\boldsymbol{\beta}^T\mathbf{x}_i + \beta_0\right) \geq 1 - \xi_i$$

- And then penalize $\sum_i \xi_i$ in the objective function:

$$\begin{array}{ll}\underset{\beta_0, \boldsymbol{\beta}, \{\xi_i\}}{\text{minimize}} & \frac{1}{2}\|\boldsymbol{\beta}\|^2 + C\sum_{i=1}^N \xi_i \\ \text{subject to} & \xi_i \geq 0,\; y_i\left(\boldsymbol{\beta}^T\mathbf{x}_i + \beta_0\right) \geq 1 - \xi_i,\; \forall i\end{array}$$

- Linearly separable case corresponds to $C = \infty$

The SVM as a Penalization Method

- Revisit the primal problem:

$$\begin{array}{ll} \underset{\beta_0,\boldsymbol{\beta},\{\xi_i\}}{\text{minimize}} & \frac{1}{2}\left\|\boldsymbol{\beta}\right\|^2 + C\sum_{i=1}^{N}\xi_i \\ \text{subject to} & \xi_i \geq 0,\ y_i\left(\boldsymbol{\beta}^T\mathbf{x}_i + \beta_0\right) \geq 1 - \xi_i,\ \forall i \end{array}$$

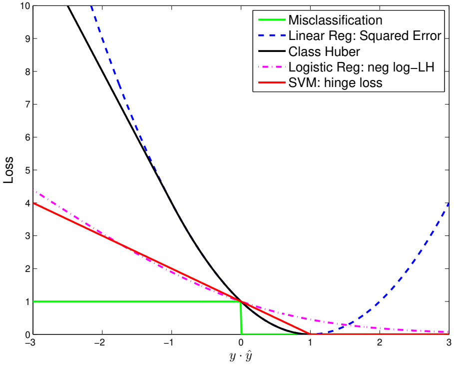
- Consider the optimization problem:

$$\begin{array}{ll} \underset{\beta_0,\boldsymbol{\beta},\{\hat{y}_i\}}{\text{minimize}} & \sum_{i=1}^{N}\left[1 - y_i\hat{y}_i\right]^+ + \frac{\lambda}{2}\left\|\boldsymbol{\beta}\right\|^2 \\ \text{subject to} & \hat{y}_i = \boldsymbol{\beta}^T\mathbf{x}_i + \beta_0,\ \forall i \end{array}$$

  where loss is $\sum_{i=1}^{N}\left[1 - y_i\hat{y}_i\right]^+$ (called hinge loss)
- Two problems are equivalent with $\lambda = 1/C$ (linearly separable case corresponds to $\lambda = 0$)

From Loss Function Point of View

- Misclassification loss $\mathbf{1}_{\{\text{sign}(\hat{y})\neq y\}}$; squared error loss $(\hat{y}-y)^2$; Class Huber $\phi_{hub\_pos}(1-y\hat{y})$; Negative log-likelihood loss $\log\left(1+e^{-y\cdot\hat{y}}\right)$; Hinge loss $[1-y_i\hat{y}_i]^+$



Solving SVM I

- The Lagrange function is

$$
\begin{aligned}
L\left(\boldsymbol{\beta}, \beta_0, \xi_i, \alpha_i, \mu_i\right) = & \frac{1}{2}\|\boldsymbol{\beta}\|^2 + C\sum_{i=1}^{N}\xi_i \\
& -\sum_{i=1}^{N}\mu_i\xi_i - \sum_{i=1}^{N}\alpha_i\left[y_i\left(\boldsymbol{\beta}^T\mathbf{x}_i + \beta_0\right) - (1-\xi_i)\right]
\end{aligned}
$$

  where $\alpha_i \geq 0$ and $\mu_i \geq 0$, $\forall i$, are dual variables
- Setting derivatives w.r.t. $\boldsymbol{\beta}, \beta_0, \{\xi_i\}$ to zero

$$
\boldsymbol{\beta} = \sum_{i=1}^{N}\alpha_i y_i \mathbf{x}_i, \quad 0 = \sum_{i=1}^{N}\alpha_i y_i, \quad \alpha_i = C - \mu_i \quad \forall i
$$

- Dual function:

$$
\begin{aligned}
g(\alpha_i, \mu_i) &= \inf_{\boldsymbol{\beta}, \beta_0, \{\xi_i\}} L(\boldsymbol{\beta}, \beta_0, \xi_i, \alpha_i, \mu_i) \\
&= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + C \sum_{i=1}^{N} \xi_i - \sum_{i=1}^{N} (\alpha_i + \mu_i) \xi_i \\
&\quad - \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - \beta_0 \sum_{i=1}^{N} \alpha_i y_i + \sum_{i=1}^{N} \alpha_i \\
&= \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j
\end{aligned}
$$

Solving SVM III

- Dual problem:

$$\begin{array}{ll} \underset{\{\alpha_i\}}{\text{maximize}} & \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j \\ \text{subject to} & 0 \leq \alpha_i \leq C, \ \forall i \\ & \sum_{i=1}^{N} \alpha_i y_i = 0. \end{array}$$

- Dual problem is a QP! (well, of course...)
- Vector-matrix representation of the objective:

$$\mathbf{1}^T \boldsymbol{\alpha} - \frac{1}{2} \boldsymbol{\alpha}^T \left(\mathrm{Diag}\left(\mathbf{y}\right) \mathbf{X}^T \mathbf{X} \mathrm{Diag}\left(\mathbf{y}\right) \right) \boldsymbol{\alpha}$$

KKT Conditions

- KKT equations characterize the solution to the primal and dual problems

$$
\begin{aligned}
\xi_i \geq 0,\ y_i\left(\boldsymbol{\beta}^T\mathbf{x}_i + \beta_0\right) \geq 1 - \xi_i,\ \forall i \\
0 = \sum_{i=1}^{N} \alpha_i y_i,\ 0 \leq \alpha_i \leq C,\ \forall i \\
\boldsymbol{\beta} = \sum_{i=1}^{N} \alpha_i y_i \mathbf{x}_i,\ \alpha_i = C - \mu_i,\ \forall i \\
0 = \mu_i \xi_i,\ \alpha_i\left[y_i\left(\boldsymbol{\beta}^T\mathbf{x}_i + \beta_0\right) - (1 - \xi_i)\right] = 0,\ \forall i
\end{aligned}
$$

Finding Decision Boundary

- Given optimal dual $\alpha_i^\star$'s, we have:

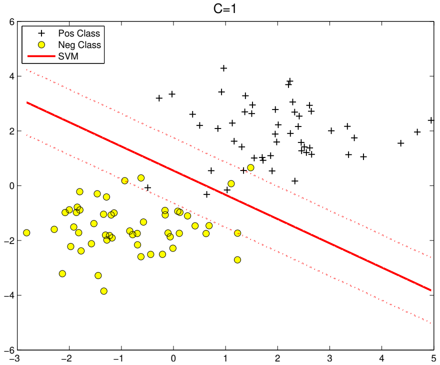
$$\boldsymbol{\beta}^\star = \sum_{i=1}^{N} \alpha_i^\star y_i \mathbf{x}_i.$$

- Support Vectors: those observations i with nonzero $\alpha_i^\star$
- For any $0 < \alpha_j^\star < C$, we have $\xi_j = 0$, and $y_j \left(\sum_{i=1}^{N} \alpha_i^\star y_i \mathbf{x}_i^T \mathbf{x}_j + \beta_0 \right) = 1$, hence we can solve for $\beta_0^\star$
- Decision function becomes:

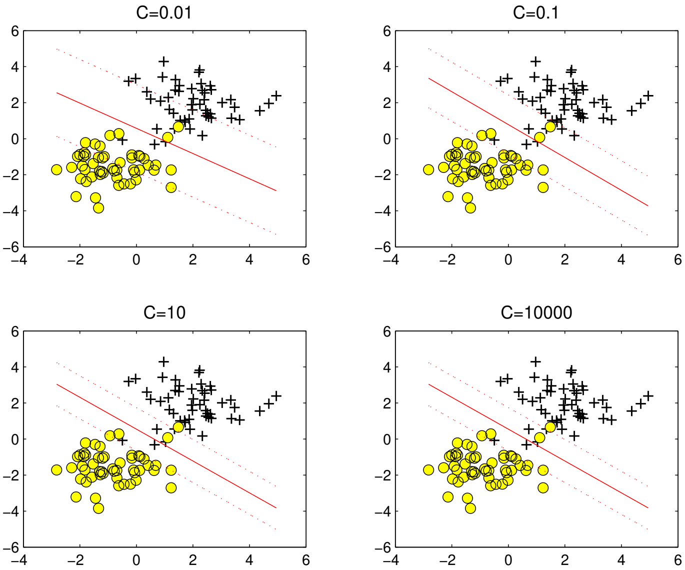
$$D\left(\mathbf{x}\right) = \mathrm{sign}\left(\hat{y}\right) = \mathrm{sign}\left(\boldsymbol{\beta}^{\star T}\mathbf{x} + \beta_0^\star\right) = \mathrm{sign}\left(\sum_{i=1}^{N} \alpha_i^\star y_i \mathbf{x}_i^T \mathbf{x} + \beta_0^\star\right)$$

- Observation: in the dual problem and the above decision function, the only operation on $\mathbf{x}_i$'s is the inner product!

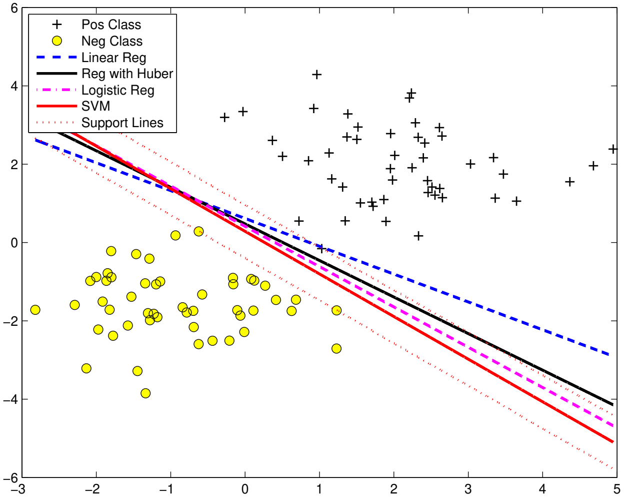
Decision Boundary



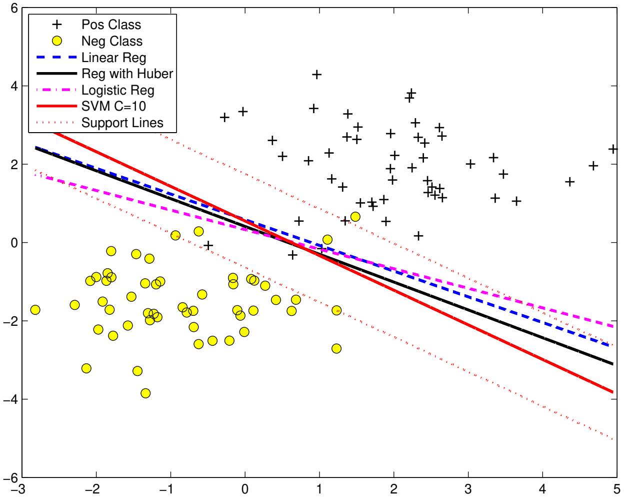
- Large C: focus attention more on points near the boundary $\Rightarrow$ small margin
- Small C: involves data further away $\Rightarrow$ large margin


C=0.01
C=0.1
C=10
C=10000
6
4
2
0
−2
−4
−6
−4
−2
0
2
4
6

Decision Boundary: Revisit Linearly Separable Case



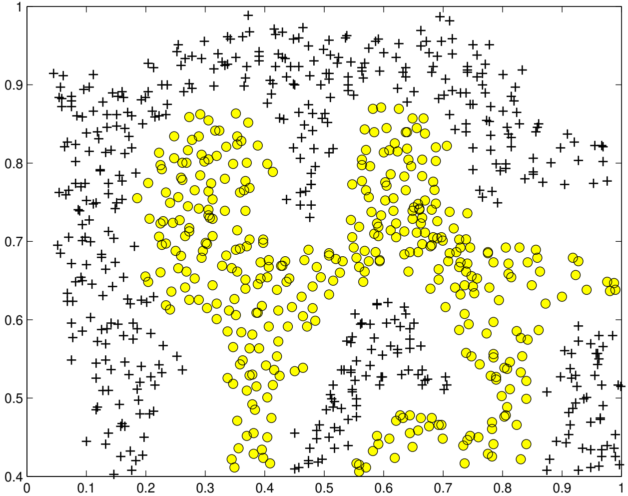
Decision Boundary: Linearly Nonseparable Case



Outline

- For the linearly separable/nonseparable cases, so far SVM works well!
- But, what if the linear decision boundary doesn't work any more?

Nonlinear Decision Surface



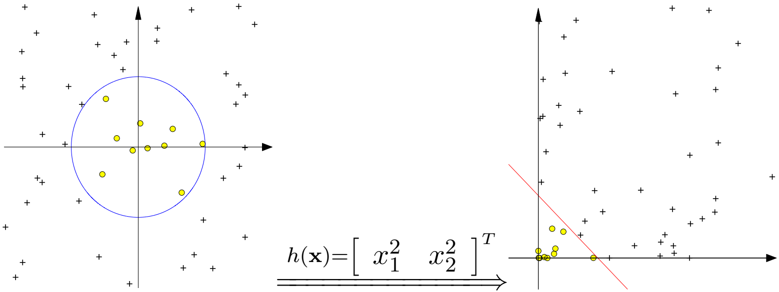
Feature Transformation

Naive method:

- use a curve instead of a line $\Rightarrow$ not efficient

Feature Transformation:

- pre-process the data with $h : \mathbb{R}^n \mapsto \mathcal{H}$, $\mathbf{x} \mapsto h(\mathbf{x})$
- $\mathbb{R}^n$: input space; $\mathcal{H}$: feature space



After Feature Transformation

- Using the features $h(\mathbf{x})$ as inputs:

$$\hat{y} = \boldsymbol{\beta}^T h(\mathbf{x}) + \beta_0$$

- Dual problem:

$$\begin{array}{ll} \underset{\{\alpha_i\}}{\text{maximize}} & \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j h(\mathbf{x}_i)^T h(\mathbf{x}_j) \\ \text{subject to} & \sum_{i=1}^{N} \alpha_i y_i = 0, \; 0 \le \alpha_i \le C, \; \forall i \end{array}$$

- The decision function:

$$D(\mathbf{x}) = \operatorname{sign}\left(\boldsymbol{\beta}^T h(\mathbf{x}) + \beta_0\right) = \operatorname{sign}\left(\sum_{i=1}^{N} \alpha_i y_i h(\mathbf{x}_i)^T h(\mathbf{x}) + \beta_0\right)$$

where β_0 can be determined, as before, by solving $y_i \hat{y}_i = 1$ for any $\mathbf{x}_i$ for which $0 < \alpha_i < C$

Kernelized SVM

- In fact, we need not specify the transformation $h(\mathbf{x})$ at all, but require only knowledge of the kernel function that computes inner products in the transformed space, i.e.:

$$K(\mathbf{x}_i, \mathbf{x}_j) \triangleq h(\mathbf{x}_i)^T h(\mathbf{x}_j)$$

- Dual problem:

$$\begin{array}{ll} \underset{\{\alpha_i\}}{\text{maximize}} & \sum_{i=1}^N \alpha_i - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j K(\mathbf{x}_i, \mathbf{x}_j) \\ \text{subject to} & \sum_{i=1}^N \alpha_i y_i = 0, \; 0 \leq \alpha_i \leq C, \; \forall i \end{array}$$

- The decision function:

$$D(\mathbf{x}) = \text{sign}\left(\sum_{i=1}^N \alpha_i y_i K(\mathbf{x}_i, \mathbf{x}) + \beta_0\right)$$

where β_0 can be determined, as before, by solving $y_i \hat{y}_i = 1$ for any $\mathbf{x}_i$ for which $0 < \alpha_i < C$

Kernel

- K should be a symmetric positive (semi-) definite function
- The previous linearly SVM corresponds to $K(\mathbf{x}_i, \mathbf{x}_j) \triangleq \mathbf{x}_i^T \mathbf{x}_j$
- Popular kernels:

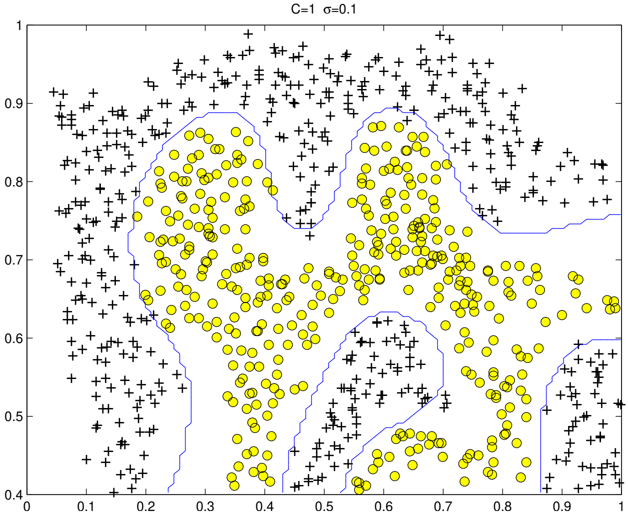
$$
\begin{aligned}
p\text{th Degree polynomial:} \quad & K(\mathbf{x}_i, \mathbf{x}_j) = \left(1 + \mathbf{x}_i^T \mathbf{x}_j\right)^p \\
\text{Gaussian radial kernel:} \quad & K(\mathbf{x}_i, \mathbf{x}_j) = e^{-\|\mathbf{x}_i - \mathbf{x}_j\|^2/\sigma^2} \\
\text{Neural network:} \quad & K(\mathbf{x}_i, \mathbf{x}_j) = \tanh\left(\kappa_1 \mathbf{x}_i^T \mathbf{x}_j + \kappa_2\right)
\end{aligned}
$$

Steps for Applying SVM

Steps:

1. In-sample training:
   1. Select the parameter C
   2. Select the kernel function $K(\mathbf{x}_i, \mathbf{x}_j)$ and related parameters
   3. Solve the dual problem to obtain $\alpha_i^\star$
   4. Compute $\beta_0^\star$, and then classify according to $\operatorname{sign}\left(\sum_{i=1}^{N} \alpha_i^\star y_i K(\mathbf{x}_i, \mathbf{x}) + \beta_0^\star\right)$
2. Out-of-sample testing:
   1. Use the trained model to test out-of-samples
   2. If the out-of-sample test is not good, adjust parameter C, or kernels, and re-train the model until the out-of-sample result is good enough (in practice, one needs a cross-validation set)

Decision Boundary by Gaussian Radial Kernel



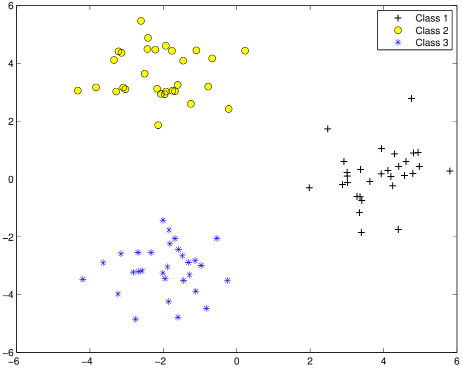
Outline

More than two classes?

- What if there are more than TWO classes?

Multiclass Classification Setup

- Labels: $\{-1, +1\} \to \{1, 2, \ldots, K\}$
- Classification decision rule: $f : \mathbf{x} \in \mathbb{R}^n \mapsto \{1, 2, \ldots, K\}$



Multiclass Classification Methods

- Main ideas:
  - Decompose the multiclass classification problem into multiple binary classification problems
  - Use the majority voting principle or a combined decision from a committee to predict the label
- Common approaches:
  - One-vs-Rest (One-vs-All) approach
  - One-vs-One (Pairwise, All-vs-All) approach

One-vs-Rest Approach

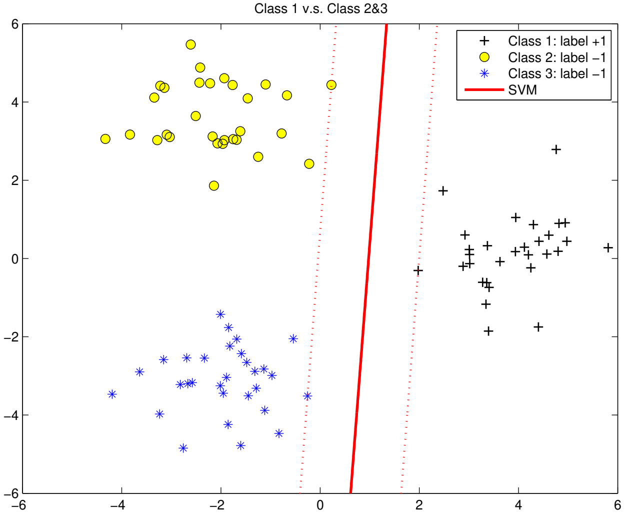
Steps:

1. Solve K different binary problems: classify class i as $+1$ versus the rest classes for all $j \in \{1, \dots, K\} \backslash i$ as -1
2. Assign a test sample to the class $\arg\max_i f_i(\mathbf{x})$, where $f_i(\mathbf{x})$ is the solution from the i-th problem

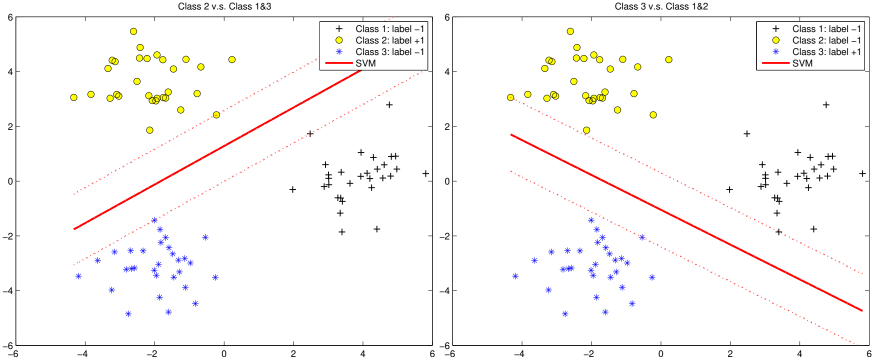
Properties:

- Simple to implement, performs well in practice
- Not optimal (asymptotically)

One-vs-Rest Example: Step 1, training

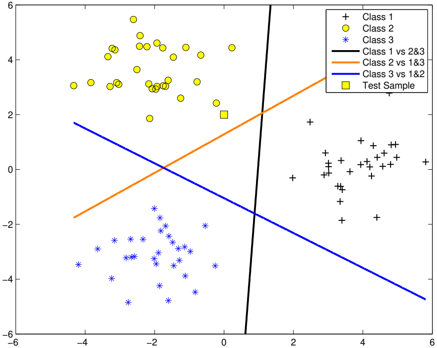


One-vs-Rest Example: Step 1, training...



One-vs-Rest Example: Step 2, test

- Test point $\mathbf{x} = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$, by $f_i(\mathbf{x}) = \mathbf{x}^T \boldsymbol{\beta}^i + \beta_0^i$, we have $f_1(\mathbf{x}) = -1.0783$, $f_2(\mathbf{x}) = 0.5545$, and $f_3(\mathbf{x}) = -2.2560$
- Assign $\begin{bmatrix} 0 & 2 \end{bmatrix}^T$ to class 2!



One-vs-One Approach

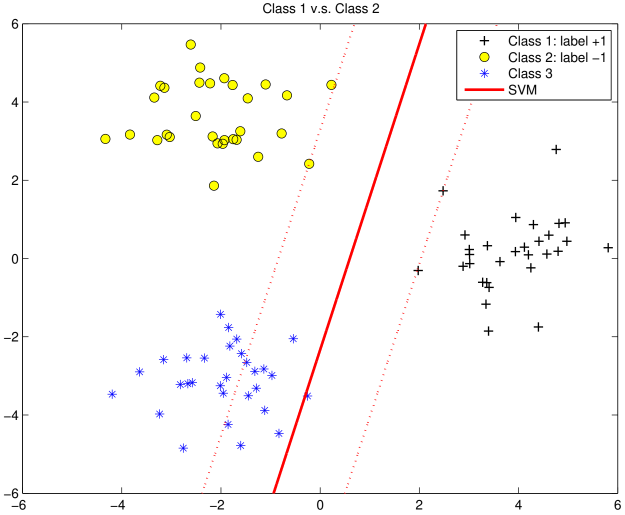
Steps:

1. Solve $\binom{K}{2}$ different binary problems: classify class i as $+1$ versus each class $j \neq i$ as -1. Each classifier is called f_{ij}
2. For prediction at a point, each classifier is queried once and issues a vote. The class with the maximum number of votes is the winner

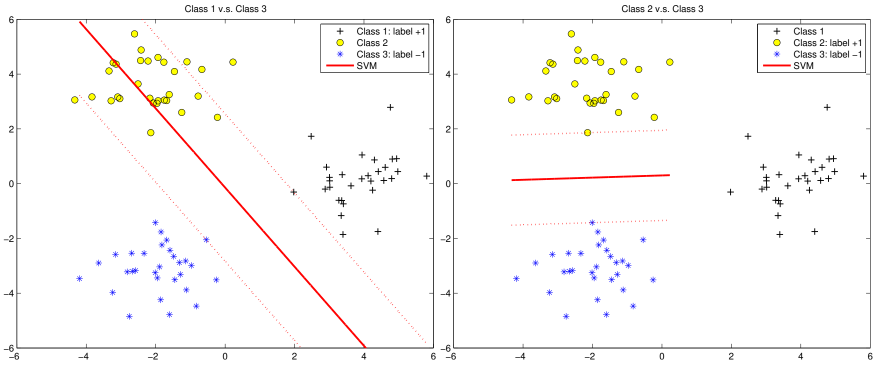
Properties:

- Training process is efficient: small binary problems
- There are too many problems when K is large (If $K = 10$, we need to train 45 binary classifiers)
- Simple to implement, performs competitively in practice

One-vs-One Example: Step 1, training

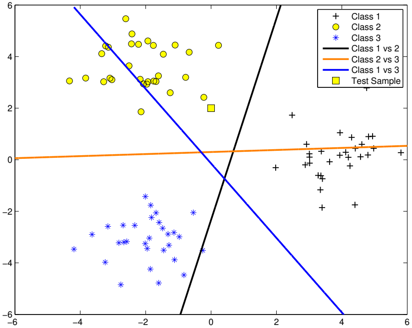


One-vs-One Example: Step 1, training...



One-vs-One Example: Step 2, test

- The same test point $\mathbf{x} = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$:
  - $f_{12}(\mathbf{x}) = -0.7710 < 0$, vote to class 2
  - $f_{23}(\mathbf{x}) = 1.0336 > 0$, vote to class 2
  - $f_{13}(\mathbf{x}) = 0.7957 > 0$, vote to class 1
- Conclusion: class 2 wins!



Outline

Application: Histogram-based Image Classification

- Multiclass: air shows, bears, Arabian horses, night scenes, and several more classes not shown here

App: Histogram-based Image Multi-Classification, Modeling

Why not put image pixels into a vector? Drawbacks:

- large size
- lack of invariance with respect to translations

Histogram-based Image representation

- color space: Hue-Saturation-Value (HSV)
- $\#$ of bins per color component $= 16$
- $\mathbf{x} \in \mathbb{R}^{4096}$: histogram of the picture, dimension $= 16^3 = 4096$
- $y \in \{airshows,\ bears, \dots\}$: the class labels

App: Histogram-based Image Multi-Classification, SVM

Applied SVMs:

- linear SVM
- Poly 2: $K(\mathbf{x}_i, \mathbf{x}_j) = \left(1 + \mathbf{x}_i^T \mathbf{x}_j\right)^2$
- Radial basis function

$$K(\mathbf{x}_i, \mathbf{x}_j) = e^{-\rho d(\mathbf{x}_i, \mathbf{x}_j)}$$

  where the distance measure $d(\mathbf{x}_i, \mathbf{x}_j)$ can be
  - Gaussian: $d(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|^2$
  - Laplacian (ℓ_1 distance): $d(\mathbf{x}_i, \mathbf{x}_j) = |\mathbf{x}_i - \mathbf{x}_j|$
  - χ^2: $d(\mathbf{x}_i, \mathbf{x}_j) = \sum_k \frac{\left(\mathbf{x}_i^{(k)} - \mathbf{x}_j^{(k)}\right)^2}{\mathbf{x}_i^{(k)} + \mathbf{x}_j^{(k)}}$

App: Histogram-based Image Multi-Classification, Result

Criteria: error rate in percentage

- Benchmark, K-nearest neighbor (KNN) method

Database	KNN ℓ_2	KNN χ^2
Corel14	47.7	26.5
Corel7	51.4	35.4

- SVM

Database	linear	Poly 2	Radial basis function		
			Gaussian	Laplacian	χ^2
Corel14	36.3	33.6	28.8	14.7	14.7
Corel7	42.7	38.9	32.2	20.5	21.6

For Further Reading

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The elements of statistical learning.
Springer New York, 2009.

Olivier Chapelle, Patrick Haffner, and Vladimir N. Vapnik,
"Support vector machines for histogram-based image classification,"
IEEE Transactions on Neural Networks. 10(5):1055–1044, 1999.

Thanks

For more information visit:

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