

# Convex Problems

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# Outline of Lecture

- Optimization problems
- Convex optimization (def., optimality conditions, reformulations)
- Quasi-convex optimization
- Classes of convex problems: LP, QP, SOCP, SDP.
- Multicriterion optimization (Pareto optimality)

(Acknowledgement to Stephen Boyd for material for this lecture.)

# Optimization Problem in Standard Form

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{array}$$

$x \in \mathbf{R}^n$  is the optimization variable

$f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$  is the objective function

$f_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ ,  $i = 1, \dots, m$  are inequality constraint functions

$h_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ ,  $i = 1, \dots, p$  are equality constraint functions.

- **Feasibility:**

- a point  $x \in \text{dom } f_0$  is feasible if it satisfies all the constraints and infeasible otherwise
- a problem is feasible if it has at least one feasible point and infeasible otherwise.

- **Optimal value:**

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p \}$$

$p^* = \infty$  if problem infeasible (no  $x$  satisfies the constraints)

$p^* = -\infty$  if problem unbounded below.

- **Optimal solution:**  $x^*$  such that  $f(x^*) = p^*$  (and  $x^*$  feasible).

# Global and Local Optimality

- A feasible  $x$  is **optimal** if  $f_0(x) = p^*$ ;  $X_{\text{opt}}$  is the set of optimal points.
- A feasible  $x$  is **locally optimal** if is optimal within a ball, i.e., there is an  $R > 0$  such that  $x$  is optimal for

$$\begin{aligned} & \underset{z}{\text{minimize}} && f_0(z) \\ & \text{subject to} && f_i(z) \leq 0, \quad i = 1, \dots, m, \quad h_i(z) = 0, \quad i = 1, \dots, p \\ & && \|z - x\|_2 \leq R \end{aligned}$$

Examples:

- $f_0(x) = 1/x$ ,  $\text{dom } f_0 = \mathbf{R}_{++}$ :  $p^* = 0$ , no optimal point
- $f_0(x) = x^3 - 3x$ :  $p^* = -\infty$ , local optimum at  $x = 1$ .

# Implicit Constraints

- The standard form optimization problem has an explicit constraint:

$$x \in \mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$$

- $\mathcal{D}$  is the domain of the problem
- The constraints  $f_i(x) \leq 0$ ,  $h_i(x) = 0$  are the explicit constraints
- A problem is unconstrained if it has no explicit constraints
- Example:

$$\underset{x}{\text{minimize}} \quad \log(b - a^T x)$$

is an unconstrained problem with implicit constraint  $b > a^T x$ .

# Feasibility Problem

- Sometimes, we don't really want to minimize any objective, just to find a feasible point:

$$\begin{array}{ll} \underset{x}{\text{find}} & x \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{array}$$

- This feasibility problem can be considered as a special case of a general problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & 0 \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{array}$$

where  $p^* = 0$  if constraints are feasible and  $p^* = \infty$  otherwise.

# Convex Optimization Problem

- Convex optimization problem in standard form:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

where  $f_0, f_1, \dots, f_m$  are convex and equality constraints are affine.

- **Local and global optima:** any locally optimal point of a convex problem is globally optimal.
- Most problems are not convex when formulated.
- Reformulating a problem in convex form is an art, there is no systematic way.



## Example

- The following problem is nonconvex (why not?):

$$\begin{aligned} & \underset{x}{\text{minimize}} && x_1^2 + x_2^2 \\ & \text{subject to} && x_1 / (1 + x_2^2) \leq 0 \\ & && (x_1 + x_2)^2 = 0 \end{aligned}$$

- The objective is convex.
- The equality constraint function is not affine; however, we can rewrite it as  $x_1 = -x_2$  which is then a linear equality constraint.
- The inequality constraint function is not convex; however, we can rewrite it as  $x_1 \leq 0$  which again is linear.
- We can rewrite it as

$$\begin{aligned} & \underset{x}{\text{minimize}} && x_1^2 + x_2^2 \\ & \text{subject to} && x_1 \leq 0 \\ & && x_1 = -x_2 \end{aligned}$$

# Global and Local Optimality

Any locally optimal point of a convex problem is globally optimal.

**Proof:** Suppose  $x$  is locally optimal (around a ball of radius  $R$ ) and  $y$  is optimal with  $f_0(y) < f_0(x)$ . We will show this cannot be.

Just take the segment from  $x$  to  $y$ :  $z = \theta y + (1 - \theta)x$ . Obviously the objective function is strictly decreasing along the segment since  $f_0(y) < f_0(x)$ :

$$\theta f_0(y) + (1 - \theta) f_0(x) < f_0(x) \quad \theta \in (0, 1].$$

Using now the convexity of the function, we can write

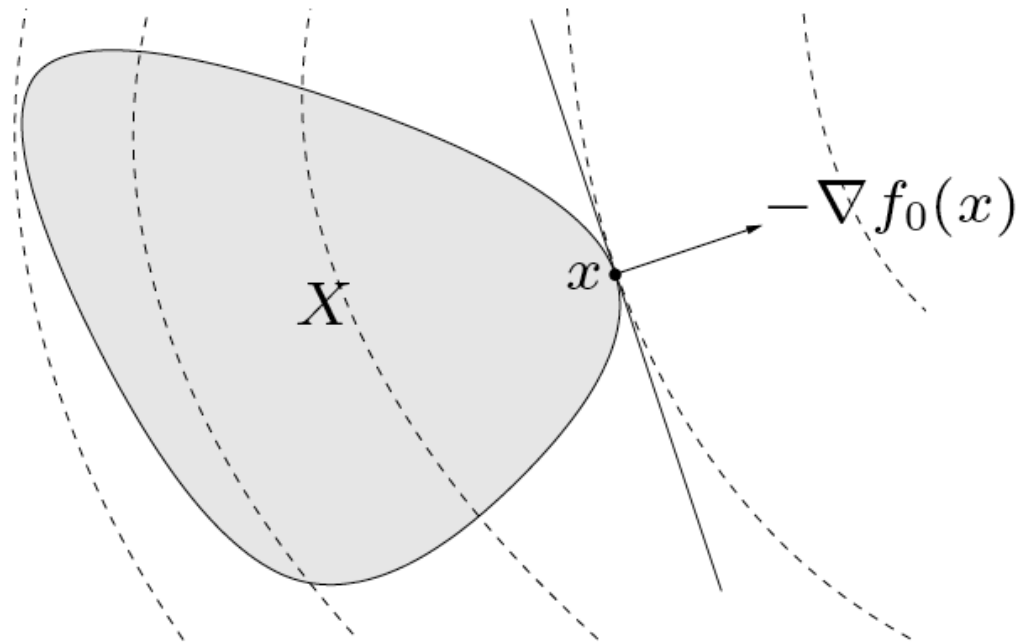
$$f_0(\theta y + (1 - \theta)x) < f_0(x) \quad \theta \in (0, 1].$$

Finally, just choose  $\theta$  sufficiently small such that the point  $z$  is in the ball of local optimality of  $x$ , arriving at a contradiction.

# Optimality Criterion for Differentiable $f_0$

**Minimum Principle:** A feasible point  $x$  is optimal if and only if

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \text{for all feasible } y$$



- **unconstrained problem:**  $x$  is optimal iff

$$x \in \text{dom } f_0, \quad \nabla f_0(x) = 0$$

- **equality constrained problem:**  $\min_x f_0(x)$  s.t.  $Ax = b$

$x$  is optimal iff

$$x \in \text{dom } f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant:**  $\min_x f_0(x)$  s.t.  $x \geq 0$

$x$  is optimal iff

$$x \in \text{dom } f_0, \quad x \geq 0, \quad \begin{cases} \nabla_i f_0(x) \geq 0 & x_i = 0 \\ \nabla_i f_0(x) = 0 & x_i > 0 \end{cases}$$

# Equivalent Reformulations

- **Eliminating/introducing equality constraints:**

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{z}{\text{minimize}} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0 \quad i = 1, \dots, m \end{array}$$

where  $F$  and  $x_0$  are such that  $Ax = b \iff x = Fz + x_0$  for some  $z$ .

- **Introducing slack variables for linear inequalities:**

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i \quad i = 1, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{x,s}{\text{minimize}} & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i \quad i = 1, \dots, m \\ & s_i \geq 0 \end{array}$$

- **Epigraph form:** a standard form convex problem is equivalent to

$$\begin{array}{ll}
 \underset{x,t}{\text{minimize}} & t \\
 \text{subject to} & f_0(x) - t \leq 0 \\
 & f_i(x) \leq 0 \quad i = 1, \dots, m \\
 & Ax = b
 \end{array}$$

- **Minimizing over some variables:**

$$\begin{array}{ll}
 \underset{x,y}{\text{minimize}} & f_0(x, y) \\
 \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m
 \end{array}$$

is equivalent to

$$\begin{array}{ll}
 \underset{x}{\text{minimize}} & \tilde{f}_0(x) \\
 \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m
 \end{array}$$

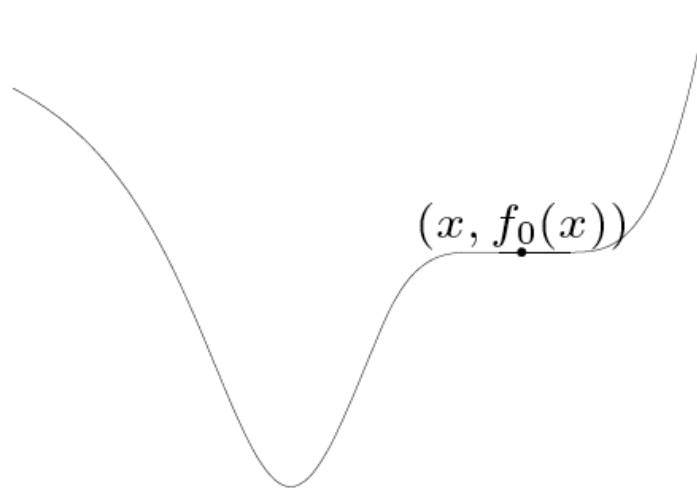
where  $\tilde{f}_0(x) = \inf_y f_0(x, y)$ .

# Quasiconvex Optimization

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

where  $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$  is quasiconvex and  $f_1, \dots, f_m$  are convex.

- Observe that it can have locally optimal points that are not (globally) optimal:





- **Convex representation** of sublevel sets of a quasiconvex function  $f_0$ : there exists a family of convex functions  $\phi_t(x)$  for fixed  $t$  such that

$$f_0(x) \leq t \iff \phi_t(x) \leq 0.$$

- **Example:**

$$f_0(x) = \frac{p(x)}{q(x)}$$

with  $p$  convex,  $q$  concave, and  $p(x) \geq 0$ ,  $q(x) > 0$  on  $\text{dom } f_0$ . We can choose:

$$\phi_t(x) = p(x) - tq(x)$$

- for  $t \geq 0$ ,  $\phi_t(x)$  is convex in  $x$
- $p(x)/q(x) \leq t$  if and only if  $\phi_t(x) \leq 0$ .

## Solving a quasiconvex problem via convex feasibility problems:

the idea is to solve the epigraph form of the problem with a sandwich technique in  $t$ :

- for fixed  $t$  the epigraph form of the original problem reduces to a feasibility convex problem

$$\phi_t(x) \leq 0, \quad f_i(x) \leq 0 \quad \forall i, \quad Ax \leq b$$

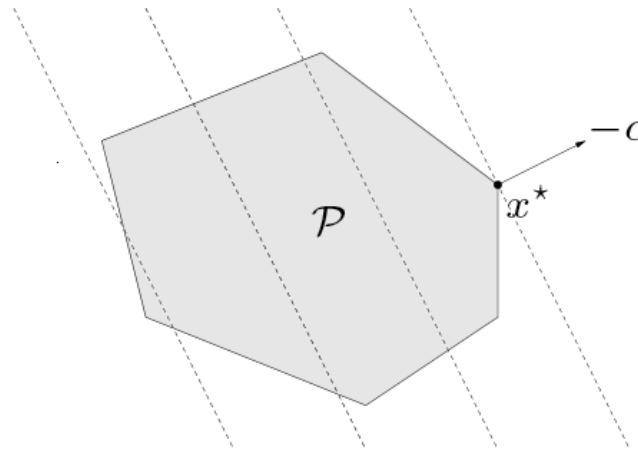
- if  $t$  is too small, the feasibility problem will be infeasible
- if  $t$  is too large, the feasibility problem will be feasible
- start with upper and lower bounds on  $t$  (termed  $u$  and  $l$ , resp.) and use a sandwich technique (bisection method): at each iteration use  $t = (l + u) / 2$  and update the bounds according to the feasibility/infeasibility of the problem.

# Classes of Convex Problems

# Linear Programming (LP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + d \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

- Convex problem: affine objective and constraint functions.
- Feasible set is a polyhedron:



# $l_1$ - and $l_\infty$ - Norm Problems as LPs

- $l_\infty$ -norm minimization:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|x\|_\infty \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

is equivalent to the LP

$$\begin{array}{ll} \underset{t,x}{\text{minimize}} & t \\ \text{subject to} & -t\mathbf{1} \leq x \leq t\mathbf{1} \\ & Gx \leq h \\ & Ax = b. \end{array}$$

- $\ell_1$ -norm minimization:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|x\|_1 \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

is equivalent to the LP

$$\begin{array}{ll} \underset{t,x}{\text{minimize}} & \sum_i t_i \\ \text{subject to} & -t \leq x \leq t \\ & Gx \leq h \\ & Ax = b. \end{array}$$

## Example: Chebyshev Center of a Polyhedron

- The Chebyshev center of a polyhedron  $\mathcal{P} = \{x \mid a_i^T x \leq b_i, i = 1, \dots, m\}$  is the center of the largest inscribed ball  $\mathcal{B} = \{x_c + u \mid \|u\| \leq r\}$ .
- Let's solve the problem:

$$\begin{array}{ll} \underset{r, x_c}{\text{maximize}} & r \\ \text{subject to} & x \in \mathcal{P} \quad \text{for all } x = x_c + u \mid \|u\| \leq r \end{array}$$

- Observe that  $a_i^T x \leq b_i$  for all  $x \in \mathcal{B}$  if and only if

$$\sup_u \{a_i^T (x_c + u) \mid \|u\| \leq r\} \leq b_i.$$

- Using Schwartz inequality, the supremum condition can be rewritten as

$$a_i^T x_c + r \|a_i\|_2 \leq b_i.$$

- Hence, the Chebyshev center can be obtained by solving:

$$\begin{array}{ll} \underset{r, x_c}{\text{maximize}} & r \\ \text{subject to} & a_i^T x_c + r \|a_i\|_2 \leq b_i, \quad i = 1, \dots, m \end{array}$$

which is an LP.



# Linear-Fractional Programming

$$\begin{aligned} & \underset{x}{\text{minimize}} && (c^T x + d) / (e^T x + f) \\ & \text{subject to} && Gx \leq h \\ & && Ax = b \end{aligned}$$

with  $\text{dom } f_0 = \{x \mid e^T x + f > 0\}$ .

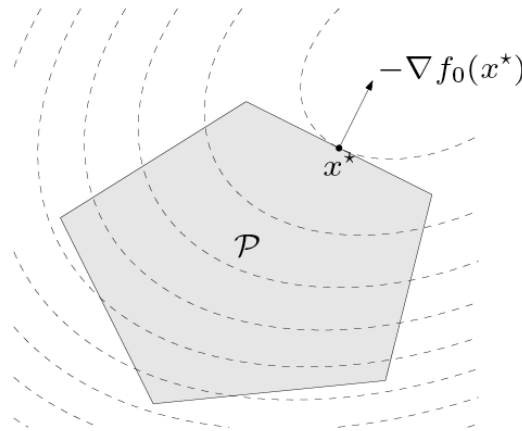
- It is a quasiconvex optimization problem (solved by bisection).
- Interestingly, the following LP is equivalent:

$$\begin{aligned} & \underset{y,z}{\text{minimize}} && c^T y + dz \\ & \text{subject to} && Gy \leq hz \\ & && Ay = bz \\ & && e^T y + fz = 1 \\ & && z \geq 0 \end{aligned}$$

# Quadratic Programming (QP)

$$\begin{aligned} & \underset{x}{\text{minimize}} && (1/2) x^T P x + q^T x + r \\ & \text{subject to} && Gx \leq h \\ & && Ax = b \end{aligned}$$

- Convex problem (assuming  $P \in \mathbf{S}^n \succeq 0$ ): convex quadratic objective and affine constraint functions.
- Minimization of a convex quadratic function over a polyhedron:



# Quadratically Constrained QP (QCQP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (1/2) x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2) x^T P_i x + q_i^T x + r_i \leq 0 \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

- Convex problem (assuming  $P_i \in \mathbf{S}^n \succeq 0$ ): convex quadratic objective and constraint functions.

# Second-Order Cone Programming (SOCP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i \quad i = 1, \dots, m \\ & Fx = g \end{array}$$

- Convex problem: linear objective and second-order cone constraints
- For  $A_i$  row vector, it reduces to an LP.
- For  $c_i = 0$ , it reduces to a QCQP.
- More general than QCQP and LP.

# Robust LP as an SOCP

- Sometimes, we don't know exactly the parameters of an optimization problem.
- Consider the robust LP:

$$\begin{aligned} & \underset{x}{\text{minimize}} && c^T x \\ & \text{subject to} && a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \dots, m \end{aligned}$$

where  $\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\| \leq 1\}$ .

- It can be rewritten as the SOCP:

$$\begin{aligned} & \underset{x}{\text{minimize}} && c^T x \\ & \text{subject to} && \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i \quad i = 1, \dots, m. \end{aligned}$$

# Generalized Inequality Constraints

- Convex problem with generalized ineq. constraints:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0 \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

where  $f_0$  is convex and  $f_i$  are  $K_i$ -convex w.r.t. proper cone  $K_i$ .

- It has the same properties as a standard convex problem.
- **Conic form problem:** special case with affine objective and constraints:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

# Semidefinite Programming (SDP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq G \\ & Ax = b \end{array}$$

- Inequality constraint is called linear matrix inequality (LMI).
- Convex problem: linear objective and linear matrix inequality (LMI) constraints.
- Observe that multiple LMI constraints can always be written as a single one.

- **LP and equivalent SDP:**

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & \text{diag}(Ax - b) \preceq 0 \end{array}$$

- **SOCP and equivalent SDP:**

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i, \quad i = 1, \dots, m \end{array}$$

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \dots, m \end{array}$$



- **Eigenvalue minimization:**

$$\underset{x}{\text{minimize}} \quad \lambda_{\max}(A(x))$$

where  $A(x) = A_0 + x_1A_1 + \cdots + x_nA_n$ , is equivalent to SDP

$$\begin{aligned} &\underset{x}{\text{minimize}} && t \\ &\text{subject to} && A(x) \preceq tI \end{aligned}$$

- It follows from

$$\lambda_{\max}(A(x)) \leq t \iff A(x) \preceq tI$$

# Vector Optimization

- General vector optimization problem:

$$\begin{aligned} & \underset{x}{\text{minimize (w.r.t. } K)} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0 \quad i = 1, \dots, m \\ & && h_i(x) = 0 \quad i = 1, \dots, p \end{aligned}$$

where the vector objective  $f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}^q$  is minimized w.r.t. proper cone  $K \subseteq \mathbf{R}^q$ .

- Convex vector optimization problem:

$$\begin{aligned} & \underset{x}{\text{minimize (w.r.t. } K)} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0 \quad i = 1, \dots, m \\ & && Ax = b \end{aligned}$$

where  $f_0$  is  $K$ -convex and  $f_1, \dots, f_m$  are convex.

# Pareto Optimality

- Set of achievable objective values:

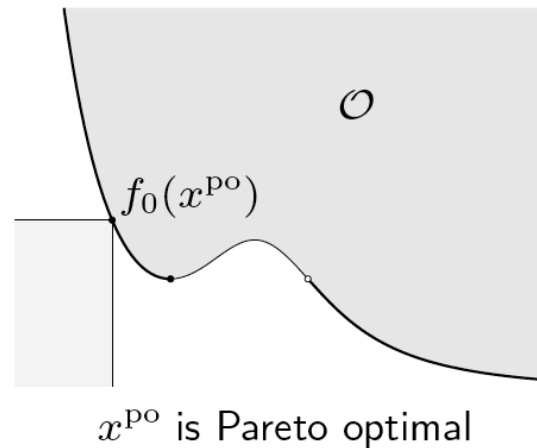
$$\mathcal{O} = \{f_0(x) \mid x \text{ is feasible}\}.$$

- A feasible  $x$  is **Pareto optimal** if  $f_0(x)$  is a minimal value of  $\mathcal{O}$ .

- A minimal value  $x$  of  $\mathcal{O}$  satisfies:

$$y \in \mathcal{O}, y \preceq_K x \implies y = x$$

(in words:  $x$  cannot be in the cone of points worse than  $y$ )



# Multicriterion Optimization

- If we now choose the proper cone  $K = \mathbf{R}_+^q$  (nonnegative orthant), then the vector optimization becomes a multicriterion optimization with  $q$  different objectives:

$$f_0(x) = (F_1(x), \dots, F_q(x)).$$

- A feasible point  $x^{\text{po}}$  is Pareto optimal if

$$y \text{ feasible, } f_0(y) \leq f_0(x^{\text{po}}) \implies f_0(x^{\text{po}}) = f_0(y).$$

- If there are multiple Pareto optimal values, there is a trade-off among the objectives.

# Scalarization for Multicriterion Problems

- To find Pareto optimal points, minimize the positive weighted sum:

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x).$$

- Example: regularized least-squares:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2.$$

# Summary

- Thus far, we have seen the basic definitions of convex sets and convex functions with examples and operations that preserve convexity.
- We have then considered convex problems in a variety of forms including quasiconvex problems, vector optimization, Pareto optimality, etc.
- We have also overviewed different classes of convex problems such as LP, QP, QCQP, SOCP, and SDP.
- We can say we have acquired the vocabulary of convex optimization.

# References

Chapter 4 of

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

<http://www.stanford.edu/~boyd/cvxbook/>