

# Convex Sets

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ELEC5470/IEDA6100A - Convex Optimization

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# Outline of Lecture

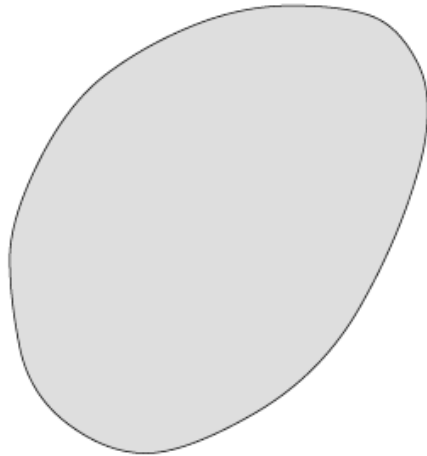
- Definition convex set
- Hyperplanes, halfspaces, polyhedra
- Balls and ellipsoids
- Convex hull
- Cones: norm cones, PSD cone
- Operations that preserve convexity
- Generalized inequalities

(Acknowledgement to Stephen Boyd for material for this lecture.)

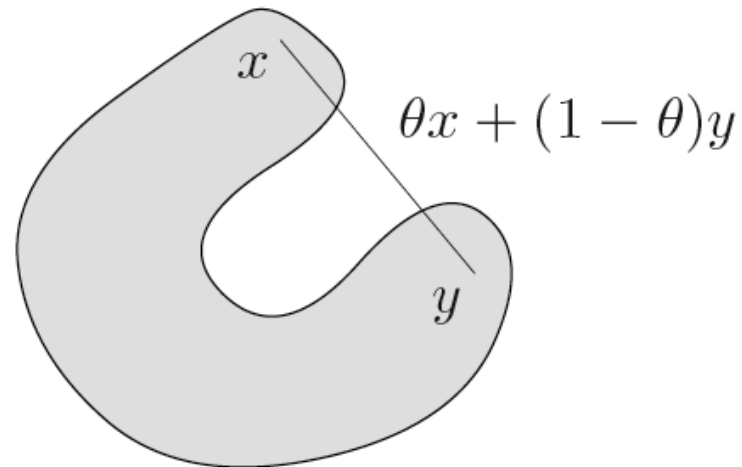
## Definition of Convex Set

- A set  $C \in \mathbf{R}^n$  is said to be **convex** if the line segment between any two points is in the set: for any  $x, y \in C$  and  $0 \leq \theta \leq 1$ ,

$$\theta x + (1 - \theta)y \in C.$$



convex



non-convex

# Examples: Hyperplanes and Halfspaces

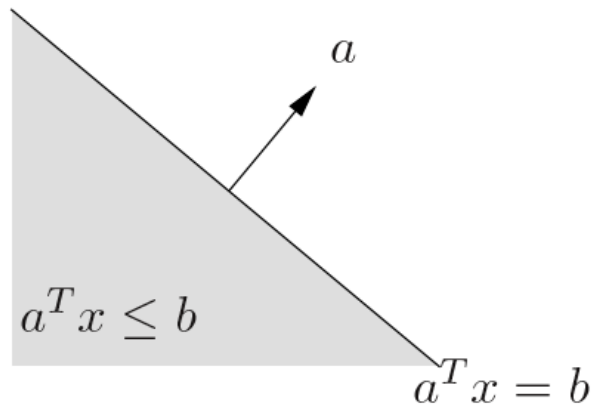
- **Hyperplane:**

$$C = \{x \mid a^T x = b\}$$

where  $a \in \mathbf{R}^n$ ,  $b \in \mathbf{R}$ .

- **Halfspace:**

$$C = \{x \mid a^T x \leq b\}$$

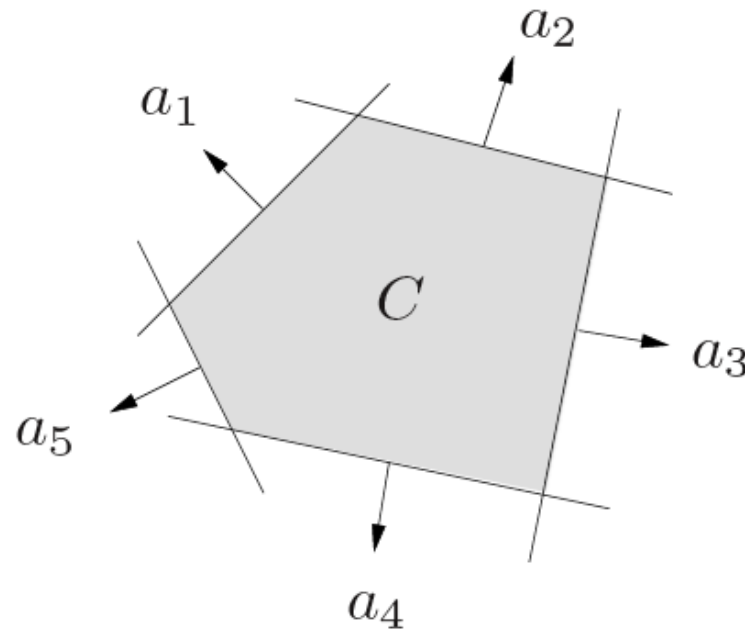


## Example: Polyhedra

- Polyhedron:

$$C = \{x \mid Ax \leq b, Cx = d\}$$

where  $A \in \mathbf{R}^{m \times n}$ ,  $C \in \mathbf{R}^{p \times n}$ ,  $b \in \mathbf{R}^m$ ,  $d \in \mathbf{R}^p$ .



# Examples: Euclidean Balls and Ellipsoids

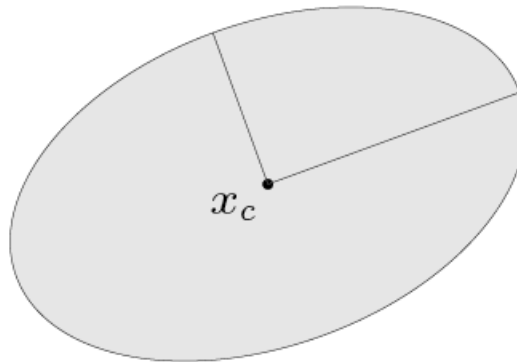
- **Euclidean ball** with center  $x_c$  and radius  $r$ :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}.$$

- **Ellipsoid:**

$$E(x_c, P) = \left\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\right\} = \{x_c + Au \mid \|u\|_2 \leq 1\}$$

with  $P \in \mathbf{R}^{n \times n} \succ 0$  (positive definite).



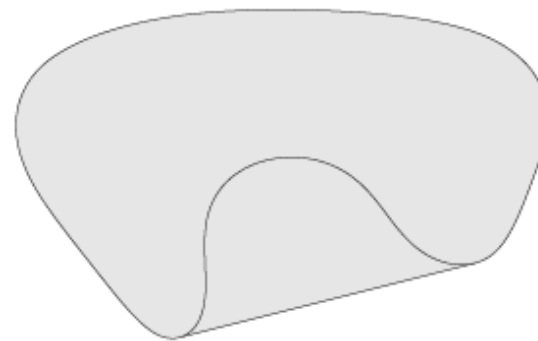
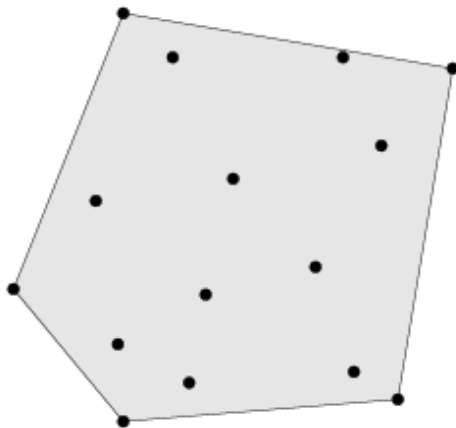
# Convex Combination and Convex Hull

- **Convex combination** of  $x_1, \dots, x_k$ : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$$

with  $\theta_1 + \dots + \theta_k = 1$ ,  $\theta_i \geq 0$ .

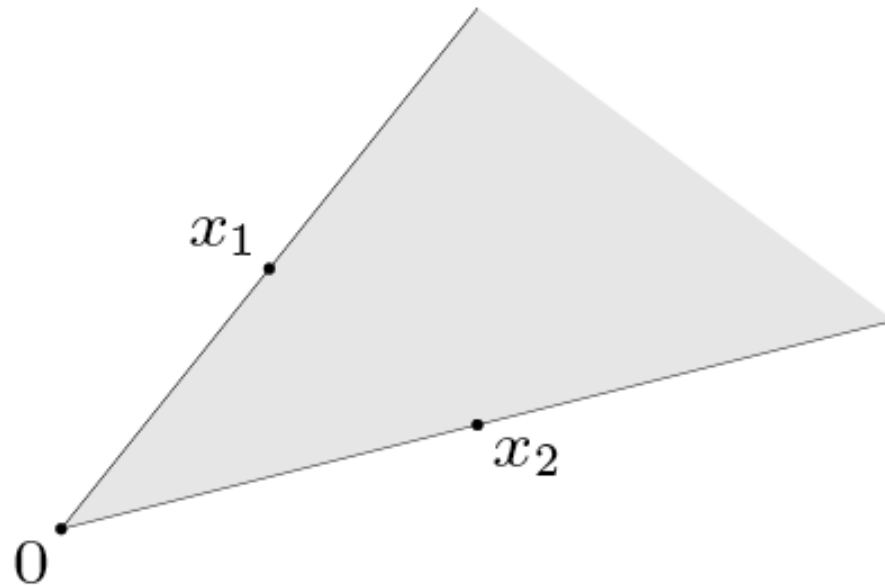
- **Convex hull** of a set: set of all convex combinations of points in the set.



# Convex Cones

- A set  $C \in \mathbf{R}^n$  is said to be a **convex cone** if the ray from each point in the set is in the set: for any  $x_1, x_2 \in C$  and  $\theta_1, \theta_2 \geq 0$ ,

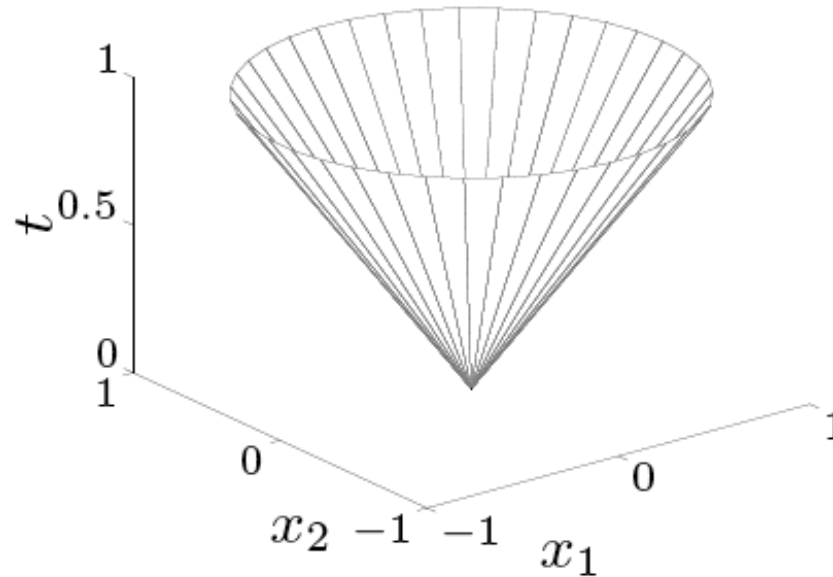
$$\theta_1 x_1 + \theta_2 x_2 \in C.$$





# Norm Balls and Norm Cones

- **Norm ball** with center  $x_c$  and radius  $r$ :  $\{x \mid \|x - x_c\| \leq r\}$  where  $\|\cdot\|$  is a norm.
- **Norm cone**:  $\{(x, t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\}$ .
- Euclidean norm cone or second-order cone (aka ice-cream cone):

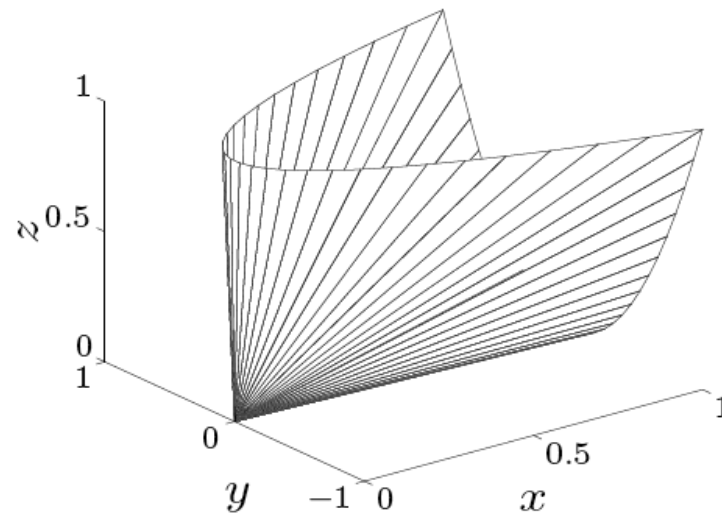


# Positive Semidefinite Cone

- Positive semidefinite (PSD) cone:

$$\mathbf{S}_+^n = \{X \in \mathbf{R}^{n \times n} \mid X = X^T \succeq 0\}.$$

- Example:  $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



# Operations that Preserve Convexity

How do we establish the convexity of a given set?

1. Applying the definition:

$$x, y \in C, 0 \leq \theta \leq 1 \implies \theta x + (1 - \theta) y \in C$$

which can be cumbersome.

2. Showing that  $C$  is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, etc.) by operations that preserve convexity:

- intersection
- affine functions
- perspective function
- linear-fractional functions

# Intersection

- **Intersection:** if  $S_1, S_2, \dots, S_k$  are convex, then  $S_1 \cap S_2 \cap \dots \cap S_k$  is convex.
- Example: a polyhedron is the intersection of halfspaces and hyperplanes.
- Example:

$$S = \{x \in \mathbf{R}^n \mid |p_x(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where  $p_x(t) = x_1 \cos t + x_2 \cos 2t + \dots + x_n \cos nt$ .

# Affine Function

- **Affine composition:** the image (and inverse image) of a convex set under an affine function  $f(x) = Ax + b$  is convex:

$$S \subseteq \mathbf{R}^n \text{ convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ convex.}$$

- Examples: scaling, translation, projection.
- Example:  $\{(x, t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\}$  is convex, so is

$$\{x \in \mathbf{R}^n \mid \|Ax + b\| \leq c^T x + d\}.$$

- Example: solution set of LMI:  $\{x \in \mathbf{R}^n \mid x_1 A_1 + \cdots + x_n A_n \preceq B\}$ .

# Perspective and Linear-Fractional Functions

**Perspective function:**  $P : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$ :

$$P(x, t) = x/t, \quad \text{dom } P = \{(x, t) \mid t > 0\}.$$

- Images and inverse images of convex sets under perspective functions are convex.

**Linear-fractional function:**  $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ :

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } P = \{x \mid c^T x + d > 0\}.$$

- Images and inverse images of convex sets under linear-fractional functions are convex.

# Generalized Inequalities

- A convex cone  $K \subseteq \mathbf{R}^n$  is a **proper cone** if it is closed, solid, and pointed.

- **Examples:**

- nonnegative orthant:

$$K = \mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$$

- positive semidefinite cone:

$$K = \mathbf{S}_+^n = \{X \in \mathbf{R}^{n \times n} \mid X = X^T \succeq 0\}$$

- nonnegative polynomials on  $[0, 1]$ :

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \dots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}.$$

- A **generalized inequality** is defined by a proper cone  $K$ :

$$y \succeq_K x \iff y - x \succeq_K 0 \text{ or } y - x \in K.$$

- **Examples:**

- componentwise inequality ( $K = \mathbf{R}_+^n$ ):

$$y \succeq_{\mathbf{R}_+^n} x \iff y_i \geq x_i, \quad i = 1, \dots, n$$

- matrix inequality ( $K = \mathbf{S}_+^n$ ):

$$Y \succeq_{\mathbf{S}_+^n} X \iff Y - X \text{ is positive semidefinite.}$$



# References

Chapter 2 of

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

[https://web.stanford.edu/~boyd/cvxbook/bv\\_cvxbook.pdf](https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf)