Regularized Robust Estimation of Mean and Covariance Matrix under Heavy Tails and Outliers

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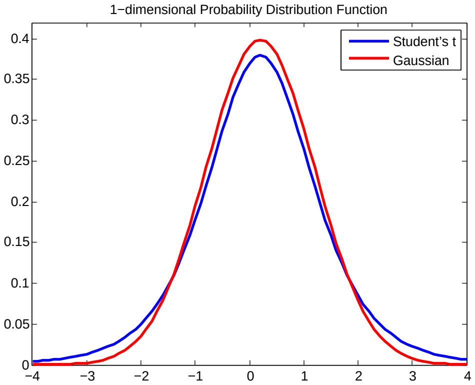
Outline

Outline

Basic Problem

- Task: estimate mean and covariance matrix from data $\{\mathbf{x}_i\}$.
- Difficulties: outlier corrupted observation (heavy-tailed underlying distribution).



Sample Average

- A straight-forward solution

$$\boldsymbol{\mu} = \mathrm{E}\left(\mathbf{x}\right) \quad \mathbf{R} = \mathrm{E}\left(\mathbf{x} - \boldsymbol{\mu}\right)\left(\mathbf{x} - \boldsymbol{\mu}\right)^T$$

$$\Downarrow$$

$$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \quad \hat{\mathbf{R}} = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{x}_i - \hat{\boldsymbol{\mu}}\right)\left(\mathbf{x}_i - \hat{\boldsymbol{\mu}}\right)^T.$$

- Works well for i.i.d. Gaussian distributed data.

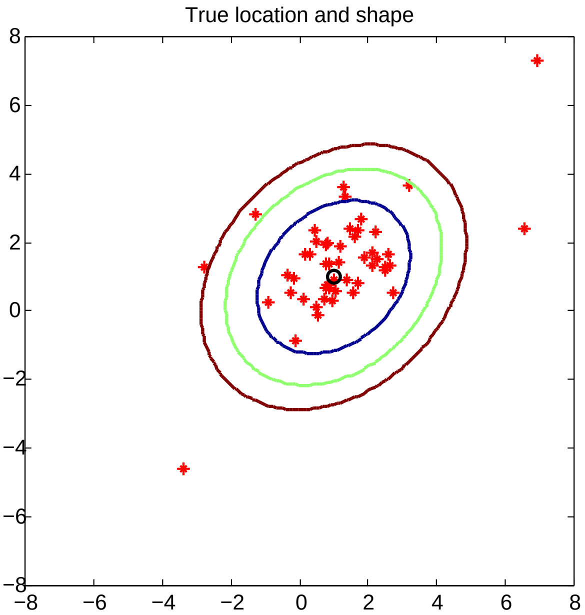
Influence of Outliers

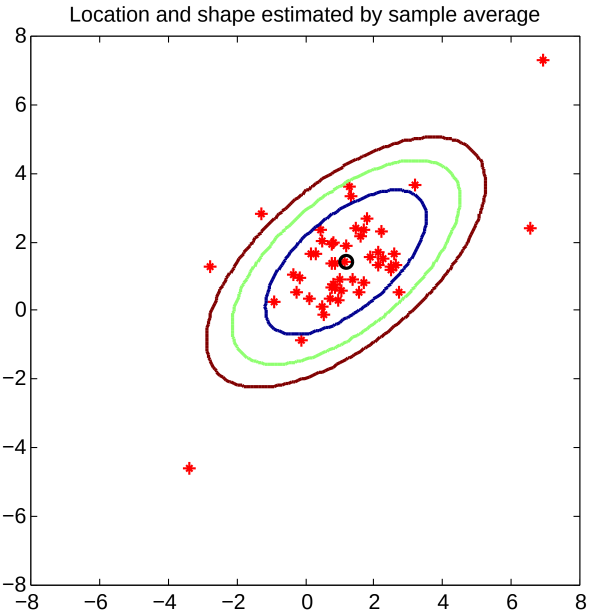
- What if the data is corrupted?
- A real-life example: Kalman filter lost track of the spacecraft during an Apollo mission because of outlier observation (caused by system noise).

Example 1: Symmetrically Distributed Outliers

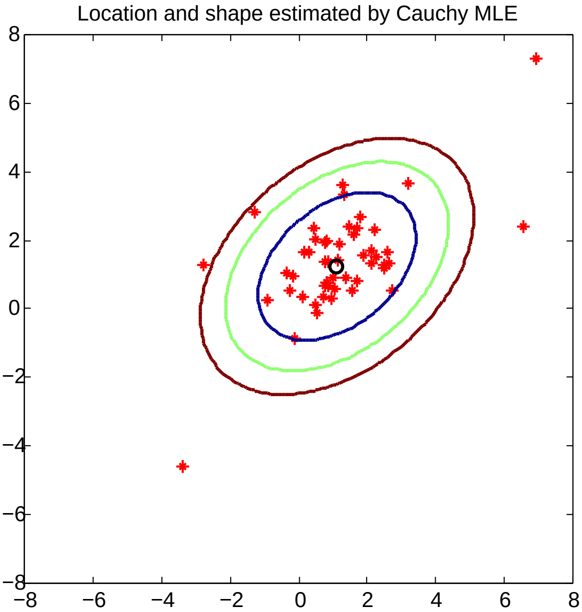
$$\mathbf{x} \sim \text{HeavyTail}\left(\mathbf{1}, \mathbf{R}\right)$$

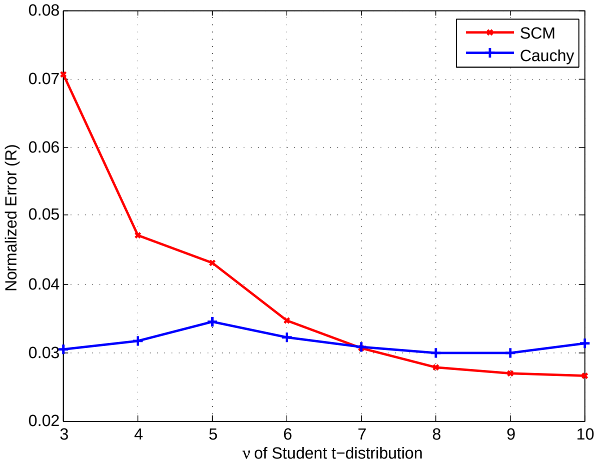
$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$


True location and shape
8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8



Location and shape estimated by sample average
8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8


Location and shape estimated by Cauchy MLE
8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8


SCM
Cauchy
Normalized Error (R)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
3
4
5
6
7
8
9
10
ν of Student t−distribution

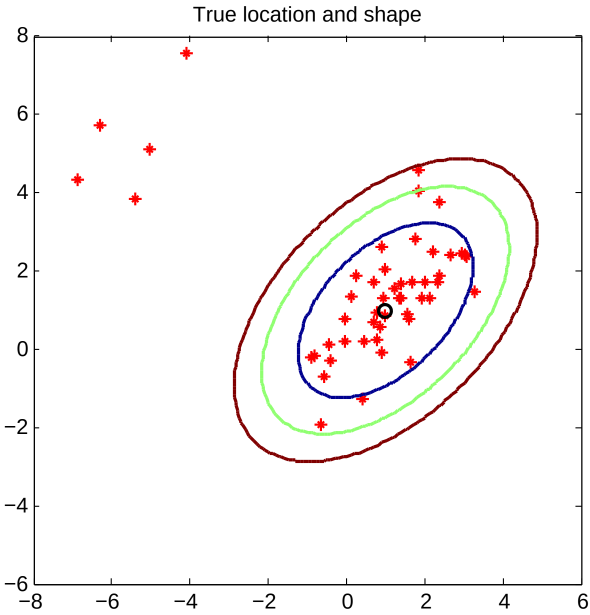
Influence of Outliers

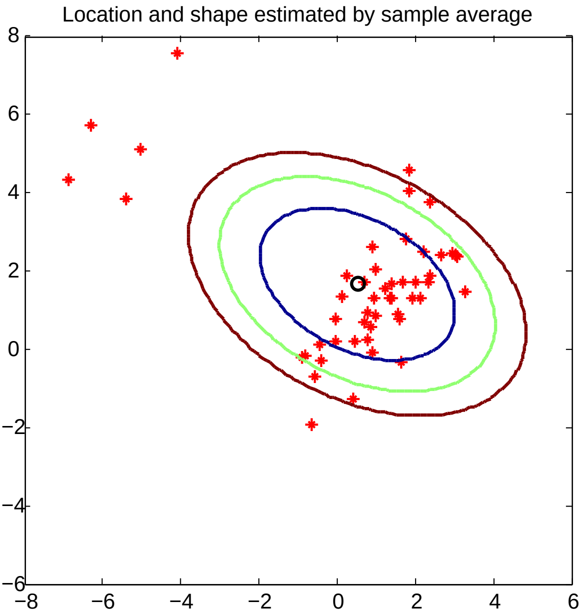
- What if the data is corrupted?

Example 2: Asymmetrically Distributed Outliers

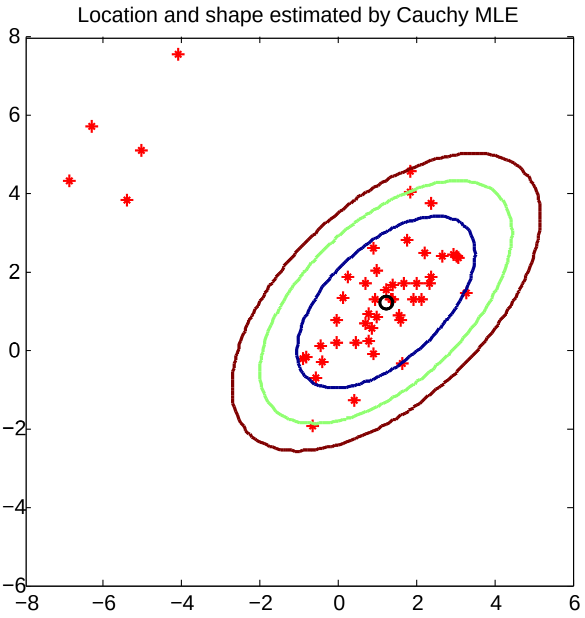
$$\mathbf{x} \sim 0.9\mathcal{N}(\mathbf{1}, \mathbf{R}) + 0.1\mathcal{N}(\boldsymbol{\mu}, \mathbf{R})$$

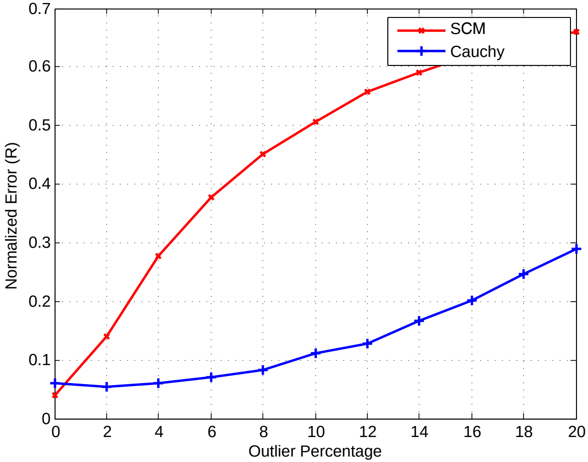
$$\boldsymbol{\mu} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$


True location and shape
8
6
4
2
0
−2
−4
−6
−8
−6
−4
−2
0
2
4
6



Location and shape estimated by sample average
8
6
4
2
0
−2
−4
−6
−8
−6
−4
−2
0
2
4
6

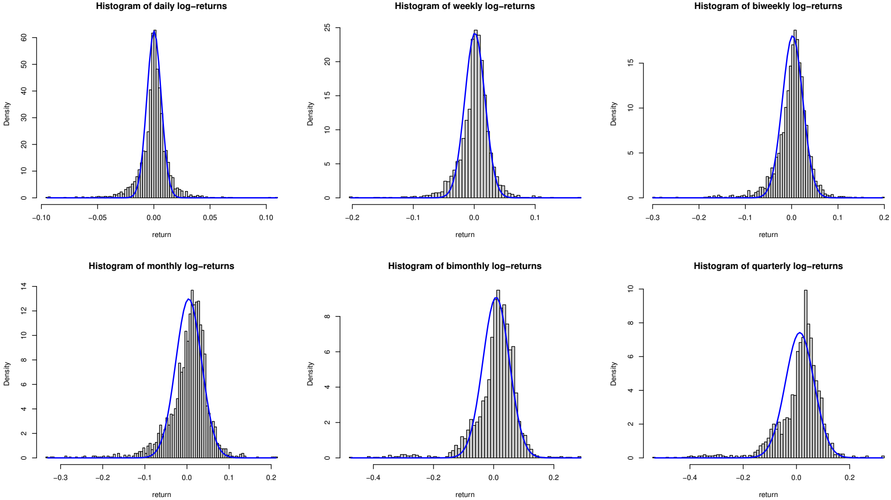

Location and shape estimated by Cauchy MLE
8
6
4
2
0
−2
−4
−6
−8
−6
−4
−2
0
2
4
6



SCM
Cauchy
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Normalized Error (R)
0
2
4
6
8
10
12
14
16
18
20
Outlier Percentage

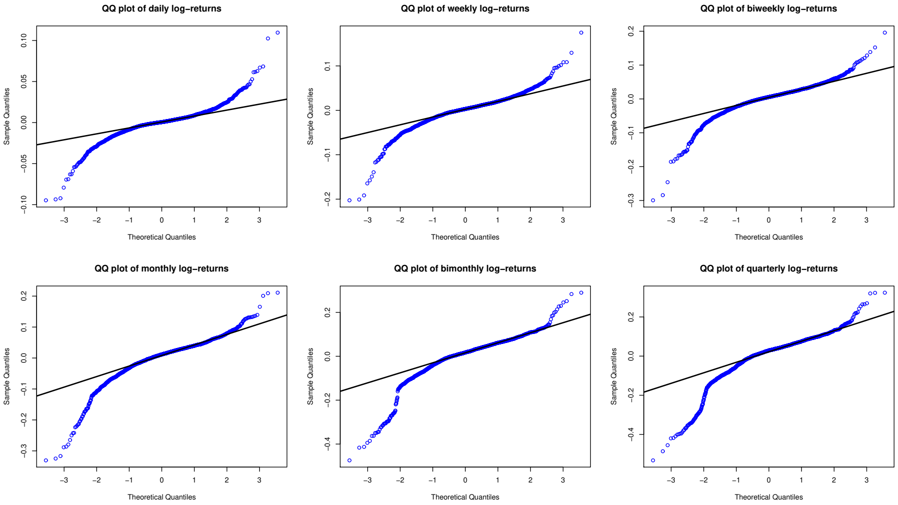
But Is Financial Data Really Heavy-Tailed?

- Histograms of S&P 500 log-returns:



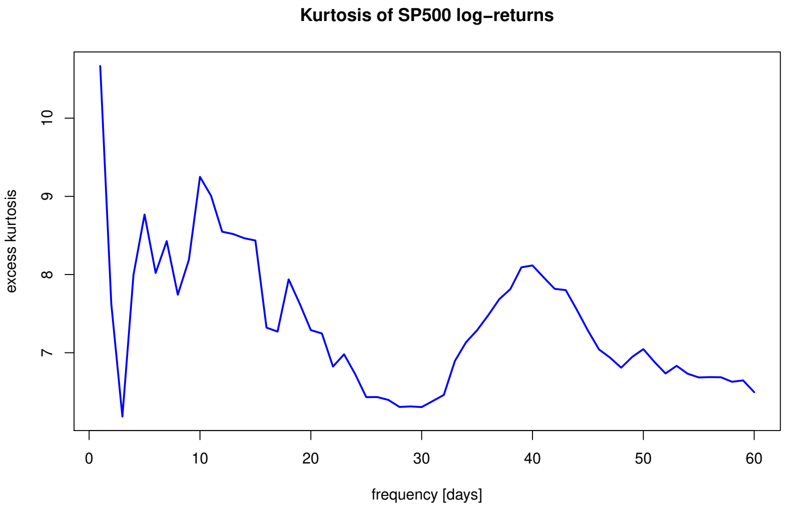
Heavy-tailness

- QQ plots of S&P 500 log-returns:



Heavy-tailness vs frequency

- Kurtosis of S&P 500 log-returns vs frequency:



Outline

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Warm-up

- Recall the Gaussian distribution

$$f(\mathbf{x}) = C \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}\right).$$

- Negative log-likelihood function

$$L(\boldsymbol{\Sigma}) = \frac{N}{2}\log\det(\boldsymbol{\Sigma}) + \frac{1}{2}\sum_{i=1}^{N}\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}.$$

- Sample covariance matrix

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i\mathbf{x}_i^T.$$

M-estimator (1960's)

- Minimizer of loss function [Mar-Mar-Yoh'06]:

$$L(\boldsymbol{\Sigma}) = \frac{N}{2} \log\det(\boldsymbol{\Sigma}) + \sum_{i=1}^{N} \rho\left(\mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i\right).$$

- Solution to fixed-point equation:

$$\boldsymbol{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} w\left(\mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i\right) \mathbf{x}_i \mathbf{x}_i^T.$$

- If ρ is differentiable

$$w = \frac{\rho'}{2}.$$

Outline

Sample Covariance Matrix

- SCM can be viewed as:

$$\hat{\boldsymbol{\Sigma}} = \sum_{i=1}^{N} w_i \mathbf{x}_i \mathbf{x}_i^T$$

with $w_i = \frac{1}{N}, \ \forall i$.

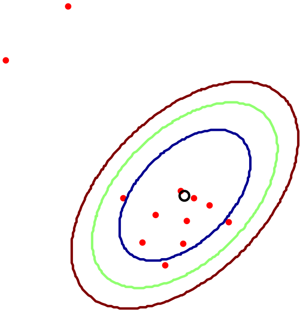
- MLE of a Gaussian distribution with loss function

$$\frac{N}{2} \log\det(\boldsymbol{\Sigma}) + \frac{1}{2} \sum_{i=1}^{N} \mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i.$$

- Why is SCM sensitive to outliers? ☹

Sample Covariance Matrix

- Consider distance
  $d_i = \sqrt{\mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i}.$
- $w_i = \frac{1}{N}$
  normal samples and outliers contribute to $\hat{\boldsymbol{\Sigma}}$ equally.
- Quadratic loss.



Tyler's M-estimator

- Given $f(\mathbf{x}) \rightarrow$ use MLE.
- $\mathbf{x}_i \sim \text{elliptical}(\mathbf{0}, \boldsymbol{\Sigma})$, what shall we do?
- Normalized sample $\mathbf{s}_i \triangleq \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|_2}$

pdf

$$f(\mathbf{s}) = C \det(\mathbf{R})^{-\frac{1}{2}} \left(\mathbf{s}^T \mathbf{R}^{-1} \mathbf{s}\right)^{-K/2}$$

Loss function

$$\frac{N}{2} \log\det(\boldsymbol{\Sigma}) + \frac{K}{2} \sum_{i=1}^{N} \log \underbrace{\left(\mathbf{s}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{s}_i\right)}_{\mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i}$$

- Tyler [Tyl'87] proposed covariance estimator $\hat{\boldsymbol{\Sigma}}$ as solution to

$$\boldsymbol{\Sigma} = \sum_{i=1}^{N} w_i \mathbf{x}_i \mathbf{x}_i^T, \quad w_i = \frac{K}{N\left(\mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i\right)}.$$

- Why is Tyler's estimator robust to outliers? ☺

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pdf

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$$\frac{N}{2} \log \det(\mathbf{\Sigma}) + \frac{K}{2} \sum_{i=1}^{N} \log \underbrace{\left(\mathbf{s}_i^T \mathbf{\Sigma}^{-1} \mathbf{s}_i\right)}_{\mathbf{x}_i^T \mathbf{\Sigma}^{-1} \mathbf{x}_i}$$

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- Why is Tyler's estimator robust to outliers? ☺

Tyler's M-estimator

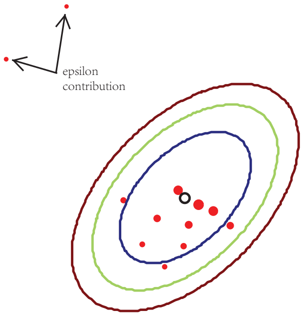
- Consider distance

  $$d_i = \sqrt{\mathbf{x}_i^T \boldsymbol{\Sigma}^{-1} \mathbf{x}_i}.$$

- $w_i \propto 1/d_i^2$

  Outliers are down-weighted.

- Logarithmic loss.



Tyler's *M*-estimator

- Tyler's *M*-estimator solves fixed-point equation

$$\boldsymbol{\Sigma} = \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\boldsymbol{\Sigma}^{-1}\mathbf{x}_i}.$$

- Existence condition: $N > K$.
- No closed-form solution.
- Iterative algorithm

$$\tilde{\boldsymbol{\Sigma}}_{t+1} = \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i}$$

$$\boldsymbol{\Sigma}_{t+1} = \tilde{\boldsymbol{\Sigma}}_{t+1}/\mathrm{Tr}\left(\tilde{\boldsymbol{\Sigma}}_{t+1}\right).$$

Outline

Unsolved Problems

Problem 1

What if the mean value is unknown?

Problem 2

How to deal with small sample scenario?

Problem 3

How to incorporate prior information?

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Outline

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Robust M-estimators for Location and Scatter

- Maronna's M-estimators [Mar'76]:

$$\frac{1}{N}\sum_{i=1}^{N} u_1\left((\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{R}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})\right)(\mathbf{x}_i-\boldsymbol{\mu})=\mathbf{0}$$

$$\frac{1}{N}\sum_{i=1}^{N} u_2\left((\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{R}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})\right)(\mathbf{x}_i-\boldsymbol{\mu})(\mathbf{x}_i-\boldsymbol{\mu})^T=\mathbf{R}.$$

- Special examples:
  - Huber's loss function.
  - MLE for Student's t-distribution.

MLE of the Student's t-distribution

- Student's t-distribution with degree of freedom ν:

$$f(\mathbf{x}) = C \det(\mathbf{R})^{-\frac{1}{2}} \left(1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{R}^{-1} (\mathbf{x}-\boldsymbol{\mu})\right)^{-\frac{K+\nu}{2}}.$$

- Negative log-likelihood

$$L^{\nu}(\boldsymbol{\mu}, \mathbf{R}) = \frac{N}{2} \log\det(\mathbf{R}) + \frac{K+\nu}{2} \sum_{i=1}^{N} \log\left(\nu + (\mathbf{x}_i-\boldsymbol{\mu})^T \mathbf{R}^{-1} (\mathbf{x}_i-\boldsymbol{\mu})\right).$$

MLE of the Student's t-distribution

- Estimating equations

$$\frac{K+\nu}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i-\boldsymbol{\mu}}{\nu+(\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{R}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}=\mathbf{0}$$

$$\frac{K+\nu}{N}\sum_{i=1}^{N}\frac{(\mathbf{x}_i-\boldsymbol{\mu})(\mathbf{x}_i-\boldsymbol{\mu})^T}{\nu+(\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{R}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}=\mathbf{R}.$$

- Weight $w_i(\nu)=\frac{K+\nu}{N}\cdot\frac{1}{\nu+(\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{R}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})}$ decreases in ν.
- Unique solution for $\nu\geq 1$.

Outline

Joint Mean-Covariance Estimation

- Assumption: $\mathbf{x}_i \sim \text{elliptical}\,(\boldsymbol{\mu}_0, \mathbf{R}_0)$.
- Goal: jointly estimate mean and covariance
  - Robust to outliers.
  - Easy to implement.
  - Provable convergence.
- A natural idea:

  MLE of heavy-tailed distributions.

Joint Mean-Covariance Estimation

- Method: fitting $\{\mathbf{x}_i\}$ to Cauchy (Student's t-distribution with $\nu = 1$) likelihood function.
  - Conservative fitting.
  - Trade-off: robustness $\Leftrightarrow$ efficiency.
  - Tractability.
- $\hat{\mathbf{R}} \rightarrow c\mathbf{R}_0$
  c depends on the unknown shape of the underlying distribution $\Longrightarrow$ estimate $\mathbf{R}/\mathrm{Tr}\,(\mathbf{R})$ instead.
- Existence condition $N > K + 1$ [Ken-Tyl'91].

Algorithm

- No closed-form solution.
- Numerical algorithm [Ken-Tyl-Var'94]:

$$\boldsymbol{\mu}_{t+1} = \frac{\sum_{i=1}^{N} w_i\left(\boldsymbol{\mu}_t, \mathbf{R}_t\right)\mathbf{x}_i}{\sum_{i=1}^{N} w_i\left(\boldsymbol{\mu}_t, \mathbf{R}_t\right)}$$

$$\mathbf{R}_{t+1} = \frac{K+1}{N}\sum_{i=1}^{N} w_i\left(\boldsymbol{\mu}_t, \mathbf{R}_t\right)\left(\mathbf{x}_i - \boldsymbol{\mu}_{t+1}\right)\left(\mathbf{x}_i - \boldsymbol{\mu}_{t+1}\right)^T$$

with

$$w_i\left(\boldsymbol{\mu}, \mathbf{R}\right) = \frac{1}{1 + \left(\mathbf{x}_i - \boldsymbol{\mu}\right)^T \mathbf{R}^{-1}\left(\mathbf{x}_i - \boldsymbol{\mu}\right)}.$$

Outline

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Regularization-Known Mean

- Problem:

  insufficient observations → estimator does not exist → algorithms fail to converge

- Methods:
  - Diagonal loading.
  - Penalized or regularized loss function.

Diagonal Loading

- Modified Tyler's iteration [Abr-Spe'07]

$$\tilde{\boldsymbol{\Sigma}}_{t+1} = \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i} + \rho\mathbf{I}$$
$$\boldsymbol{\Sigma}_{t+1} = \tilde{\boldsymbol{\Sigma}}_{t+1}/\mathrm{Tr}\left(\tilde{\boldsymbol{\Sigma}}_{t+1}\right).$$

- Provable convergence [Che-Wie-Her'11].
- Systematic way of choosing parameter ρ [Che-Wie-Her'11].
- But without a clear motivation.

Penalized Loss Function I

- Wiesel's penalty [Wie'12]

$$h(\boldsymbol{\Sigma}) = \log\det(\boldsymbol{\Sigma}) + K \log \mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}\right),$$

  $\boldsymbol{\Sigma} \propto \mathbf{T}$ minimizes $h(\boldsymbol{\Sigma})$.

- Penalized loss function

$$L^{\text{Wiesel}}(\boldsymbol{\Sigma}) = \frac{N}{2}\log\det(\boldsymbol{\Sigma}) + \frac{K}{2}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^T\boldsymbol{\Sigma}^{-1}\mathbf{x}_i\right) + \alpha\left(\log\det(\boldsymbol{\Sigma}) + K\log\mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}\right)\right).$$

- Algorithm

$$\boldsymbol{\Sigma}_{t+1} = \frac{N}{N+2\alpha}\frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i} + \frac{2\alpha}{N+2\alpha}\frac{K\mathbf{T}}{\mathrm{Tr}\left(\boldsymbol{\Sigma}_t^{-1}\mathbf{T}\right)}.$$

Penalized Loss Function II

- Alternative penalty: KL-divergence

$$h(\boldsymbol{\Sigma}) = \log\det(\boldsymbol{\Sigma}) + \mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}\right),$$

  $\boldsymbol{\Sigma} = \mathbf{T}$ minimizes $h(\boldsymbol{\Sigma})$.

- Penalized loss function

$$L^{\mathrm{KL}}(\boldsymbol{\Sigma}) = \frac{N}{2}\log\det(\boldsymbol{\Sigma}) + \frac{K}{2}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^T\boldsymbol{\Sigma}^{-1}\mathbf{x}_i\right)$$
$$+\alpha\left(\log\det(\boldsymbol{\Sigma}) + \mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}\right)\right).$$

- Algorithm?

Questions

Existence & Uniqueness?

Which one is better?

Algorithm convergence?

Questions

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Which one is better?

Algorithm convergence?

Questions

Existence & Uniqueness?

Which one is better?

Algorithm convergence?

Existence and Uniqueness for Wiesel's Shrinkage Estimator

Theorem [Sun-Bab-Pal'14]

Wiesel's shrinkage estimator exists a.s., and is also unique up to a positive scale factor, if and only if the underlying distribution is continuous and $N > K - 2\alpha$.

- Existence condition for Tyler's estimator: $N > K$
  - Regularization relaxes the requirement on the number of samples.
  - Setting $\alpha = 0$ (no regularization) reduces to Tyler's condition.
  - Stronger confidence on the prior information $\Rightarrow$ less number of samples required.

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Existence and Uniqueness for KL-Shrinkage Estimator

Theorem [Sun-Bab-Pal'14]

KL-shrinkage estimator exists a.s., and is also unique, if and only if the underlying distribution is continuous and $N > K - 2\alpha$

Compared with Wiesel's shrinkage estimator:

- Share the same existence condition.
- Without scaling ambiguity.

Any connection? Which one is better?

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Equivalence

Theorem [Sun-Bab-Pal'14]

Wiesel's shrinkage estimator and KL-shrinkage estimator are equivalent.

- Fixed-point equation for KL-shrinkage estimator

$$\mathbf{\Sigma} = \frac{N}{N+2\alpha}\frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\mathbf{\Sigma}^{-1}\mathbf{x}_i} + \frac{2\alpha}{N+2\alpha}\mathbf{T}.$$

- The solution satisfies equality

$$\mathrm{Tr}\left(\mathbf{\Sigma}^{-1}\mathbf{T}\right) = K.$$

- Fixed-point equation for Wiesel's shrinkage estimator

$$\mathbf{\Sigma} = \frac{N}{N+2\alpha}\frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\mathbf{\Sigma}^{-1}\mathbf{x}_i} + \frac{2\alpha}{N+2\alpha}\frac{K\mathbf{T}}{\mathrm{Tr}\left(\mathbf{\Sigma}^{-1}\mathbf{T}\right)}.$$

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- Fixed-point equation for Wiesel's shrinkage estimator

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Majorization-Minimization (MM)

- Problem:

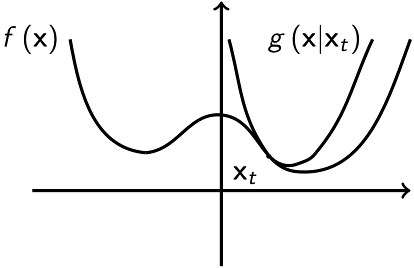
$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{X} \end{array}$$

- Majorization-minimization:

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x}|\mathbf{x}_t)$$

with

$$\begin{array}{l} f(\mathbf{x}_t) = g(\mathbf{x}_t|\mathbf{x}_t) \\ f(\mathbf{x}) \leq g(\mathbf{x}|\mathbf{x}_t) \ \ \forall \mathbf{x} \in \mathcal{X} \\ f'(\mathbf{x}_t;\mathbf{d}) = g'(\mathbf{x}_t;\mathbf{d}|\mathbf{x}_t) \ \ \forall \mathbf{x}_t + \mathbf{d} \in \mathcal{X} \end{array}$$



Modified Algorithm for Wiesel's Shrinkage Estimator

- Surrogate function

$$g\left(\boldsymbol{\Sigma}|\boldsymbol{\Sigma}_t\right) = \frac{N}{2}\log\det\left(\boldsymbol{\Sigma}\right) + \frac{K}{2}\sum_{i=1}^{N}\frac{\mathbf{x}_i^T\boldsymbol{\Sigma}^{-1}\mathbf{x}_i}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i} + \alpha\left(\log\det\left(\boldsymbol{\Sigma}\right) + K\frac{\mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}\right)}{\mathrm{Tr}\left(\boldsymbol{\Sigma}_t^{-1}\mathbf{T}\right)}\right)$$

- Update

$$\tilde{\boldsymbol{\Sigma}}_{t+1} = \frac{N}{N+2\alpha}\frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i} + \frac{2\alpha}{N+2\alpha}\frac{K\mathbf{T}}{\mathrm{Tr}\left(\boldsymbol{\Sigma}_t^{-1}\mathbf{T}\right)}$$

- Normalization

$$\boldsymbol{\Sigma}_{t+1} = \tilde{\boldsymbol{\Sigma}}_{t+1}/\mathrm{Tr}\left(\tilde{\boldsymbol{\Sigma}}_{t+1}\right)$$

Theorem [Sun-Bab-Pal'14]

Under the existence conditions, the modified algorithm for Wiesel's shrinkage estimator converges to the unique solution.

Algorithm for KL-Shrinkage Estimator

- Surrogate function

$$g\left(\boldsymbol{\Sigma}|\boldsymbol{\Sigma}_t\right) = \frac{N}{2}\log\det\left(\boldsymbol{\Sigma}\right) + \frac{K}{2}\sum_{i=1}^{N}\frac{\mathbf{x}_i^T\boldsymbol{\Sigma}^{-1}\mathbf{x}_i}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i}$$
$$+\alpha\left(\log\det\left(\boldsymbol{\Sigma}\right) + \mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}\right)\right)$$

- Update

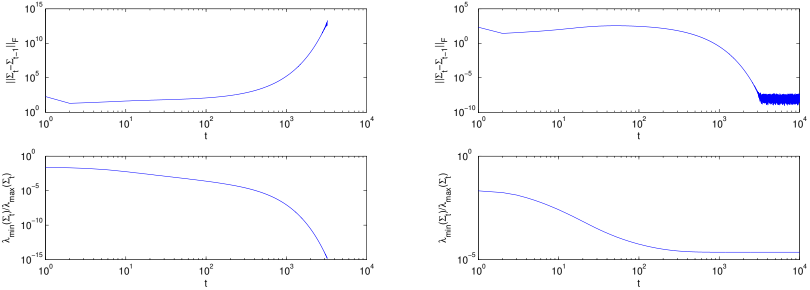
$$\boldsymbol{\Sigma}_{t+1} = \frac{N}{N+2\alpha}\frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\boldsymbol{\Sigma}_t^{-1}\mathbf{x}_i} + \frac{2\alpha}{N+2\alpha}\mathbf{T}$$

Theorem [Sun-Bab-Pal'14]

Under the existence conditions, the algorithm for KL-shrinkage estimator converges to the unique solution.

Algorithm Convergence

- Parameters: $K = 10$, $N = 8$.



(A) (B)

FIGURE: (a) when the existence conditions are not satisfied with $\alpha_0 = 0.96$, and (b) when the existence conditions are satisfied with $\alpha_0 = 1.04$.

Outline

Regularization-Unknown Mean

- Problem: $\boldsymbol{\mu}_0$ is unknown!
- A simple solution: plug-in $\hat{\boldsymbol{\mu}}$
  - Sample mean
  - Sample median
- But...
  - Two-step estimation, not jointly optimal.
  - Estimation error of $\hat{\boldsymbol{\mu}}$ propagates.
- To be done: shrinkage estimator for joint mean-covariance estimation with target $(\mathbf{t}, \mathbf{T})$.

Regularization-Unknown Mean

- Method: adding shrinkage penalty $h(\boldsymbol{\mu}, \mathbf{R})$ to loss function (negative log-likelihood of Cauchy distribution).
- Design criteria:
  - $h(\boldsymbol{\mu}, \mathbf{R})$ attains minimum at prior $(\mathbf{t}, \mathbf{T})$.
  - $h(\mathbf{t}, \mathbf{T}) = h(\mathbf{t}, r\mathbf{T}), \ \forall r > 0$.
- Reason:
  - $\mathbf{R}$ can be estimated up to an unknown scale factor.
  - $\mathbf{T}$ is a prior for the parameter $\mathbf{R}/\mathrm{Tr}(\mathbf{R})$.

Regularization-Unknown Mean

Proposed penalty function

$$h(\boldsymbol{\mu}, \mathbf{R}) = \alpha \left(K \log \left(\mathrm{Tr}\left(\mathbf{R}^{-1}\mathbf{T} \right) \right) + \log\det(\mathbf{R}) \right) + \gamma \log \left(1 + (\boldsymbol{\mu} - \mathbf{t})^T \mathbf{R}^{-1} (\boldsymbol{\mu} - \mathbf{t}) \right)$$

Proposition [Sun-Bab-Pal'15]

$(\mathbf{t}, r\mathbf{T})$, $\forall r > 0$ are the minimizers of $h(\boldsymbol{\mu}, \mathbf{R})$.

Regularization-Unknown Mean

Proposed penalty function

$$h(\boldsymbol{\mu}, \mathbf{R}) = \alpha \left(K \log \left(\mathrm{Tr} \left(\mathbf{R}^{-1} \mathbf{T} \right) \right) + \log \det (\mathbf{R}) \right) + \gamma \log \left(1 + (\boldsymbol{\mu} - \mathbf{t})^T \mathbf{R}^{-1} (\boldsymbol{\mu} - \mathbf{t}) \right)$$

Proposition [Sun-Bab-Pal'15]

$(\mathbf{t}, r\mathbf{T})$, $\forall r > 0$ are the minimizers of $h(\boldsymbol{\mu}, \mathbf{R})$.

Regularization-Unknown Mean

- Resulting optimization problem:

$$\begin{aligned}\underset{\boldsymbol{\mu},\mathbf{R}\succ\mathbf{0}}{\text{minimize}} \quad & \frac{(K+1)}{2}\sum_{i=1}^{N}\log\left(1+(\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{R}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})\right) \\ & +\alpha\left(K\log\left(\mathrm{Tr}\left(\mathbf{R}^{-1}\mathbf{T}\right)\right)+\log\det\left(\mathbf{R}\right)\right) \\ & +\gamma\log\left(1+(\boldsymbol{\mu}-\mathbf{t})^T\mathbf{R}^{-1}(\boldsymbol{\mu}-\mathbf{t})\right)+\frac{N}{2}\log\det\left(\mathbf{R}\right).\end{aligned}$$

- A minimum satisfies the stationary condition $\frac{\partial L^{\text{shrink}}(\boldsymbol{\mu},\mathbf{R})}{\partial\boldsymbol{\mu}}=\mathbf{0}$ and $\frac{\partial L^{\text{shrink}}(\boldsymbol{\mu},\mathbf{R})}{\partial\mathbf{R}}=\mathbf{0}$.

Regularization-Unknown Mean

- $d_i(\boldsymbol{\mu}, \mathbf{R}) = \sqrt{(\mathbf{x}_i - \boldsymbol{\mu})^T \mathbf{R}^{-1} (\mathbf{x}_i - \boldsymbol{\mu})}$, $d_{\mathbf{t}}(\boldsymbol{\mu}, \mathbf{R}) = \sqrt{(\mathbf{t} - \boldsymbol{\mu})^T \mathbf{R}^{-1} (\mathbf{t} - \boldsymbol{\mu})}$.
- $w_i(\boldsymbol{\mu}, \mathbf{R}) = \frac{1}{1+d_i^2(\boldsymbol{\mu}, \mathbf{R})}$, $w_{\mathbf{t}}(\boldsymbol{\mu}, \mathbf{R}) = \frac{1}{1+d_{\mathbf{t}}^2(\boldsymbol{\mu}, \mathbf{R})}$.
- Stationary condition:

$$\mathbf{R} = \frac{K+1}{N+2\alpha} \sum_{i=1}^{N} w_i(\boldsymbol{\mu}, \mathbf{R}) (\mathbf{x}_i - \boldsymbol{\mu}) (\mathbf{x}_i - \boldsymbol{\mu})^T + \frac{2\gamma}{N+2\alpha} w_{\mathbf{t}}(\boldsymbol{\mu}, \mathbf{R}) (\boldsymbol{\mu} - \mathbf{t}) (\boldsymbol{\mu} - \mathbf{t})^T + \frac{2\alpha K}{N+2\alpha} \frac{\mathbf{T}}{\mathrm{Tr}(\mathbf{R}^{-1}\mathbf{T})}$$

$$\boldsymbol{\mu} = \frac{(K+1)\sum_{i=1}^{N} w_i(\boldsymbol{\mu}, \mathbf{R}) \mathbf{x}_i + 2\gamma w_{\mathbf{t}}(\boldsymbol{\mu}, \mathbf{R}) \mathbf{t}}{(K+1)\sum_{i=1}^{N} w_i(\boldsymbol{\mu}, \mathbf{R}) + 2\gamma w_{\mathbf{t}}(\boldsymbol{\mu}, \mathbf{R})}$$

Existence and Uniqueness

Theorem [Sun-Bab-Pal'15]

Assuming continuous underlying distribution, the estimator exists under either of the following conditions:

(i) if $\gamma > \gamma_1$, then $\alpha > \alpha_1$,

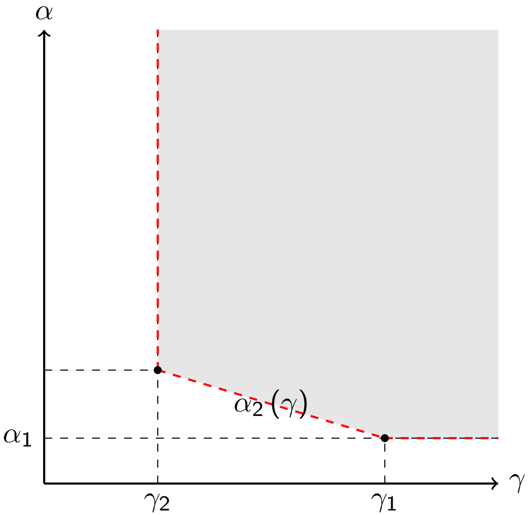
(ii) if $\gamma_2 < \gamma \leq \gamma_1$, then $\alpha > \alpha_2(\gamma)$,

where

$$\alpha_1 = \frac{1}{2}(K - N),$$

$$\alpha_2(\gamma) = \frac{1}{2}\left(K + 1 - N - \frac{2\gamma + N - K - 1}{N - 1}\right),$$

and $\gamma_1 = \frac{1}{2}(K + 1)$, $\gamma_2 = \frac{1}{2}(K + 1 - N)$.

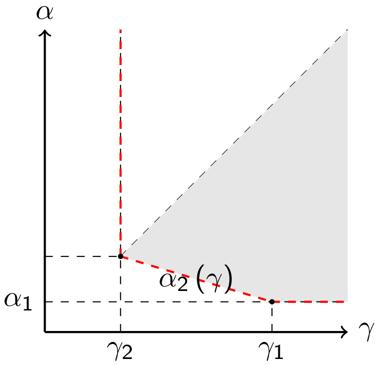


α
$\alpha_2(\gamma)$
α_1
γ
γ_2
γ_1

Existence and Uniqueness

Theorem [Sun-Bab-Pal'15]

The shrinkage estimator is unique if $\gamma \geq \alpha$.



Algorithm in μ and R

- Surrogate function

$$\begin{aligned} L\left(\boldsymbol{\mu},\mathbf{R}|\boldsymbol{\mu}_t,\mathbf{R}_t\right) &= \tfrac{K+1}{2}\sum w_i\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)\left(\mathbf{x}_i-\boldsymbol{\mu}\right)^T\mathbf{R}^{-1}\left(\mathbf{x}_i-\boldsymbol{\mu}\right) \\ &+\gamma w_{\mathbf{t}}\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)\left(\mathbf{t}-\boldsymbol{\mu}\right)^T\mathbf{R}^{-1}\left(\mathbf{t}-\boldsymbol{\mu}\right) \\ &+\left(\tfrac{N}{2}+\alpha\right)\log\det\left(\mathbf{R}\right)+\alpha K\frac{\mathrm{Tr}\left(\mathbf{R}^{-1}\mathbf{T}\right)}{\mathrm{Tr}\left(\mathbf{R}_t^{-1}\mathbf{T}\right)} \end{aligned}$$

- Update

$$\boldsymbol{\mu}_{t+1}=\frac{(K+1)\sum_{i=1}^{N}w_i\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)\mathbf{x}_i+2\gamma w_{\mathbf{t}}\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)\mathbf{t}}{(K+1)\sum_{i=1}^{N}w_i\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)+2\gamma w_{\mathbf{t}}\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)}$$

$$\begin{aligned} \mathbf{R}_{t+1} &= \frac{K+1}{N+2\alpha}\sum_{i=1}^{N}w_i\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)\left(\mathbf{x}_i-\boldsymbol{\mu}_{t+1}\right)\left(\mathbf{x}_i-\boldsymbol{\mu}_{t+1}\right)^T \\ &+\frac{2\gamma}{N+2\alpha}w_{\mathbf{t}}\left(\boldsymbol{\mu}_t,\mathbf{R}_t\right)\left(\mathbf{t}-\boldsymbol{\mu}_{t+1}\right)\left(\mathbf{t}-\boldsymbol{\mu}_{t+1}\right)^T+\frac{2\alpha K}{N+2\alpha}\frac{\mathbf{T}}{\mathrm{Tr}\left(\mathbf{R}_t^{-1}\mathbf{T}\right)} \end{aligned}$$

Algorithm in μ and R

Theorem [Sun-Bab-Pal'15]

Under the existence and uniqueness conditions, the algorithm in $\boldsymbol{\mu}$ and $\mathbf{R}$ for the proposed shrinkage estimator converges to the unique solution.

Algorithm in $\boldsymbol{\Sigma}$

- Consider case $\alpha = \gamma$, apply transform

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{R} + \boldsymbol{\mu}\boldsymbol{\mu}^T & \boldsymbol{\mu} \\ \boldsymbol{\mu}^T & 1 \end{bmatrix}$$

$$\bar{\mathbf{x}}_i = [\mathbf{x}_i; 1], \quad \bar{\mathbf{t}} = [\mathbf{t}; 1]$$

- Equivalent loss function

$$L^{\text{shrink}}(\boldsymbol{\Sigma}) = \left(\frac{N}{2} + \alpha\right) \log\det(\boldsymbol{\Sigma}) + \frac{K+1}{2} \sum_{i=1}^{N} \log\left(\bar{\mathbf{x}}_i^T \boldsymbol{\Sigma}^{-1} \bar{\mathbf{x}}_i\right)$$
$$+ \alpha K \log\left(\mathrm{Tr}\left(\mathbf{S}^T \boldsymbol{\Sigma}^{-1} \mathbf{S}\mathbf{T}\right)\right) + \alpha \log\left(\bar{\mathbf{t}}^T \boldsymbol{\Sigma}^{-1} \bar{\mathbf{t}}\right)$$

with $\mathbf{S} = \begin{bmatrix} \mathbf{I}_K \\ \mathbf{0}_{1\times K} \end{bmatrix}$.

- $L^{\text{shrink}}(\boldsymbol{\Sigma})$ is scale-invariant.

Algorithm in $\boldsymbol{\Sigma}$

- Surrogate function

$$L\left(\boldsymbol{\Sigma}|\boldsymbol{\Sigma}_t\right) = \left(\frac{N}{2}+\alpha\right)\log\det\left(\boldsymbol{\Sigma}\right) + \frac{K+1}{2}\sum_{i=1}^{N}\frac{\bar{\mathbf{x}}_i^T\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}_i}{\bar{\mathbf{x}}_i^T\boldsymbol{\Sigma}_t^{-1}\bar{\mathbf{x}}_i}$$
$$+\alpha\left(K\frac{\mathrm{Tr}\left(\mathbf{S}^T\boldsymbol{\Sigma}^{-1}\mathbf{S}\mathbf{T}\right)}{\mathrm{Tr}\left(\mathbf{S}^T\boldsymbol{\Sigma}_t^{-1}\mathbf{S}\mathbf{T}\right)} + \frac{\bar{\mathbf{t}}^T\boldsymbol{\Sigma}^{-1}\bar{\mathbf{t}}}{\bar{\mathbf{t}}^T\boldsymbol{\Sigma}_t^{-1}\bar{\mathbf{t}}}\right)$$

- Update

$$\tilde{\boldsymbol{\Sigma}}_{t+1} = \frac{K+1}{N+2\alpha}\sum_{i=1}^{N}\frac{\bar{\mathbf{x}}_i\bar{\mathbf{x}}_i^T}{\bar{\mathbf{x}}_i^T\boldsymbol{\Sigma}_t^{-1}\bar{\mathbf{x}}_i}$$
$$+\frac{2\alpha}{N+2\alpha}\left(\frac{K\mathbf{S}\mathbf{T}\mathbf{S}^T}{\mathrm{Tr}\left(\mathbf{S}^T\boldsymbol{\Sigma}_t^{-1}\mathbf{S}\mathbf{T}\right)} + \frac{\bar{\mathbf{t}}\bar{\mathbf{t}}^T}{\bar{\mathbf{t}}^T\boldsymbol{\Sigma}_t^{-1}\bar{\mathbf{t}}}\right)$$

$$\boldsymbol{\Sigma}_{t+1} = \tilde{\boldsymbol{\Sigma}}_{t+1}/\left(\tilde{\boldsymbol{\Sigma}}_{t+1}\right)_{K+1,K+1}$$

Algorithm in $\boldsymbol{\Sigma}$

Theorem [Sun-Bab-Pal'15]

Under the existence conditions, which simplifies to $N > K + 1 - 2\alpha$ for $\alpha = \gamma$, the algorithm in $\boldsymbol{\Sigma}$ for the proposed shrinkage estimator converges to the unique solution.

Simulations

- Parameters: $K = 100$

$$\boldsymbol{\mu}_0 = \mathbf{1}_{K\times 1}$$

$$(\mathbf{R}_0)_{ij} = 0.8^{|i-j|}$$

- Error measurement: KL-distance

$$\mathrm{err}\left(\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}}\right) = E\left\{D_{KL}\left(\mathcal{N}\left(\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}}\right)\|\mathcal{N}\left(\boldsymbol{\mu}_0, \mathbf{R}_0\right)\right)\right.$$
$$\left. + D_{KL}\left(\mathcal{N}\left(\boldsymbol{\mu}_0, \mathbf{R}_0\right)\middle\|\mathcal{N}\left(\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}}\right)\right)\right\}$$

Performance Comparison for Gaussian

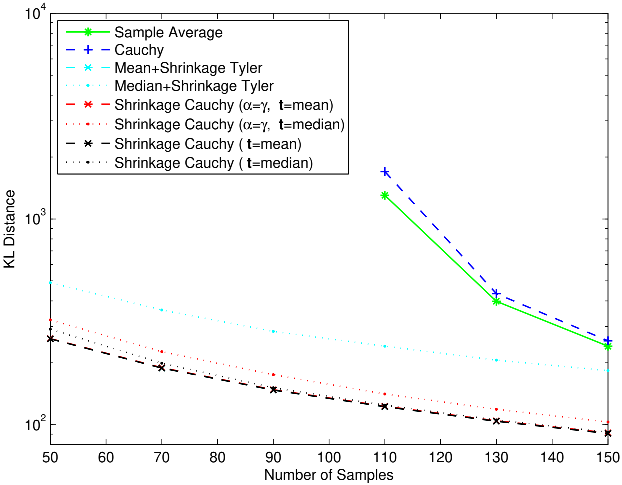


FIGURE: $\mathbf{x}_i \sim \mathcal{N}(\boldsymbol{\mu}_0, \mathbf{R}_0)$

Performance Comparison for t-distribution ($\nu = 3$)

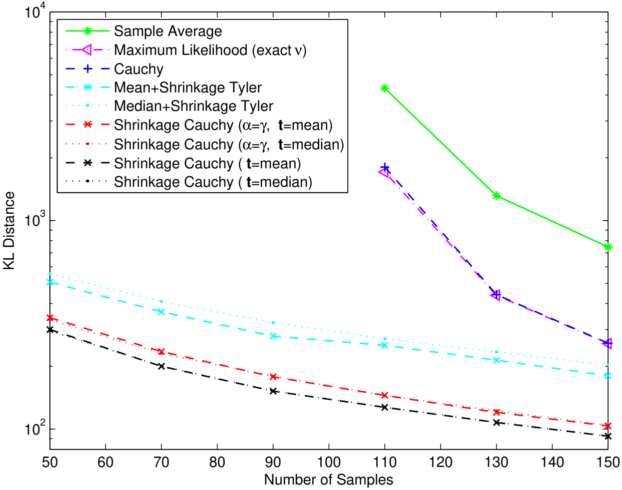


FIGURE: $\mathbf{x}_i \sim t_3\left(\boldsymbol{\mu}_0, \mathbf{R}_0\right)$

Performance Comparison for Elliptical Distribution

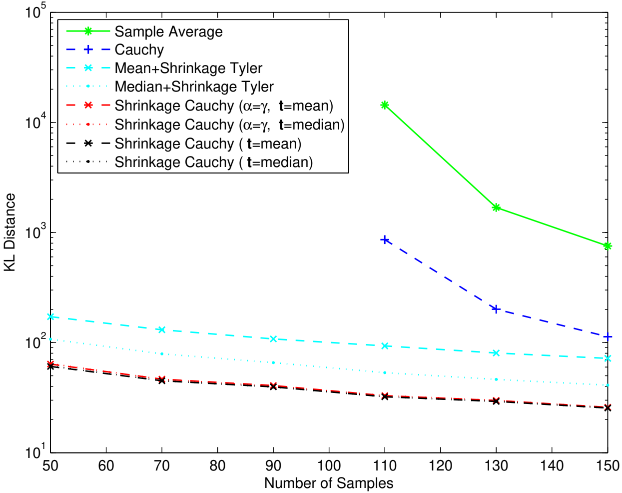
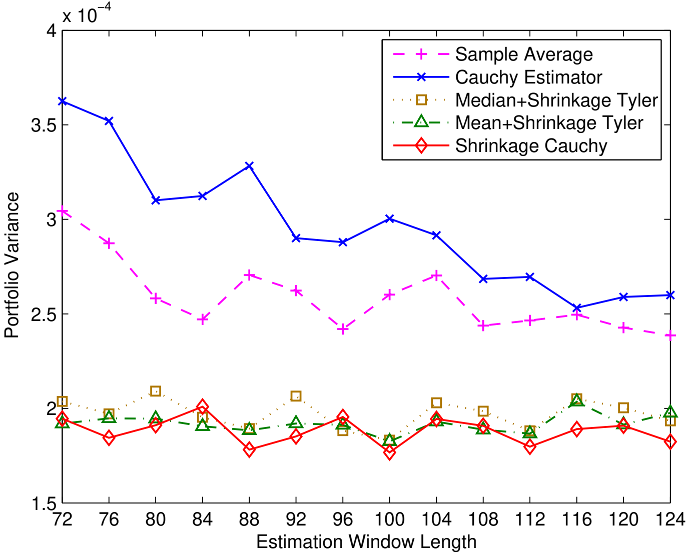


FIGURE: $\mathbf{x}_i \sim \boldsymbol{\mu}_0 + \sqrt{\tau}\mathbf{u}$, $\tau \sim \chi^2$, $\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_0)$

Real Data Simulation

- Minimum variance portfolio.
- Training : S&P 500 index components weekly log-returns, $K = 40$.
  - Estimate $\mathbf{R}$
  - Construct portfolio weights $\mathbf{w}$
- Parameter selection: choose α yields minimum variance on validation set.
- Collect half a year portfolio returns.

$\cdots$	train	validate	test	$\cdots$


x 10^-4
4
3.5
3
2.5
2
1.5
Portfolio Variance
72
76
80
84
88
92
96
100
104
108
112
116
120
124
Estimation Window Length
Sample Average
Cauchy Estimator
Median+Shrinkage Tyler
Mean+Shrinkage Tyler
Shrinkage Cauchy

Summary

- In this lecture, we have discussed:
  - Robust mean-covariance estimation for heavy-tailed distributions via Tyler estimator
  - Shrinkage estimation in small sample scenario.

  - Robust mean-covariance estimation for heavy-tailed distributions via Cauchy's MLE estimator
  - Shrinkage estimation in small sample scenario.

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Thanks

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