Designing MIMO Communication Systems: Constellation Choice and Linear Transceiver Design

Daniel Pérez Palomar, *Member, IEEE,* and Sergio Barbarossa, *Member, IEEE*

***Abstract*—This papers considers the design of a mutiple-input multiple-output (MIMO) communication system with channel knowledge at the transmitter and receiver. The design methods available in the literature have addressed the following two aspects of the problem: a) choice of the symbol constellations for a given transmission scheme or b) choice of the optimal (linear) precoder and equalizer for a given choice of the constellations. More specifically, the choice of the constellations has been made enforcing a diagonal, or parallel, transmission. However, in practice, the two problems of choosing the constellations and the linear precoder/equalizer are clearly coupled, and the diagonal structure may not be necessarily the best. This paper attempts to provide a global view of the problem by bridging the gap between the existing results on the selection of the constellations and on the design of the signal processing in the form of a linear transceiver (i.e., precoding at the transmitter and equalization at the receiver).**

***Index Terms*—Constellation choice, diagonal structure, gap approximation, linear precoding, linear transceiver, MIMO system, parallel transmission.**

I. Introduction

MULTIPLE-INPUT multiple-output (MIMO) channels provide a general framework for modeling a series of different communication systems of diverse physical nature, ranging from single-antenna frequency-selective channels [1], to wireless multiantenna systems [2], [3], and to wireline digital subscriber line (DSL) systems [4]. This abstract modeling allows for a unified treatment using a very convenient vector-matrix notation.

The focus of this paper is on point-to-point MIMO communication systems with perfect channel state information (CSI) at both sides of the link. In such a case, the system can adapt to each channel realization to improve the spectral efficiency and/or reliability of the communication. From a fundamental point of view, the optimal design, in terms of maximum information rate, is well known: The MIMO channel has to be diagonalized and ideal Gaussian codes have to be transmitted through the channel eigenmodes with a waterfilling power profile [2], [5]–[7]. In practice, the ideal Gaussian codes are substituted with finite order constellations [such as quadrature amplitude modulation (QAM) constellations] and practical coding schemes. Furthermore, to simplify the design of such a system, it is customary to divide it into an inner uncoded part, which transmits symbols drawn from given constellations, and an outer coded part that adds redundancy in order to include error correction capabilities. Although the ultimate system performance depends on the combination of both parts (in fact, for some systems, such a division does not even apply), it is convenient, from the mathematical tractability point of view, to concentrate on the uncoded part, independently of the error correction block, to simplify the analysis and design.

The uncoded part of a system can be divided into two blocks: the constellation mapping and the signal processing. The constellation mapping refers to how the data bits are mapped into points of a constellation, whereas the signal processing refers to any additional processing in the form of precoding at the transmitter and equalization at the receiver that modifies the channel into an equivalent channel. In particular, the focus will be on linear signal processing, which is also termed linear transceiver, for complexity reasons (this choice is also supported from the fact that a linear transceiver is optimal from an information- theoretic viewpoint). Linear processing is typically used, for example, to convert the given MIMO channel into a set of independent channels. The overall system can then be schematized as shown in Fig. 1. Roughly speaking, the existing results in the literature for MIMO channels include, on the one hand, the choice of the constellations under a diagonal or parallel transmission and, on the other hand, the optimization of the linear transceiver without explicitly referring to the constellations used. However, a unified view of the two problems is missing (an exception is [8], where the joint design of the constellations and linear transceiver is considered under a perfect reconstruction criterion, and the corresponding extension to multiservice communications in [9]).

As far as the choice of the constellations is concerned, the gap approximation method gives approximately the best constellations to be used on a set of parallel subchannels to guarantee a given error probability on each of the subchannels [10]–[13]. Such a result is typically used on MIMO channels by first diagonalizing the channel matrix and then using the channel eigenmodes as parallel subchannels (following the guidelines dictated by the capacity-achieving solution). However, such combination of channel diagonalization and employment of the gap approximation over the channel eigenmodes has never been proved to be optimal. This paper sheds some light into this problem by clarifying when it is indeed optimal and quantifying how suboptimal it can be.

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D. P. Palomar was with the Technical University of Catalonia (UPC), Barcelona, Spain, and was also with the University of Rome "La Sapienza," Rome, Italy. He is now with the Department of Electrical Engineering, Princeton University, Princeton, NJ 08544 (e-mail: danielp@princeton.edu).

S. Barbarossa is with the INFOCOM Department, University of Rome "La Sapienza," 00184 Rome, Italy (e-mail: sergio@infocom.uniroma1.it).

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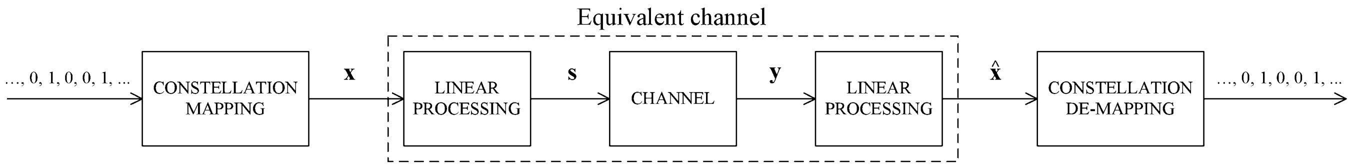

Fig. 1. Division of the uncoded part of a communication system into two blocks: the constellation mapping and the linear processing (which modifies the channel into an equivalent channel).

Regarding the design of the linear MIMO transceiver, many results are available, but either without making any comment about the constellations or assuming that they have been previously chosen, regardless of the linear transceiver scheme. The classical approach refers to the minimization of the sum of the mean square error (MSE) of all subchannels or, equivalently, the trace of the MSE matrix with a fixed transmit power [14], [1], [15]. Some others results consider the maximization of the signal to interference-plus-noise ratio (SINR) [1]. In [16], a general unifying framework was developed to consider a wide range of different design criteria; in particular, the optimal design was obtained for the family of Schur-concave and Schur-convex cost functions. However, rather than the MSE or the SINR, the ultimate performance of a system is given by the bit error rate (BER), which is more difficult to handle. In [17], the minimization of the BER (and also of the Chernoff upper bound) averaged over the subchannels was treated in detail when a diagonal structure was imposed. Recently, the minimum BER design without the diagonal structure constraint was independently obtained in [16] and [18] for the case of equal constellations, resulting in an optimal nondiagonal structure. The generalization to different constellations was obtained in [19]. In any case, the constellations are always assumed to be known (previously chosen somehow). Note that as opposed to fixing the transmit power and optimizing some measure of quality, it is also possible to minimize the transmit power subject to some minimum global quality of the system or, even better, subject to an independent quality for each of the subchannels [20].

This paper addresses the problem of designing both the constellations and the linear transceiver (for a given BER) of a point-to-point MIMO communication system with CSI. It attempts to provide a global view of the problem by bridging the gap between the existing independent results on both problems. In other words, it deals with the multiobjective optimization problem involving the three fundamental parameters: BER, rate, and power. After an overview of different methods to select the constellations, the optimality of the diagonal transmission strategy is assessed for each of the methods, showing when it is an optimal transmission depending on the BER requirements. Since the diagonal structure is adopted in many current systems, we also quantify the performance loss resulting from its use when it is not optimal. A special emphasis will be placed on the well-known gap approximation as a sounded method to select the constellations; in particular, in Theorem 1, an approximate optimal solution is given for the joint design of the constellations and linear transceiver.

The paper is structured as follows. In Section II, the signal model is introduced, the problem is formulated, and some preliminary results are reviewed. Section III gives an overview of different methods for choosing the constellations. The main results of the paper are obtained in Section IV, where the diagonal structure is analyzed in detail. Section V confronts the results of the two previous sections and summarizes the whole paper. Finally, Section VI concludes the paper.

The following notation is used. Boldface upper-case letters denote matrices, boldface lower-case letters denote column vectors, and italics denote scalars. $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ represent the set of $m \times n$ matrices with real- and complex-valued entries, respectively. The superscripts $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ denote transpose, complex conjugate, and Hermitian operations, respectively. $[\mathbf{X}]_{i,j}$ (also $[\mathbf{X}]_{ij}$) denotes the (ith, jth) element of matrix $\mathbf{X}$. Tr$(\cdot)$ and $\det(\cdot)$ denote the trace and the determinant of a matrix, respectively. The operator $(x)^+ \triangleq \max(0, x)$ is the projection on the nonnegative orthant.

II. System Model and Problem Formulation

In this section, first, the basic signal model for MIMO channels and linear transceivers is introduced; then, the problem addressed in the paper is explicitly formulated; finally, some preliminary results on the design of linear MIMO transceivers are given.

A. MIMO Signal Model

The signal model corresponding to a transmission through a general MIMO communication channel with n_T transmit and n_R receive dimensions is

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n} \tag{1}$$

where $\mathbf{s} \in \mathbb{C}^{n_T\times 1}$ is the transmitted vector, $\mathbf{H} \in \mathbb{C}^{n_R\times n_T}$ is the channel matrix, $\mathbf{y} \in \mathbb{C}^{n_R\times 1}$ is the received vector, and $\mathbf{n} \in \mathbb{C}^{n_R\times 1}$ is a zero-mean circularly symmetric complex Gaussian interference-plus-noise vector with arbitrary covariance matrix $\mathbf{R}_n$, i.e., $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_n)$. In some situations (such as in multicarrier systems), it may be useful to model the system as a set of N parallel and noninterfering MIMO channels, for which the results of this paper also hold.

A linear transceiver is composed of a linear precoding at the transmitter and a linear equalization at the receiver. The transmitted vector can be written as (see Fig. 2)

$$\mathbf{s} = \mathbf{B}\mathbf{x} \tag{2}$$

where $\mathbf{B} \in \mathbb{C}^{n_T\times L}$ is the transmit matrix (linear precoder) and $\mathbf{x} \in \mathbb{C}^{L\times 1}$ is the data vector that contains the L symbols to be transmitted (zero-mean,[1] normalized, and uncorrelated, i.e.,

[1]If a constellation does not have zero mean, the receiver can always remove the mean and then proceed as if the mean was zero, resulting in a loss of transmitted power. Indeed, the mean of the signal does not carry any information and can always be set to zero saving power at the transmitter.

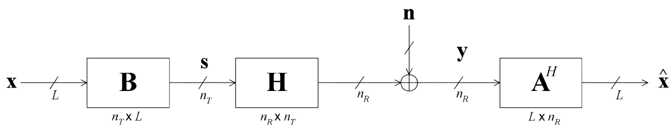

Fig. 2. Scheme of a general MIMO communication system with a linear transceiver.

$\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}$) drawn from a set of finite-order constellations $\{\mathcal{C}_i\}_{i=1}^L$, i.e., $x_i \in \mathcal{C}_i$. Note that different symbols may be drawn from different constellations. For the sake of notation, it is assumed that $L \leq \min(n_R, n_T)$. The total average transmitted power (in units of energy per transmission) is

$$\mathbb{E}[\|\mathbf{s}\|^2] = \mathrm{Tr}(\mathbf{B}\mathbf{B}^H). \tag{3}$$

Similarly, the estimated data vector at the receiver is (see Fig. 2)

$$\hat{\mathbf{x}} = \mathbf{A}^H \mathbf{y} \tag{4}$$

where $\mathbf{A}^H \in \mathbb{C}^{L \times n_R}$ is the receive matrix (linear equalizer).

B. Problem Formulation

This paper attempts to characterize the Pareto optimal solutions[2] of the multiobjective optimization problem corresponding to the uncoded part of the system with the three following objectives: (uncoded) BER, (uncoded) rate, and transmit power. To be more specific, given the signal model $\hat{\mathbf{x}} = \mathbf{A}^H(\mathbf{H}\mathbf{B}\mathbf{x} + \mathbf{n})$, where the symbols $x_i \in \mathcal{C}_i$, the optimization of the uncoded part of the system corresponds to the design of the linear transceiver $(\mathbf{B}, \mathbf{A})$ and to the choice of the constellations $\{\mathcal{C}_i\}_{i=1}^L$. We now comment on each of the objectives or parameters to be optimized.

- The global average BER of the system is perhaps the best way to characterize the quality of the system with a single parameter:

$$\mathrm{BER}_0 = \frac{1}{L}\sum_{i=1}^{L} \mathrm{BER}_i \tag{5}$$

  where BER_i is the BER of the ith subchannel. Another equally good approach is to consider the same BER constraint on each of the subchannels:

$$\mathrm{BER}_0 = \mathrm{BER}_i \quad 1 \leq i \leq L. \tag{6}$$

  In fact, as was obtained in [19], both types of BER constraints are essentially the same and there is no significant difference. The reason is that the average BER is strongly dominated by the minimum of the individual BERs and, hence, an optimized system must necessarily have almost equal BERs on the subchannels. For the rest of the paper, equal BERs on each subchannel as in (6) are considered [since it is analytically more convenient than (5)].

- The rate of the system is trivially defined as the number of transmitted bits

$$R_0 = \sum_{i=1}^{L} \log_2 |\mathcal{C}_i| \tag{7}$$

  where $|\mathcal{C}_i|$ denotes the size of the constellation $\mathcal{C}_i$. For a given R_0, it is not clear what is the best combination of the constellations $\{\mathcal{C}_i\}$. In principle, a full search should be done over all the combinations that give the desired rate; however, as will be seen later in Section III-B, there are simple methods to choose quasioptimal combinations such as the gap approximation.
- Finally, the required transmit power is given by $P_0 = \mathrm{Tr}(\mathbf{B}\mathbf{B}^H)$.

Since the system is characterized with three parameters, a simple way to obtain Pareto optimal solutions is to fix two of the competing parameters and optimize the third one with respect to the optimization variables $(\mathbf{B}, \mathbf{A})$ and $\{\mathcal{C}_i\}_{i=1}^L$. To be more specific, we will fix the BER and the rate to finally minimize the required transmit power. The main purpose of the paper is to obtain a simple and quasioptimal way to choose the constellations and to design the linear transceiver.

C. Preliminaries on the Design of Linear MIMO Transceivers

In this section, we will briefly review the design of the linear transceiver assuming that the constellations have already been chosen. In particular, we will consider the minimization of the transmitted power subject to independent quality constraints on each subchannel; for example, BER constraints as in (6), SINR constraints, or MSE constraints. As will be argued later, all three types of constraints essentially reduce to the same problem [20]. For simplicity of exposition, the problem is formulated with MSE constraints of the form $\mathrm{MSE}_i \leq \rho_i$, where the ρ_i's are the MSE requirements.

Defining the MSE matrix as the covariance matrix of the error between the transmitted and estimated vectors

$$\begin{aligned}\mathbf{E}(\mathbf{A}, \mathbf{B}) &\triangleq \mathbb{E}\left[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^H\right] \\ &= (\mathbf{A}^H\mathbf{H}\mathbf{B} - \mathbf{I})(\mathbf{B}^H\mathbf{H}^H\mathbf{A} - \mathbf{I}) \\ &\quad + \mathbf{A}^H\mathbf{R}_n\mathbf{A}\end{aligned} \tag{8}$$

and noting that the MSE of the ith subchannel is obtained as the ith diagonal element of $\mathbf{E}$, i.e., $\mathrm{MSE}_i = [\mathbf{E}]_{ii}$, the problem can be mathematically formulated as

$$\begin{aligned}\min_{\mathbf{A}, \mathbf{B}} \quad & \mathrm{Tr}(\mathbf{B}\mathbf{B}^H) \\ \text{s.t.} \quad & [\mathbf{E}(\mathbf{A}, \mathbf{B})]_{ii} \leq \rho_i \quad 1 \leq i \leq L.\end{aligned} \tag{9}$$

1) Linear Receiver: If the transmitter $\mathbf{B}$ is fixed, the optimal receiver is obtained as the well-known linear MMSE receiver, also termed Wiener filter, since it minimizes simultaneously all MSEs [22], [16], [20]. In addition, the ZF constraint $\mathbf{A}^H\mathbf{H}\mathbf{B} = \mathbf{I}$ can be imposed to avoid crosstalk among the different links established through the MIMO channel; in such a case, the well-known ZF receiver is obtained [19]. Both the MMSE and ZF

[2]A Pareto optimal solution is an optimal solution to a multiobjective optimization problem; it is defined as any solution that cannot be improved with respect to any of the objectives without worsening the others [21].

receivers, along with the corresponding MSE matrices, can be written in a compact way as follows:

$$\mathbf{A} = \mathbf{R}_n^{-1}\mathbf{H}\mathbf{B}(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1} \tag{10}$$
$$\mathbf{E} = (\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1} \tag{11}$$

where ν is a parameter defined as

$$\nu \triangleq \begin{cases} 1, & \text{for the MMSE receiver} \\ 0, & \text{for the ZF receiver} \end{cases}$$

and $\mathbf{R}_H \triangleq \mathbf{H}^H\mathbf{R}_n^{-1}\mathbf{H}$ is the squared whitened channel matrix. Note that we can always consider that a receiver is composed of an MMSE or ZF stage plus some other stage without loss of optimality (since the MMSE and ZF receivers are capacity-lossless[3]). It important to remark that the SINRs and the MSEs are easily related for the MMSE/ZF receiver by $\text{SINR}_i = \text{MSE}_i^{-1} - \nu$ [23, Prob. 6.5], [16], [20]. In addition, the BER can be analytically expressed as a function of the SINR, assuming that the crosstalk can be well modeled as a Gaussian noise [23]–[25]. Hence, given a BER requirement and a constellation, we can straightforwardly obtain the equivalent SINR or MSE requirement; this is why the formulation with MSE constraints in (9) is without loss of generality.

2) Linear Transmitter: Now, in order to design the transmitter $\mathbf{B}$, the following problem has to be solved:

$$\begin{array}{ll} \min\limits_{\mathbf{B}} & \text{Tr}(\mathbf{B}\mathbf{B}^H) \\ \text{s.t.} & \left[(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}\right]_{ii} \leq \rho_i, \quad 1 \leq i \leq L \end{array} \tag{12}$$

where the ρ_i's are the MSE requirements assumed in decreasing order $\rho_i \geq \rho_{i+1}$, without loss of generality. This is a nonconvex and very complicated problem since, in general, the MSE matrix is not diagonal, i.e., $\mathbf{B}$ does not diagonalize $\mathbf{R}_H$. For the sake of simplicity, one can impose a diagonal structure on $\mathbf{B}^H\mathbf{R}_H\mathbf{B}$ by choosing the following suboptimal solution for $\mathbf{B}$ [20]:

$$\mathbf{B} = \mathbf{U}_{H,1}\boldsymbol{\Sigma}_B \tag{13}$$

where $\mathbf{U}_{H,1} \in \mathbb{C}^{n_T \times L}$ has as columns the eigenvectors of $\mathbf{R}_H$ corresponding to the L largest eigenvalues λ_i in increasing order $\lambda_i \leq \lambda_{i+1}$ and $\boldsymbol{\Sigma}_B \in \mathbb{R}^{L\times L}$ is a diagonal matrix with squared-diagonal elements corresponding to the power p_i allocated on each subchannel, i.e., $\boldsymbol{\Sigma}_B = \text{diag}\left(\left\{\sqrt{p_i}\right\}\right)$. The problem is then reduced to obtaining the power allocation as

$$\begin{array}{ll} \min\limits_{\{p_i\}} & \sum\limits_{i=1}^{L} p_i \\ \text{s.t.} & \dfrac{1}{\nu + p_i\lambda_i} \leq \rho_i \quad 1 \leq i \leq L \\ & p_i \geq 0 \end{array} \tag{14}$$

[3]It is straightforward to verify that the MMSE and ZF receivers in (10) are capacity lossless simply by checking that the mutual information (for a given transmitter $\mathbf{B}$) after the receiver $\mathbf{A}$, $\log\det\left(\mathbf{I} + \mathbf{B}^H\mathbf{H}^H\mathbf{A}(\mathbf{A}^H\mathbf{R}_n\mathbf{A})^{-1}\mathbf{A}^H\mathbf{H}\mathbf{B}\right)$ is equal to the mutual information of the channel $\log\det(\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})$.

which is a simple convex problem with solution given by

$$p_i = \lambda_i^{-1}\left(\rho_i^{-1} - \nu\right) \quad 1 \leq i \leq L. \tag{15}$$

Note that each MSE constraint is satisfied with equality: $\text{MSE}_i = \rho_i$ for $1 \leq i \leq L$.

However, the choice (13) is suboptimal. The optimal solution to problem (12) was obtained in [20] as

$$\mathbf{B} = \mathbf{U}_{H,1}\boldsymbol{\Sigma}_B\mathbf{Q} \tag{16}$$

where $\mathbf{Q} \in \mathbb{C}^{L\times L}$ is an additional unitary matrix with respect to the suboptimal solution (13), also termed "rotation" matrix, which can be easily computed [20]. Note that by using the optimal $\mathbf{B}$, the optimal power allocation for (16) is no longer given by (15) but by a waterfilling solution with multiple waterlevels (cf. [20]).

3) Diagonal versus Nondiagonal Transmission: To better understand the underlying structure of the communication when using an MMSE/ZF receiver and a transmitter of the form $\mathbf{B} = \mathbf{U}_{H,1}\boldsymbol{\Sigma}_B\mathbf{Q}$, write the global transmit-receive process $\hat{\mathbf{x}} = \mathbf{A}^H(\mathbf{H}\mathbf{B}\mathbf{x} + \mathbf{n})$ explicitly as

$$\hat{\mathbf{x}} = \mathbf{Q}^H\left(\nu\mathbf{I} + \boldsymbol{\Sigma}_B^H\mathbf{D}_{H,1}\boldsymbol{\Sigma}_B\right)^{-1} \times\boldsymbol{\Sigma}_B^H\mathbf{D}_{H,1}^{1/2}\left(\mathbf{D}_{H,1}^{1/2}\boldsymbol{\Sigma}_B\mathbf{Q}\mathbf{x} + \mathbf{w}\right) \tag{17}$$

where $\mathbf{w}$ is an equivalent normalized white noise, and $\mathbf{D}_{H,1} = \mathbf{U}_{H,1}^H\mathbf{R}_H\mathbf{U}_{H,1}$ is the diagonalized squared whitened channel matrix. For the ZF receiver ($\nu = 0$), the previous expression simplifies to

$$\hat{\mathbf{x}} = \mathbf{x} + \mathbf{Q}^H\left(\boldsymbol{\Sigma}_B^H\mathbf{D}_{H,1}\boldsymbol{\Sigma}_B\right)^{-1/2}\mathbf{w} \tag{18}$$

which clearly satisfies the condition $\mathbf{A}^H\mathbf{H}\mathbf{B} = \mathbf{I}$ (by definition) but has, in general, a correlated noise among the subchannels. In other words, when using the ZF receiver, the global transmission is not really diagonal or parallel since the noise is colored.

In fact, the fully diagonal or parallel transmission does not depend on whether the ZF or the MMSE receivers are used but on the choice of the "rotation" $\mathbf{Q}$. Indeed, by setting $\mathbf{Q} = \mathbf{I}$, the global transmit-receive process (17) is fully diagonalized:

$$\hat{\mathbf{x}} = \left(\nu\mathbf{I} + \boldsymbol{\Sigma}_B^H\mathbf{D}_{H,1}\boldsymbol{\Sigma}_B\right)^{-1}\boldsymbol{\Sigma}_B^H\mathbf{D}_{H,1}^{1/2}\left(\mathbf{D}_{H,1}^{1/2}\boldsymbol{\Sigma}_B\mathbf{x} + \mathbf{w}\right) \tag{19}$$

which can be rewritten as

$$\hat{x}_i = \alpha_i\left(\sqrt{p_i\lambda_i}x_i + w_i\right), \quad 1 \leq i \leq L \tag{20}$$

where $\alpha_i = \sqrt{p_i\lambda_i}/(\nu + p_i\lambda_i)$ (see Fig. 3). Interestingly, by choosing $\mathbf{Q} = \mathbf{I}$, the MMSE receiver also results in a diagonal transmission (which is never the case in the traditional approach, where only the receiver is optimized). This is all summarized in the following.

Remark 1: The suboptimal solution (13) leads to a diagonal transmission or diagonal structure, whereas the optimal solution

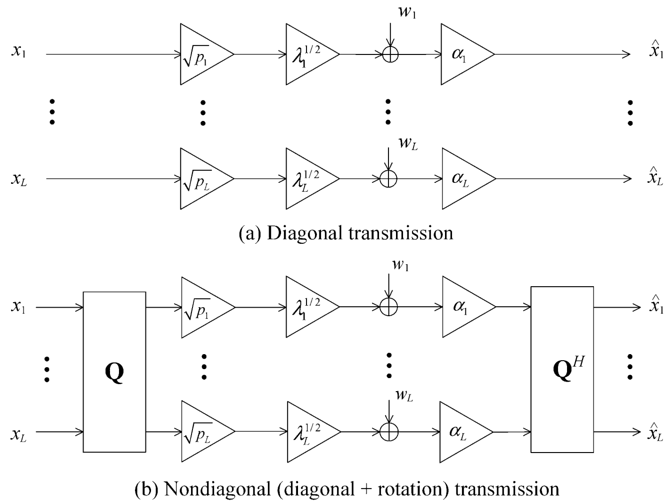

Fig. 3. Scheme of diagonal and nondiagonal (due to the rotation) transmissions.

(16) yields a nondiagonal scheme composed of an inner diagonal structure placed between a "pre-rotation" and a "post-rotation" operators, as shown in Fig. 3.

III. On the Choice of the Constellations

In this section, we consider the choice of the constellations (and, equivalently, of the allocated power) for transmission over a set of parallel subchannels:

$$y_i = \sqrt{p_i \lambda_i} x_i + w_i, \quad 1 \le i \le L \tag{21}$$

where $\mathbb{E}\left[x_i x_j^*\right] = \delta_{ij}$ and $\mathbb{E}\left[w_i w_j^*\right] = \delta_{ij}$. The SNR of the ith subchannel is given by $\text{SNR}_i = p_i \lambda_i$, which is a combination of the subchannel gain λ_i and the allocated power p_i.

A. Capacity-Achieving Solution

We start the discussion with the well-known capacity-achieving solution since the following methods are strongly based on it. From the landmark work by Shannon in 1948 [26], [27], the achievable information rate through a channel with a given SNR is given by $\log_2(1 + \text{SNR})$ bits/transmission. For the set of parallel subchannels in (21), the achievable information rate is given by the sum $\sum_{i=1}^{L} \log_2(1 + p_i \lambda_i)$, and the capacity is given by the maximum achievable rate over all possible power allocation strategies $\{p_i\}$. The optimum power distribution is the well-known waterfilling solution [28], [5]

$$p_i = \left(\mu - \frac{1}{\lambda_i}\right)^+ \quad 1 \le i \le L \tag{22}$$

where μ is the waterlevel chosen to satisfy the power constraint with equality (the waterlevel can be alternatively chosen to satisfy a given required rate with minimum power).

To achieve the channel capacity, however, it is necessary to use ideal Gaussian codes [5], which are not practical for real systems; hence the need to employ simpler and more practical constellations such as QAM or pulse amplitude modulation (PAM).

B. Gap Approximation

The basic idea of the gap approximation is to avoid the need for ideal Gaussian codes inherent in the capacity-achieving solution. Instead, a family of practical constellations, such as QAM or PAM, is employed. The number of bits that can be transmitted for a given family of constellations and a given probability of detection error P_e is approximately given by $\log_2(1 + \text{SNR}/\Gamma)$, where $\Gamma \ge 1$ is the gap which depends only on the family of constellations and on P_e [10]–[13]. Interestingly, there is a constant gap Γ between the Shannon capacity and the spectral efficiency of realistic constellations, which can be interpreted as a penalty for not using ideal Gaussian codes. This observation was initially reported for channels with intersymbol interference with decision-feedback equalization [29]. The gap approximation is a widely used technique to select the constellations in wireline communication systems such as DSL [30].

The derivation of the gap approximation method is straightforward as we now show. For QAM constellations, for example, the symbol error probability can be approximated (upper bounded, to be exact) by $P_e(\text{SNR}) \simeq 2\exp(-((3/2)/(M-1))\text{SNR})$ [31], [24], from which the constellation size that achieves a given P_e in the ith parallel subchannel of (21) is

$$M_i = 1 + \frac{\frac{3}{2}\text{SNR}_i}{\log_e(2/P_e)} = 1 + \frac{\text{SNR}_i}{\Gamma} \tag{23}$$

where $\Gamma = (2/3)\log_e(2/P_e)$ is the gap. If, instead, the error probability is approximated as $P_e(\text{SNR}) \simeq 4\mathcal{Q}\left(\sqrt{(3/(M-1))\text{SNR}}\right)$, where $\mathcal{Q}(x) \triangleq (1/\sqrt{2\pi}) \int_x^\infty e^{-\lambda^2/2} d\lambda$ [31], [23], then the resulting gap is $\Gamma = (\mathcal{Q}^{-1}(P_e/4))^2/3$. Hence, the total number of bits that can be transmitted with a given P_e is

$$\sum_{i=1}^{L} \log_2(M_i) = \sum_{i=1}^{L} \log_2\left(1 + p_i \frac{\lambda_i}{\Gamma}\right) \tag{24}$$

where we have used $\text{SNR}_i = p_i \lambda_i$. The optimum power distribution that maximizes the total number of bits also follows a waterfilling solution

$$p_i = \left(\mu - \frac{\Gamma}{\lambda_i}\right)^+ \quad 1 \le i \le L \tag{25}$$

and the number of bits to be transmitted on the ith subchannel is $(\log_2(\mu\lambda_i/\Gamma))^+$.

Although the gap approximation method is designed to achieve the same symbol error probability P_e (and not bit error probability) on each subchannel, it can still be approximately used when the same BER is desired (because the BER and P_e differ approximately by the factor $\log_2 M$ which is not in the exponent and therefore is not very relevant). The more exact expression for the gap $\Gamma = (2/3)\log_e(0.2/P_e)$ was proposed in [13] based on a better fit of the BER for $M \ge 4$. As a final comment, it is important to point out that the gap approximation can easily incorporate other factors such as a margin gain γ_m (which is an additional gain included to make the system more robust) and a coding gain γ_c (to account for the gain given by an outer code) with the following gap expression: $\Gamma = (2/3)\log_e(0.2/P_e) \times \gamma_m/\gamma_c$.

C. Gap Approximation + Distortion

In practice, however, it is not possible to implement the continuous bit distribution (also termed bit loading) as given by the gap approximation due mainly to two sources of distortion: the *granularity* and the *bit cap* [30]. The granularity refers to the smallest incremental unit of information that can be transmitted, which means that the bit distribution dictated by the gap approximation has to be somehow rounded [30]; for example, it is customary to use constellations that correspond to an integer number of bits. The bit cap is a distortion phenomenon that arises because the maximum constellation-size is generally limited in real systems [30].

The distortion introduced in the bit distribution implies a distortion on the power allocation necessary to achieve the desired error probability and can be conveniently written as

$$p_i = \left(\left(\mu - \frac{\Gamma}{\lambda_i}\right) - \mu\Delta_i\right)^+ = \left(\mu\left(1-\Delta_i\right) - \frac{\Gamma}{\lambda_i}\right)^+ \quad (26)$$

where $\mu\Delta_i$ denotes the distortion. The number of bits to be transmitted on the ith subchannel is then $(\log_2(\mu(1-\Delta_i)\lambda_i/\Gamma))^+$.

Next, we characterize the distortion resulting from two different types of rounding and from the bit cap.

1) Rounding to the Closest Smaller Integer: Rounding to the closest smaller integer satisfies the condition

$$0 \leq \log_2\left(\mu\frac{\lambda_i}{\Gamma}\right) - \log_2\left(\mu(1-\Delta_i)\frac{\lambda_i}{\Gamma}\right) < 1$$

which implies

$$0 \leq \Delta_i < \frac{1}{2}. \quad (27)$$

2) Rounding to the Closest Integer: Rounding to the closest smaller integer satisfies the condition

$$-\frac{1}{2} \leq \log_2\left(\mu\frac{\lambda_i}{\Gamma}\right) - \log_2\left(\mu(1-\Delta_i)\frac{\lambda_i}{\Gamma}\right) < \frac{1}{2}$$

which implies

$$1-\sqrt{2} \leq \Delta_i < 1 - \frac{1}{\sqrt{2}}. \quad (28)$$

3) Bit Cap: For the active subchannels, the power allocation is $p_i = \mu(1-\Delta_i) - (\Gamma/\lambda_i)$ with a limitation on the maximum constellation size or, equivalently, on the maximum gain $p_i\lambda_i/\Gamma \leq \gamma_{\max}$. This implies the following bounds on the distortion:

$$1 - \frac{\Gamma}{\mu\lambda_i}(1+\gamma_{\max}) \leq \Delta_i < 1 - \frac{\Gamma}{\mu\lambda_i}. \quad (29)$$

D. Equal Power

The power allocation obtained from the gap approximation $p_i = (\mu - \Gamma/\lambda_i)^+$ can be approximated in practice with a flat power allocation or equal power distribution $p_i = \mu$ over the best subchannels.[4] Such a solution is approximately valid for sufficiently high SNR; to be exact, it suffices to have $\mu\lambda_i/\Gamma \gg 1$ for the used subchannels such that $\log_2(1+\mu\lambda_i/\Gamma) \approx \log_2(\mu\lambda_i/\Gamma)$ (indeed, due to the logarithmic dependence, the number of transmitted bits is insensitive to the exact power allocation). This observation was empirically made in [32] (called "on/off" distribution) and further analyzed in [33] using the duality gap.

[4]One possible way to choose the best subchannels for the flat power allocation is to select the subchannels that would be used by the waterfilling solution.

In practice, as happened with the gap approximation, the uniform power allocation is distorted due to the rounding of the allocated bits.

E. Equal Constellations

In order to reduce the complexity of a system employing different constellations and codes, it can be constrained to use the same constellation and code in all subchannels (possibly optimizing the utilized bandwidth to transmit only over those subchannels with a sufficiently high gain), i.e., an equal-rate transmission. Examples of this pragmatic and simple solution are found in the European standard HIPERLAN/2 [34] and in the US standard IEEE 802.11 [35] for Wireless Local Area Networks (WLAN).

In order to achieve the same BER on each subchannel (with equal constellations), the SINRs must be the same and the power allocation is $p_i = \mu/\lambda_i$ such that resulting gains of the subchannels $p_i\lambda_i$ are equal.

F. Numerical Comparison

In this subsection, we compare the following methods for choosing the constellations: i) *best-constellations* for the subchannels, which is obtained through a full search over all combinations of constellations; ii) *gap approximation*; iii) *equal-power*, which optimizes the number of subchannels used for each channel realization; iv) *equal-constellations*, which also optimizes the number of subchannels used for each channel realization; v) *best fixed-equal-constellations*, which are equal constellations over the best subchannels but with a fixed number of subchannels used for all channel realizations; vi) *wrong fixed-equal-constellations*, which are equal constellations again over a fixed number of subchannels but, in this case, wrongly chosen.

The gains of the subchannels are obtained as the eigenvalues of a randomly generated 4×4 channel matrix with independent and identically distributed (i.i.d.) complex Gaussian elements of zero mean and unit variance. To design the system, we impose a BER of 10^{-3} on each subchannel (which is basically equivalent to fixing the global averaged BER to 10^{-3}, cf. Section II-C) and compute the minimum transmit power for each desired rate.

In Fig. 4, the required normalized transmit power (defined as P/σ_n^2 where σ_n^2 is the noise power) is plotted as a function of the desired rate for the six methods for choosing the constellations; to be exact, the outage power with an outage probability of 5% (computed with 1000 channel realizations) is plotted. The full search over all constellations gives, as expected, the best solution; in fact, it gives a Pareto optimal solution in terms of BER, rate, and power. The gap approximation, as could be argued from its analytical derivation, is almost indistinguishable from the full search. The equal-power method approximates the gap approximation extremely well. The equal-constellation approach has some performance loss (about 1 or 2 dB) but still performs quite well. Surprisingly, the best fixed-equal-constellation method performs very close to its adaptive counterpart, with the

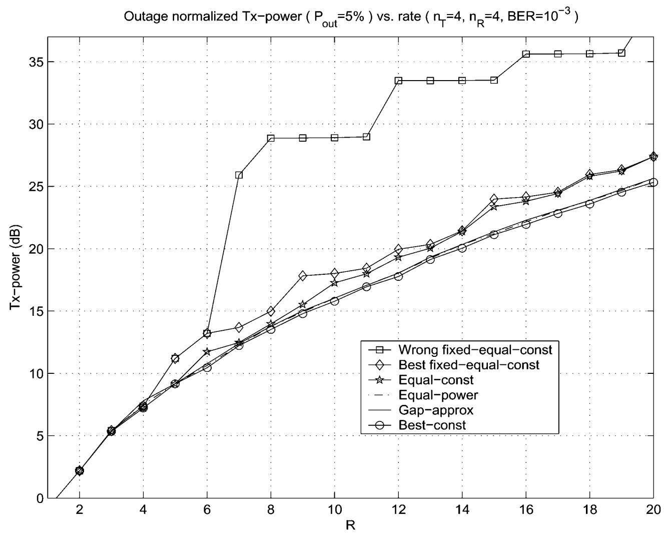

Fig. 4. Tradeoff curves of power versus rate (for a given BER of 10^{-3}) for different methods of choosing the constellations.

advantage that not only the constellations but, in addition, the number of subchannels are fixed for all channel realizations. In particular, for a 4×4 MIMO channel, the best choice is $L = 3$ (for other configurations such as 6×6 or 8×8 MIMO channels, the best choice is $L = 4$ and $L = 5$, respectively; this amounts to using approximately 65% of the subchannels). However, if the number of subchannels is not properly chosen (in particular, we use all the subchannels $L = 4$ in Fig. 4), the performance degrades dramatically. Similar curves as those plotted in Fig. 4 are obtained for other channel configurations with different number of transmit and receive dimensions, for which the same observations hold.

Summarizing, if the system can adapt the constellations to the channel realization, then the gap approximation and the equal-power methods are virtually optimal. If, instead, the system cannot change the constellations, the loss can still be kept small provided that the (possibly fixed) number of subchannels to be used is properly selected such that the bad subchannels are always discarded.

IV. On the Design of Linear MIMO Transceivers

The purpose of this section is to shed some light into the choice between diagonal and nondiagonal transmissions. We first obtain some general results regarding the optimality and suboptimality of the diagonal structure and then apply these results to the different methods for choosing the constellations described in Section III. In particular, Theorem 1 reveals an approximate optimal solution for the joint design of the constellations and linear transceiver.

We restate the following result from [16], which characterizes the transmitter when the performance of the system is measured by a Schur-concave/convex function [36].

Proposition 1: [16] Consider the following two constrained optimization problems:

$$\begin{aligned} \min_{\mathbf{B}} \quad & f_0\left(\left\{\left[(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}\right]_{ii}\right\}\right) \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{B}\mathbf{B}^H) \le P_0 \end{aligned}$$

and

$$\begin{aligned} \min_{\mathbf{B}} \quad & \mathrm{Tr}(\mathbf{B}\mathbf{B}^H) \\ \text{s.t.} \quad & f_0\left(\left\{\left[(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}\right]_{ii}\right\}\right) \le \alpha_0 \end{aligned}$$

where P_0 and α_0 are the power and quality requirements, respectively, and $f_0 : \mathbb{R}^L \longrightarrow \mathbb{R}$ is an arbitrary cost function (increasing in each variable) of the MSEs. Then, it follows that there is an optimal solution $\mathbf{B}$ with the following structure:

$$\mathbf{B} = \begin{cases} \mathbf{U}_{H,1}\mathbf{\Sigma}_B, & \text{if } f_0 \text{ is Schur-concave} \\ \mathbf{U}_{H,1}\mathbf{\Sigma}_B\mathbf{Q}, & \text{if } f_0 \text{ is Schur-convex} \end{cases} \tag{30}$$

where $\mathbf{U}_{H,1}$ and $\mathbf{\Sigma}_B$ are defined as in (13), and $\mathbf{Q}$ is a unitary matrix such that the MSE matrix $(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}$ has equal diagonal elements (e.g., the unitary Fourier matrix).

An immediate consequence of Proposition 1 is that the diagonal structure is optimal for Schur-concave functions; whereas for Schur-convex ones, it is never optimal (unless the used channel eigenvalues are all equal).

In many cases, however, the system cannot be measured by a Schur-concave/convex function. The following result gives the necessary and sufficient conditions under which the diagonal transmission is optimal for any system.

Proposition 2: The diagonal transmission, given by (13) and (15), is the optimal solution of problem (12) if and only if

$$\lambda_{i-1}^{1/2}\mathrm{MSE}_{i-1} \ge \lambda_i^{1/2}\mathrm{MSE}_i \quad 1 < i \le L \tag{31}$$

where the channel eigenvalues λ_i are in increasing order $\lambda_i \le \lambda_{i+1}$, and the MSEs are in decreasing order $\mathrm{MSE}_i \ge \mathrm{MSE}_{i+1}$, where $\mathrm{MSE}_i \triangleq \rho_i = 1/(\nu + p_i\lambda_i)$.

Proof: This result was given in [20] for the case of an MMSE receiver ($\nu = 1$) and is easily obtained from the Karush–Kuhn–Tucker (KKT) optimality conditions of the problem [37], [38] (see Appendix A for a proof). ■

What is important to remark here is that, depending on the constellation chosen for each subchannel (cf. Section III), the MSEs required to guarantee a given BER with these constellations change. As a consequence, the system may or may not satisfy the conditions in (31). This means that for some constellations the diagonal transmission may be optimal, whereas for others it may not.

Proposition 2 is useful to obtain a binary answer to the question of whether the diagonal transmission is optimal or not. In practice, when Proposition 2 is not satisfied but a suboptimal diagonal structure is still employed, it is important to quantify the loss resulting from the use of the diagonal structure. The answer is given by the following result.

Proposition 3: The power loss ΔP incurred by using a diagonal transmission, given by (13) and (15), with respect to the optimal transmission (possibly including an additional "rotation" in (13), as in (16)) is upper bounded as

$$\Delta P \le \sum_{i \in \mathcal{I}} \mathrm{MSE}_i \times \left(\frac{1}{\min_{j \le i}\left\{\lambda_j^{1/2}\mathrm{MSE}_j\right\}} - \frac{1}{\lambda_i^{1/2}\mathrm{MSE}_i}\right)^2 \tag{32}$$

where the channel eigenvalues λ_i are in increasing order $\lambda_i \le \lambda_{i+1}$, the MSEs are in decreasing order $\mathrm{MSE}_i \ge \mathrm{MSE}_{i+1}$, and $\mathcal{I}$ is the set of subchannels that do not satisfy the diagonality condition in (31), i.e., $\mathcal{I} \triangleq \left\{i : \lambda_i^{1/2}\mathrm{MSE}_i > \min_{j<i}\left\{\lambda_j^{1/2}\mathrm{MSE}_j\right\}\right\}$.

Proof: The loss of optimality of the diagonal transmission can be easily quantified with the concept of duality gap arising in convex optimization theory [37], [38] (see Appendix B for a detailed proof). ■

In words, Proposition 3 says that if the optimality conditions of the diagonal structure (obtained in Proposition 2) are not satisfied by not too much, then the diagonal transmission is almost optimal, i.e., the performance degrades gracefully.

Interestingly, the previous conditions on the optimality of the diagonal structure can be checked before actually designing the transceiver so that one is able to check, *a priori*, the maximum power loss. In the following, we particularize the previous results to different methods for choosing the constellations, as described in Section III.

A. Capacity-Achieving Solution

For convenience of exposition, we start by briefly recalling the well-known diagonality result of the capacity-achieving solution. It suffices to show that the mutual information is always increased when the whitened channel matrix is diagonalized:

$$\begin{aligned}\log\det\left(\mathbf{I}+\mathbf{R}_n^{-1}\mathbf{H}\mathbf{R}_x\mathbf{H}^H\right) &= \log\det\left(\mathbf{I}+\mathbf{D}_H\mathbf{U}_H^H\mathbf{R}_x\mathbf{U}_H\right)\\ &= \log\det\left(\mathbf{I}+\mathbf{D}_H\tilde{\mathbf{R}}_x\right)\\ &\leq \sum_i \log\left(1+\lambda_i\left[\tilde{\mathbf{R}}_x\right]_{ii}\right) \quad (33)\end{aligned}$$

where $\mathbf{R}_x$ is the transmit covariance matrix, we have used the eigendecomposition $\mathbf{H}^H\mathbf{R}_n^{-1}\mathbf{H} = \mathbf{U}_H\mathbf{D}_H\mathbf{U}_H^H$, $\tilde{\mathbf{R}}_x \triangleq \mathbf{U}_H^H\mathbf{R}_x\mathbf{U}_H$ (note that the power constraint $\mathrm{Tr}(\mathbf{R}_x) \leq P$ is similarly given in terms of $\tilde{\mathbf{R}}_x$ by $\mathrm{Tr}(\tilde{\mathbf{R}}_x) \leq P$), and the inequality comes from Hadamard's inequality $\det(\mathbf{R}) \leq \prod_i [\mathbf{R}]_{ii}$ (with equality if and only if $\mathbf{R}$ is diagonal) [5]. Therefore, the transmit covariance matrix should be of the form $\mathbf{R}_x = \mathbf{U}_H\mathrm{diag}(\{p_i\})\mathbf{U}_H^H$ (so that $\tilde{\mathbf{R}}_x$ is diagonal), where the optimal power allocation follows the waterfilling form as in (22): $p_i = (\mu - 1/\lambda_i)^+$.

This is a well-known result obtained from information-theoretic arguments, e.g., [5]. Interestingly, we now obtain the optimality of the diagonal structure when using the waterfilling power allocation from a pure signal processing perspective invoking Proposition 2.

Corollary 1: The diagonal transmission is an optimal structure when the power allocation over the channel eigenvalues is given by the classical capacity-achieving waterfilling solution $p_i = (\mu - 1/\lambda_i)^+$.

Proof: It follows from Proposition 2 (see Appendix C setting $\Gamma = 1$). ■

B. Gap Approximation

As explained in Section III, the gap approximation is an approximately optimal way to choose the constellations for a set of parallel subchannels (in the sense of maximizing the rate with a fixed transmit power or minimizing the power with a fixed rate). When dealing with a MIMO channel, one can always diagonalize the channel matrix and treat it as a set of parallel subchannels. However, the optimality of the diagonal structure for the gap approximation (as happens with the capacity-achieving solution) has never been proved (to the best of the authors' knowledge) even though it is commonly used in practice. An exception is [8], where the joint design of the constellations and linear transceiver is considered under a perfect reconstruction criterion (i.e., for a ZF receiver) and the diagonal structure is found to be optimal (the choice of constellations derived happens to be essentially equivalent to the gap approximation method although not explicitly mentioned). The following result shows that indeed the diagonal structure is optimal when using the gap approximation (for MMSE and ZF receivers).

Theorem 1: The diagonal transmission is an optimal structure when the performance of the system is measured by the aggregate rate of the subchannels $\sum_i \log_2(1+\mathrm{SINR}_i/\Gamma)$ according to the gap approximation method with gap Γ.

Proof: It suffices to show that the function that measures the performance of the system is Schur-concave and then invoke Proposition 1. Recalling the relation $\mathrm{SINR}_i = \mathrm{MSE}_i^{-1} - \nu$ as given in Section II, the cost function of the system (to be minimized) can be written as

$$f_0(\{\mathrm{MSE}_i\}) = -\sum_i \log_2\left(1+\frac{\left(\mathrm{MSE}_i^{-1}-\nu\right)}{\Gamma}\right)$$

which is Schur-concave since it is the sum of identical concave functions (given by $-\log_2(1+(x^{-1}-\nu)/\Gamma)$) [36, 3.H.2]. ■

Theorem 1 states that the combination of the gap approximation method plus a diagonal structure is in fact an almost optimal way (due to the nonexact nature of the gap approximation itself) to jointly design the constellations and the linear transceiver.

For illustration purposes (since the following subsections proceed in the same way), we characterize the optimality of the diagonal structure invoking Proposition 2 instead.

Corollary 2: The diagonal transmission is an optimal structure when the power allocation over the channel eigenvalues is given by the gap approximation waterfilling solution $p_i = (\mu - \Gamma/\lambda_i)^+$.

Proof: It follows from Proposition 2 (see Appendix C for details). ■

C. Gap Approximation + Distortion

As treated in detail in Section III, a practical implementation of the gap approximation requires rounding the number of allocated bits on each subchannel, which can be written in terms of power as $p_i = (\mu(1-\Delta_i) - \Gamma/\lambda_i)^+$. The following result gives the conditions under which the diagonal structure is optimal for transmission.

Corollary 3: The diagonal transmission is an optimal structure when the power allocation over the channel eigenvalues is given by the gap approximation plus a distortion $p_i = (\mu(1-\Delta_i) - \Gamma/\lambda_i)^+$ if the following conditions are satisfied:

$$\Delta_{i+1} \leq \left(1-\sqrt{\frac{\lambda_i}{\lambda_{i+1}}}\right) + \Delta_i\sqrt{\frac{\lambda_i}{\lambda_{i+1}}} \qquad (34)$$

or, roughly speaking

$$\Delta_{i+1} \leq \begin{cases}\Delta_i, & \lambda_{i+1} \approx \lambda_i\\ 1, & \lambda_{i+1} \gg \lambda_i.\end{cases} \qquad (35)$$

Proof: The proof follows from Proposition 2 (see Appendix D for details). ∎

Since the optimality conditions of Corollary 3 are not satisfied in many cases, it is important to quantify then the suboptimality of the diagonal structure. We now obtain a very simple upper bound on the increase of power required due to the use of the suboptimal diagonal structure.

Corollary 4: The loss incurred by using a diagonal transmission with respect to the optimal transmission, when the gap approximation with distortion is used, is upper bounded as

$$\Delta P \leq \mu^2 \sum_{i \in \mathcal{I}} \mathrm{MSE}_i \times \left(\Delta_i \lambda_i^{1/2} - \min_{j \leq i} \left\{ \Delta_j \lambda_j^{1/2} \right\} \right)^2 \quad (36)$$

where $\mathcal{I}$ is the set of subchannels that do not satisfy the diagonality condition in (31), as defined in Proposition 3. In case of rounding to the closest smaller integer $(0 \leq \Delta_i < 1/2)$, for example, the following more illustrative bound is obtained:

$$\Delta P \leq \mu^2 \sum_{i \in \mathcal{I}} \Delta_i^2 \lambda_i \mathrm{MSE}_i. \quad (37)$$

Proof: It follows from Proposition 3 (see Appendix E for details). ∎

From the previous result, it is clear that the loss of performance increases gracefully with the distortion on each subchannel [see, for example, (37)]. Hence, we can expect the gap approximation plus distortion to have a negligible loss in practice (this is supported by the numerical results in Section IV-F).

D. Equal Power

In the simple case of using a uniform power allocation, the diagonal structure is almost always optimal as the following result shows.

Corollary 5: The diagonal transmission is an optimal structure when an equal power allocation is used if

$$\frac{\nu}{P_0/L} \leq \lambda_1 \quad (38)$$

which is always true for the ZF receiver $(\nu = 0)$ and for the MMSE receiver $(\nu = 1)$ if the total power P_0 is sufficiently high.

Proof: It follows from Proposition 2 (see Appendix F for details). ∎

The following corollary quantifies the loss in case that the minimum used channel eigenvalue λ_1 is sufficiently small so that the condition of Corollary 5 is not satisfied.

Corollary 6: The loss incurred by using a diagonal transmission with respect to the optimal transmission, when an equal power allocation is used, is upper bounded as

$$\Delta P \leq \sum_{i:\lambda_i < \nu/(P_0/L)} \left(\frac{\nu}{\lambda_i} + \frac{P_0}{L} \right). \quad (39)$$

Proof: It follows from Proposition 3 (see Appendix G for details). ∎

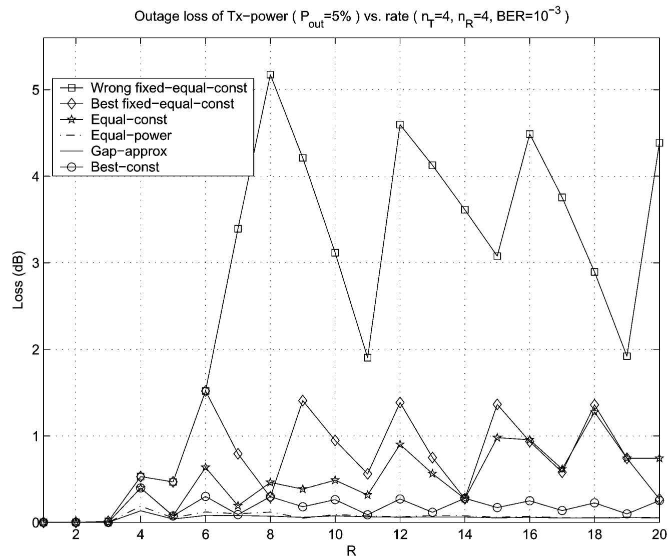

Fig. 5. Loss of performance of the diagonal transmission with respect to the optimal structure (in terms of required transmit power to achieve a given BER) as a function of the rate for different ways of choosing the constellations.

E. Equal Constellations

In the simple case of using the same constellation on each subchannel, the diagonal structure is never optimal with probability one (in case of a randomly chosen channel).

Corollary 7: The diagonal transmission is an optimal structure when equal constellations are used if and only if the channel eigenvalues used are identical.

Proof: It follows from Proposition 2 (Appendix H for details). ∎

We now quantify the loss in case that equal constellations are used with a diagonal transmission.

Corollary 8: The loss incurred by using a diagonal transmission with respect to the optimal transmission, when equal constellations are used, is upper bounded as

$$\Delta P \leq \frac{1}{\mathrm{MSE}_0} \left(\sum_{i=1}^{L} \lambda_i^{-1} - \frac{1}{L} \left(\sum_{i=1}^{L} \lambda_i^{-1/2} \right)^2 \right) \leq \frac{1}{\mathrm{MSE}_0} \sum_{i=1}^{L} \lambda_i^{-1} \quad (40)$$

where MSE_0 is the required MSE for each of the subchannels to satisfy the required BER constraint.

Proof: It follows from Proposition 3 (see Appendix I for details). ∎

F. Numerical Comparison

To support and illustrate the analytical results obtained in this section, the loss of the diagonal transmission with respect to the optimal structure is numerically evaluated for the different methods of choosing the constellations (cf. Section III-F). A 4×4 channel matrix is randomly generated with i.i.d. complex Gaussian elements of zero mean and unit variance. The loss is measured in terms of increase of transmit power required to

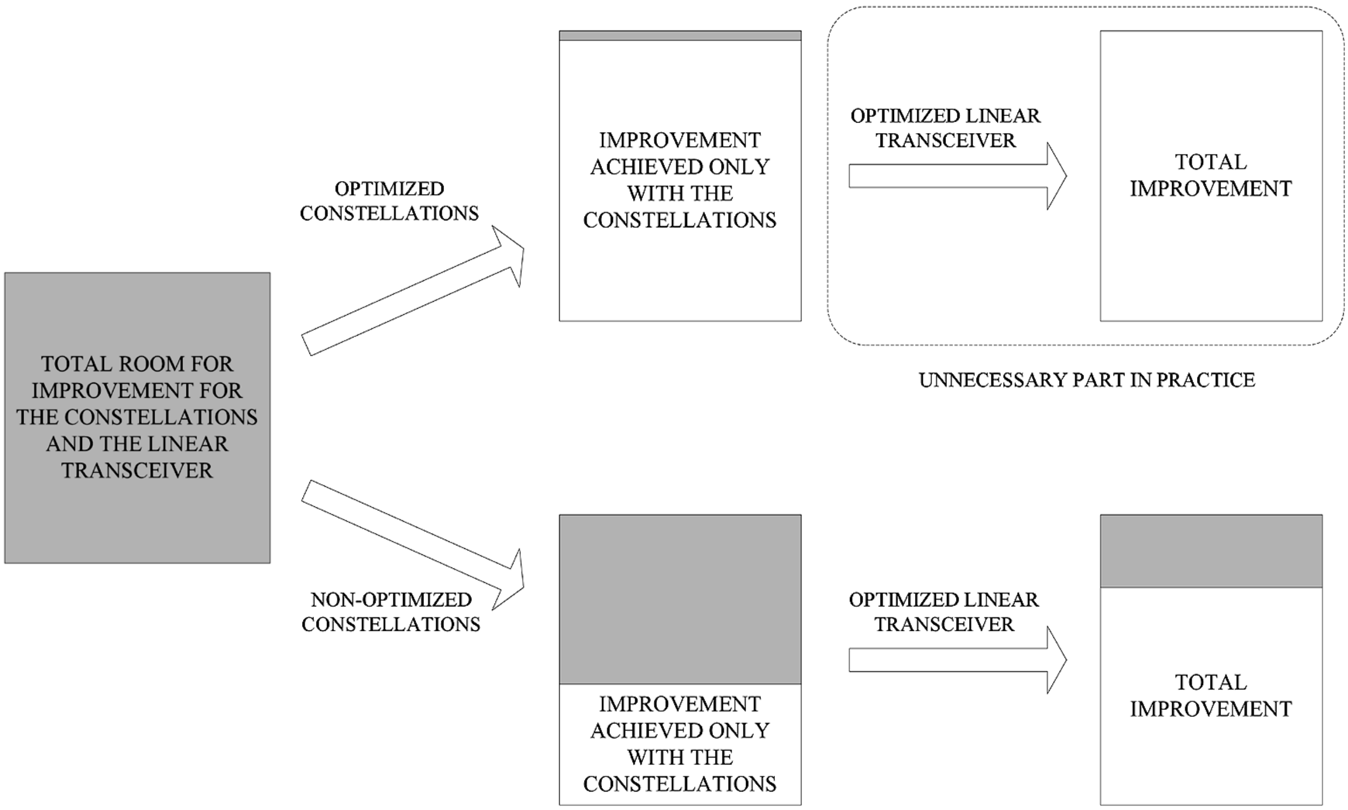

Fig. 6. Potential improvement that can be achieved by optimizing the constellations and/or the linear transceiver.

achieve a given BER of 10^{-3} on each subchannel; it can be defined as a percentage $\Delta P(\%) = 100(P^{\mathrm{diag}} - P^{\min})/P^{\mathrm{diag}}$ or as a difference in decibels $\Delta P(\mathrm{dB}) = P^{\mathrm{diag}}(\mathrm{dB}) - P^{\min}(\mathrm{dB})$, where P^{diag} is the power needed with the diagonal structure and $P^{\min}$ the minimum power required (assuming the optimal structure).[5]

In Fig. 5, the power loss of the diagonal transmission is plotted as a function of the rate. It can be observed that when the constellations are properly selected (best-constellation, gap approximation, and equal-power), the loss is insignificant (less than 0.2 dB). For the equal-constellation approach, the loss is still very small (0.5–1 dB). Interestingly, if the constellations are chosen equal and fixed for all channel realizations (using $L = 3$), the loss is also small (around 1 dB). However, if the choice of the constellations is not properly done (fixed equal-constellations with $L = 4$), then the loss can be quite significant (around 4 dB).

Numerical results for the upper bounds on the loss of performance of Corollaries 4, 6, and 8 (and also of the general upper bound in Proposition 3) are not reported in the plots for the sake of space; however, they can be easily summarized as follows. For the gap approximation with rounding, the general bound in Proposition 3 is indistinguishable from the real loss (0.05 dB of difference), whereas the bound in Corollary 4 is very loose (2 dB over the real loss in Fig. 5); hence, Corollary 4 is mainly a theoretical result to observe that the loss of performance increases gracefully. For an equal power allocation, both upper bounds in Proposition 3 and Corollary 6 are very tight and are almost indistinguishable from the real loss (0.05 dB of difference). For equal constellations, both upper bounds in Proposition 3 and Corollary 8 (only the tighter bound) are reasonably tight (the loss in decibels is approximately doubled, i.e., if the real loss is about 0.5 or 1 dB, the upper bound would be around 1 or 2 dB, respectively).

[5]The loss in percentage or in decibels is easily related as $\Delta P(\%)/100 = 1 - 10^{-\Delta P(\mathrm{dB})/10}$ or $\Delta P(\mathrm{dB})/10 = -\log_{10}(1 - \Delta P(\%)/100)$.

V. Summary of Constellation Choice and Linear Transceiver Design

In this section, we first list some of the results obtained in the paper and then summarize the lesson learned and illustrate it with a numerical result.

The main analytical results of the paper can be paraphrased as follows.

- The combination of the (ideal) gap approximation method plus a diagonal structure on the MIMO channel is an almost optimal way to jointly design the constellations and the linear transceiver.
- With the gap approximation method plus distortion (the case in practice), the diagonal transmission either is optimal or incurs in a really small loss. The same comment holds for the more pragmatic equal power allocation.
- With equal constellations, the diagonal transmission is never optimal. However, if the number of subchannels used is properly selected (even if it is independent on the channel realization), the loss is small; otherwise, the loss can be significant.

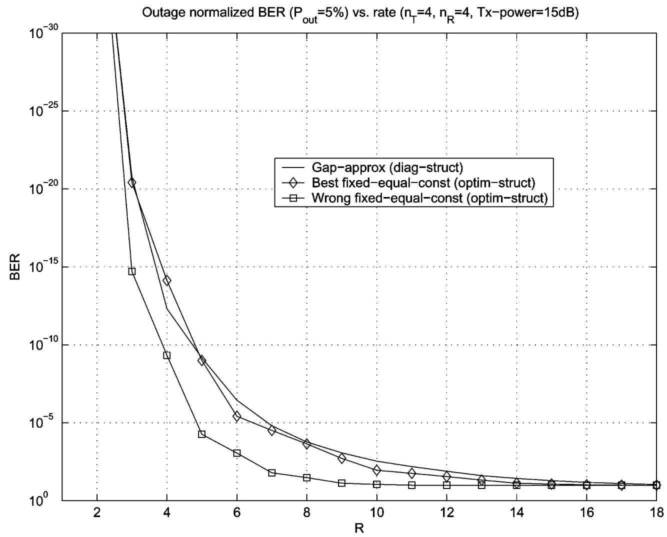

Fig. 7. Tradeoff curve of BER versus rate (for a given transmit power of 15 dB) corresponding to the two extreme approaches: gap-approximation + diagonal structure and wrong fixed equal-constellations + optimal structure; and also the intermediate approach: best fixed equal-constellations + optimal structure.

Hence, we can say that the better the constellations are selected, the less effort is needed in designing the linear transceiver, as is illustrated in Fig. 6.

In Fig. 7, the tradeoff curve of BER versus rate (for a given normalized transmitted power of 15 dB) is plotted for the two extreme approaches: gap approximation (including rounding) with a diagonal structure (which is virtually optimal) and wrongly fixed equal constellations with the optimal structure; and also for the intermediate approach: best fixed equal constellations with the optimal structure.

VI. Conclusions

This paper has considered the design of the uncoded part of a point-to-point MIMO communication system with respect to three parameters: BER, rate, and transmit power. As opposed to the existing results that deal either with the choice of the constellations (imposing a diagonal transmission) or with the design of the linear transceiver (with given fixed constellations), this paper has approached both problems in a unified way. Among other results, an almost optimal way to jointly design the constellations and the linear transceiver happens to be the well-known combination of the gap approximation method plus a diagonal structure on the MIMO channel (which is a widely used approach in practice).

The final conclusion is that, if possible, the constellations should be optimized for each channel realization, for example, with the gap approximation plus rounding; in such a case, the diagonal structure has been shown to be almost optimal (it is indeed optimal if the exact gap approximation without rounding is used). On the other hand, if the constellations are not properly chosen, the performance decreases, and something can be gained back by optimizing the transmission structure rather than diagonalizing the channel.

One issue not addressed in this paper that may be interesting for future research is the effect of nonperfect CSI, which is clearly relevant for real systems.

Appendix A
Proof of Proposition 2

The original problem is

$$\begin{aligned} \min_{\mathbf{B}} \quad & \mathrm{Tr}\left(\mathbf{B}\mathbf{B}^H\right) \\ \text{s.t.} \quad & \left[\left(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B}\right)^{-1}\right]_{ii} \leq \rho_i \quad 1 \leq i \leq L. \end{aligned} \tag{41}$$

As was shown in [20], this problem can be rewritten in convex form as

$$\begin{aligned} \min_{\{p_i\}} \quad & \sum_{i=1}^{L} p_i \\ \text{s.t.} \quad & \sum_{j=i}^{L} \frac{1}{\nu + p_j\lambda_j} \leq \sum_{j=i}^{L} \rho_j, \quad 1 \leq i \leq L \\ & p_i \geq 0. \end{aligned} \tag{42}$$

Now, since (42) is a convex problem, we can analyze it using the existing tools for convex optimization [37], [38]. In particular, we first form the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{L} p_i + \sum_{i=1}^{L} \mu_i \left(\sum_{j=i}^{L} \frac{1}{\nu + p_j\lambda_j} - \sum_{j=i}^{L} \rho_j \right) - \sum_{i=1}^{L} \gamma_i p_i \tag{43}$$

where the μ_i's and the γ_i's are the dual variables or Lagrange multipliers. Then, we obtain the sufficient and necessary KKT optimality conditions (the problem satisfies the Slater's condition and therefore strong duality holds) [37], [38]:

$$\sum_{j=i}^{L} \frac{1}{\nu + p_j\lambda_j} \leq \sum_{j=i}^{L} \rho_j, \quad p_i \geq 0 \tag{44}$$

$$\mu_i \geq 0, \qquad \gamma_i \geq 0 \tag{45}$$

$$\left(\sum_{j=1}^{i} \mu_j \right) \frac{\lambda_i}{(\nu + p_i\lambda_i)^2} + \gamma_i = 1 \tag{46}$$

$$\mu_i \left(\sum_{j=i}^{L} \frac{1}{\nu + p_j\lambda_j} - \sum_{j=i}^{L} \rho_j \right) = 0, \qquad \gamma_i p_i = 0. \tag{47}$$

At this point, it suffices to plug the diagonal solution of (15) into the KKT conditions and observe under which circumstances they are satisfied. Recalling the diagonal solution $p_i = \lambda_i^{-1}\left(\rho_i^{-1} - \nu\right)$, it is straightforward to see that $1/(\nu + p_i\lambda_i) = \rho_i$, which implies that $p_i > 0$ and $\gamma_i = 0$. Therefore, the only KKT conditions that are not trivially satisfied are

$$\begin{aligned} 1 &= \left(\sum_{j=1}^{i} \mu_j \right) \frac{\lambda_i}{(\nu + p_i\lambda_i)^2} \\ &= \left(\sum_{j=1}^{i} \mu_j \right) \lambda_i \rho_i^2 \end{aligned} \tag{48}$$

where $\mu_i \geq 0$. Since $\sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i+1} \mu_j$ for $1 \leq i < L$, it follows that such a set of μ_i's exists if and only if $\lambda_i\rho_i^2 \geq \lambda_{i+1}\rho_{i+1}^2$. ■

APPENDIX B
PROOF OF PROPOSITION 3

A useful concept in convex optimization theory is the duality gap since, among other things, it provides an upper bound on the loss of optimality for an arbitrary (in general suboptimal) solution to the problem. The gap is defined as the difference between the primal and the dual objective functions [37], [38]. In the present problem [see (42))], the primal objective is

$$f(\mathbf{p}) = \sum_{i=1}^{L} p_i \tag{49}$$

and the constraints are

$$\sum_{j=i}^{L} \frac{1}{\nu + p_j\lambda_j} \le \sum_{j=i}^{L} \rho_j, \quad 1 \le i \le L$$
$$p_i \ge 0.$$

The corresponding Lagrangian is

$$\begin{aligned}\mathcal{L}(\mathbf{p}, (\boldsymbol{\mu}, \boldsymbol{\gamma})) &= \sum_{i=1}^{L} p_i + \sum_{i=1}^{L} \mu_i \left(\sum_{j=i}^{L} \frac{1}{\nu + p_j\lambda_j} - \sum_{j=i}^{L} \rho_j \right) \\ &\quad - \sum_{i=1}^{L} \gamma_i p_i \\ &= \sum_{i=1}^{L} (1-\gamma_i)\, p_i + \sum_{i=1}^{L} \tilde{\mu}_i \left(\frac{1}{\nu + p_i\lambda_i} - \rho_i \right)\end{aligned} \tag{50}$$

where $\tilde{\mu}_i = \sum_{j=1}^{i} \mu_j$. The dual function is defined as $g(\boldsymbol{\mu}, \boldsymbol{\gamma}) = \inf_{\mathbf{p}} \mathcal{L}(\mathbf{p}, (\boldsymbol{\mu}, \boldsymbol{\gamma}))$. Setting the gradient of the Lagrangian to zero, we obtain

$$1 = \tilde{\mu}_i \frac{\lambda_i}{(\nu + p_i\lambda_i)^2} + \gamma_i.$$

The dual function is then

$$\begin{aligned}g(\boldsymbol{\mu}, \boldsymbol{\gamma}) &= \sum_{i=1}^{L} (1-\gamma_i) \frac{1}{\lambda_i} \left(\sqrt{\frac{\tilde{\mu}_i\lambda_i}{1-\gamma_i}} - \nu \right) \\ &\quad + \sum_{i=1}^{L} \tilde{\mu}_i \left(\sqrt{\frac{1-\gamma_i}{\tilde{\mu}_i\lambda_i}} - \rho_i \right) \\ &= \sum_{i=1}^{L} \left(\sqrt{\frac{\tilde{\mu}_i(1-\gamma_i)}{\lambda_i}} - (1-\gamma_i) \frac{1}{\lambda_i}\nu \right) \\ &\quad + \sum_{i=1}^{L} \left(\sqrt{\frac{\tilde{\mu}_i(1-\gamma_i)}{\lambda_i}} - \tilde{\mu}_i\rho_i \right) \\ &= 2\sum_{i=1}^{L} \sqrt{\frac{\tilde{\mu}_i(1-\gamma_i)}{\lambda_i}} \\ &\quad - \sum_{i=1}^{L} \left(\frac{\nu(1-\gamma_i)}{\lambda_i} + \tilde{\mu}_i\rho_i \right).\end{aligned} \tag{51}$$

The gap, as a function of the primal variable $\mathbf{p}$ and of the dual variables $(\boldsymbol{\mu}, \boldsymbol{\gamma})$, is given by

$$\Gamma(\mathbf{p}, (\boldsymbol{\mu}, \boldsymbol{\gamma})) = f(\mathbf{p}) - g(\boldsymbol{\mu}, \boldsymbol{\gamma}). \tag{52}$$

Evaluating the gap at the suboptimal solution corresponding to the diagonal transmission $p_i = \lambda_i^{-1}\left(\rho_i^{-1} - \nu\right)$ gives

$$\begin{aligned}\Gamma(\boldsymbol{\mu}, \boldsymbol{\gamma}) &= \sum_{i=1}^{L} \frac{\rho_i^{-1} - \nu\gamma_i}{\lambda_i} - 2\sum_{i=1}^{L} \sqrt{\frac{\tilde{\mu}_i(1-\gamma_i)}{\lambda_i}} + \sum_{i=1}^{L} \tilde{\mu}_i\rho_i \\ &= \sum_{i=1}^{L} \left(\sqrt{\frac{\rho_i^{-1} - \nu\gamma_i}{\lambda_i}} - \sqrt{\tilde{\mu}_i\rho_i} \right)^2 \\ &\quad + 2\left(\sqrt{\frac{\rho_i^{-1} - \nu\gamma_i}{\lambda_i}\tilde{\mu}_i\rho_i} - \sqrt{\frac{\tilde{\mu}_i(1-\gamma_i)}{\lambda_i}} \right).\end{aligned} \tag{53}$$

Although the exact loss of optimality is given by $\inf_{(\boldsymbol{\mu}, \boldsymbol{\gamma})} \Gamma(\boldsymbol{\mu}, \boldsymbol{\gamma})$, we can obtain an upper bound on the loss by choosing some other convenient value for the dual variables $(\boldsymbol{\mu}, \boldsymbol{\gamma})$. In particular, by choosing $\gamma_i = 0$, the gap simplifies to

$$\Gamma(\boldsymbol{\mu}) = \sum_{i=1}^{L} \rho_i \left(\sqrt{\frac{1}{\lambda_i\rho_i^2}} - \sqrt{\tilde{\mu}_i} \right)^2. \tag{54}$$

If we could choose now $\tilde{\mu}_i = 1/\left(\lambda_i\rho_i^2\right)$, the gap would become zero, but we can only choose feasible dual variables $\mu_i \ge 0$ or, equivalently, $\tilde{\mu}_i \ge \tilde{\mu}_{i-1}$ ($\tilde{\mu}_0 = 0$). In other words, the gap becomes zero if and only if $\lambda_i\rho_i^2 \le \lambda_{i-1}\rho_{i-1}^2$, which agrees with the result obtained in Proposition 2. Choosing now

$$\tilde{\mu}_i = \max\left(\frac{1}{\lambda_i\rho_i^2}, \tilde{\mu}_{i-1} \right) = \max_{j \le i} \left\{ \frac{1}{\lambda_j\rho_j^2} \right\}, \quad 1 \le i \le L \tag{55}$$

the gap simplifies to

$$\Gamma = \sum_{i=1}^{L} \rho_i \left(\sqrt{\frac{1}{\lambda_i\rho_i^2}} - \sqrt{\max_{j \le i} \left\{ \frac{1}{\lambda_j\rho_j^2} \right\}} \right)^2 \tag{56}$$

from which (32) is readily obtained.

APPENDIX C
PROOF OF COROLLARY 2

It suffices to note that the MSEs achieved by the gap approximation solution in the active subchannels are given by $\text{MSE}_i = 1/(\mu\lambda_i + (\nu - \Gamma))$ (from $\text{MSE}_i = (\text{SINR}_i + \nu)^{-1}$ and $\text{SINR}_i = p_i\lambda_i$) and to invoke Proposition 2.

Since $(\nu - \Gamma) \le \Gamma$ ($\Gamma \ge 1$ and $\nu \in \{0, 1\}$) and $\Gamma < \mu\sqrt{\lambda_i\lambda_{i+1}}$ ($\Gamma < \mu\lambda_i$ and $\Gamma < \mu\lambda_{i+1}$ in the active subchannels), it follows that $(\nu - \Gamma) < \mu\sqrt{\lambda_i\lambda_{i+1}}$. Now, using $\lambda_{i+1} \ge \lambda_i$, we can write

$$\begin{aligned}(\nu - \Gamma) \frac{1}{\sqrt{\lambda_i\lambda_{i+1}}} \left(\lambda_{i+1}^{1/2} - \lambda_i^{1/2} \right) &\le \mu \left(\lambda_{i+1}^{1/2} - \lambda_i^{1/2} \right) \\ \mu\lambda_i^{1/2} + (\nu - \Gamma)\lambda_i^{-1/2} &\le \mu\lambda_{i+1}^{1/2} + (\nu - \Gamma)\lambda_{i+1}^{-1/2} \\ \lambda_i^{1/2} \frac{1}{\mu\lambda_i + (\nu - \Gamma)} &\ge \lambda_{i+1}^{1/2} \frac{1}{\mu\lambda_{i+1} + (\nu - \Gamma)}\end{aligned}$$

where the last inequality corresponds to $\lambda_i^{1/2}\text{MSE}_i \ge \lambda_{i+1}^{1/2}\text{MSE}_{i+1}$ and, hence, Proposition 2 can be invoked. ■

APPENDIX D
PROOF OF COROLLARY 3

It suffices to note that the MSEs achieved by the gap approximation solution plus a distortion in the active subchannels are given by $\mathrm{MSE}_i = 1/(\mu(1-\Delta_i)\lambda_i - (\Gamma-\nu))$ (from $\mathrm{MSE}_i = (\mathrm{SINR}_i + \nu)^{-1}$ and $\mathrm{SINR}_i = p_i\lambda_i$) and to invoke Proposition 2.

The condition of Proposition 2, $\lambda_i^{1/2}\mathrm{MSE}_i \geq \lambda_{i+1}^{1/2}\mathrm{MSE}_{i+1}$, can be written in this case as

$$\sqrt{\frac{\lambda_i}{\lambda_{i+1}}} \leq \frac{\mu(1-\Delta_{i+1}) - (\Gamma-\nu)/\lambda_{i+1}}{\mu(1-\Delta_i) - (\Gamma-\nu)/\lambda_i}$$

or, equivalently, as

$$\Delta_{i+1} \leq \left(1 - \sqrt{\frac{\lambda_i}{\lambda_{i+1}}}\right) + \Delta_i\sqrt{\frac{\lambda_i}{\lambda_{i+1}}} + \frac{(\Gamma-\nu)}{\mu}\left(\sqrt{\frac{1}{\lambda_i\lambda_{i+1}}} - \frac{1}{\lambda_{i+1}}\right).$$

Since $\sqrt{1/(\lambda_i\lambda_{i+1})} - 1/\lambda_{i+1} \geq 0$, a sufficient condition so that the previous inequality is satisfied is (34).

APPENDIX E
PROOF OF COROLLARY 4

From Corollary 2, the diagonal structure is optimal for the gap approximation without distortion, i.e., $\lambda_{i-1}^{1/2}\mathrm{MSE}_{i-1} \geq \lambda_i^{1/2}\mathrm{MSE}_i$ or, equivalently

$$\max_{j\leq i}\left\{\frac{1}{\lambda_j^{1/2}\mathrm{MSE}_j}\right\} = \frac{1}{\lambda_i^{1/2}\mathrm{MSE}_i}$$

which can be rewritten [using $\mathrm{MSE}_i = (\mu\lambda_i - (\Gamma-\nu))^{-1}$] as

$$\max_{j\leq i}\left\{\mu\lambda_j^{1/2} - (\Gamma-\nu)\lambda_j^{-1/2}\right\} = \mu\lambda_i^{1/2} - (\Gamma-\nu)\lambda_i^{-1/2}.$$

With the distortion, however, the MSE is $\mathrm{MSE}_i = (\mu(1-\Delta_i)\lambda_i - (\Gamma-\nu))^{-1}$, and we can write

$$\begin{aligned}
&\max_{j\leq i}\left\{\frac{1}{\lambda_j^{1/2}\mathrm{MSE}_j}\right\} \\
&= \max_{j\leq i}\left\{\mu\lambda_j^{1/2} - (\Gamma-\nu)\lambda_j^{-1/2} - \mu\Delta_j\lambda_j^{1/2}\right\} \\
&\leq \max_{j\leq i}\left\{\mu\lambda_j^{1/2} - (\Gamma-\nu)\lambda_j^{-1/2}\right\} + \max_{j\leq i}\left\{-\mu\Delta_j\lambda_j^{1/2}\right\} \\
&= \mu\lambda_i^{1/2} - (\Gamma-\nu)\lambda_i^{-1/2} - \min_{j\leq i}\left\{\mu\Delta_j\lambda_j^{1/2}\right\} \\
&= \frac{1}{\lambda_i^{1/2}\mathrm{MSE}_i} + \mu\left(\Delta_i\lambda_i^{1/2} - \min_{j\leq i}\left\{\Delta_j\lambda_j^{1/2}\right\}\right).
\end{aligned}$$

The loss incurred by using a diagonal transmission can then be upper bounded (using Proposition 3) as

$$\Delta P \leq \mu^2\sum_{i\in\mathcal{I}}\mathrm{MSE}_i \times \left(\Delta_i\lambda_i^{1/2} - \min_{j\leq i}\left\{\Delta_j\lambda_j^{1/2}\right\}\right)^2. \quad \blacksquare$$

APPENDIX F
PROOF OF COROLLARY 5

The MSEs achieved by the equal power allocation $p_i = P_0/L$ are $\mathrm{MSE}_i = 1/(\nu + P_0\lambda_i/L)$. We can now invoke Proposition 2 to obtain the condition for the optimality of the diagonal structure.

The condition of Proposition 2 $\lambda_i^{1/2}\mathrm{MSE}_i \geq \lambda_{i+1}^{1/2}\mathrm{MSE}_{i+1}$ can be written in this case as

$$\lambda_i^{1/2}\frac{1}{\nu + \frac{P_0\lambda_i}{L}} \geq \lambda_{i+1}^{1/2}\frac{1}{\nu + \frac{P_0\lambda_{i+1}}{L}}$$

or, after some manipulations (recalling that $\lambda_{i+1} \geq \lambda_i$), as

$$\left(\frac{\nu}{P_0/L}\right)^2 \leq \lambda_i\lambda_{i+1}.$$

Since the eigenvalues are in increasing order, the previous conditions for all subchannels is equivalent to

$$\left(\frac{\nu}{P_0/L}\right)^2 \leq \lambda_1\lambda_2.$$

A sufficient condition is given in (38). ■

APPENDIX G
PROOF OF COROLLARY 6

To upper bound the loss, instead of using the final result given in Proposition 3, we use the similar expression in (54). Noting that the term $1/\left(\lambda_i^{1/2}\mathrm{MSE}_i\right)$ is increasing in λ_i for $\lambda_i \geq \nu/(P_0/L)$ (and decreasing otherwise), we can particularize the upper bound on the loss of power in (54) with the choice

$$\sqrt{\tilde{\mu}_i} = \begin{cases} 0, & \lambda_i < \frac{\nu}{P_0/L} \\ \frac{1}{\lambda_i^{1/2}\mathrm{MSE}_i}, & \text{otherwise} \end{cases}$$

since the condition $\tilde{\mu}_i \leq \tilde{\mu}_{i+1}$ is then satisfied as required in (54). The bound on the power loss is then

$$\Delta P \leq \sum_{i:\lambda_i<\nu/(P_0/L)}\frac{1}{\lambda_i\mathrm{MSE}_i}$$

from which (39) is obtained using $\mathrm{MSE}_i = 1/(\nu + P_0\lambda_i/L)$.

APPENDIX H
PROOF OF COROLLARY 7

If the same constellations are used each with the same BER, then the MSEs are all equal: $\mathrm{MSE}_i = \text{const}$. The condition of Proposition 2 is then

$$\lambda_i \geq \lambda_{i+1}.$$

However, since the channel eigenvalues are assumed in increasing order $\lambda_i \leq \lambda_{i+1}$, this can only be true if and only if $\lambda_i = \lambda_{i+1}$. ■

APPENDIX I
PROOF OF COROLLARY 8

To upper bound the loss, instead of using the final result given in Proposition 3, we use the similar expression in (54). Noting that all the MSEs are equal $\mathrm{MSE}_i = \mathrm{MSE}_0 \; \forall\, i$, the upper bound on the loss of power is

$$\Delta P \leq \sum_{i=1}^{L} \mathrm{MSE}_0^{-1} \left(\lambda_i^{-1/2} - \mathrm{MSE}_0 \sqrt{\tilde{\mu}_i} \right)^2 .$$

Since the λ_i's are in increasing order and the $\tilde{\mu}_i$'s have to be in increasing order, the best choice is $\mathrm{MSE}_0 \sqrt{\tilde{\mu}_i} = \mu \; \forall\, i$, obtaining

$$\Delta P \leq \mathrm{MSE}_0^{-1} \sum_{i=1}^{L} \left(\lambda_i^{-1/2} - \mu \right)^2$$

where the minimizing μ is obtained as $\mu = (1/L) \sum_{i=1}^{L} \lambda_i^{-1/2}$. The bound is finally given by

$$\Delta P \leq \mathrm{MSE}_0^{-1} \left(\sum_{i=1}^{L} \lambda_i^{-1} - \frac{1}{L} \left(\sum_{i=1}^{L} \lambda_i^{-1/2} \right)^2 \right)$$

from which the upper bound in (40) is obtained. ■

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Daniel Pérez Palomar (S'99–M'03) received the Electrical Engineering and Ph.D. degrees from the Technical University of Catalonia (UPC), Barcelona, Spain, in 1998 and 2003, respectively.

During 1998, he was with the Department of Electronic Engineering, King's College London (KCL), London, U.K. From January 1999 to December 2003, he was a Research Assistant with the Department of Signal Theory and Communications, UPC. From April to November 2001, he held a visiting research appointment at the Department of Electrical Engineering, Stanford University, Stanford, CA. From January to December 2002, he was a visiting researcher with the Telecommunications Technological Center of Catalonia (CTTC), Barcelona. From August to November 2003, he was a Guest Researcher with the Department of Signals, Sensors, and Systems, Royal Institute of Technology (KTH), Stockholm, Sweden. From November 2003 to February 2004, he was a Visiting Researcher with the INFOCOM Department, University of Rome "La Sapienza," Rome, Italy. He is currently a Fulbright Research Fellow at Princeton University, Princeton, NJ. He has participated in several European projects such as ACTS-SUNBEAM (1999), IST-METRA (2000–2001), IST I-METRA (2001–2003), and IST ROMANTIK (2002–2004). His primary research interests include information-theoretic and communication aspects of wireless MIMO channels and array signal processing, with special emphasis on convex optimization theory applied to communications systems.

Dr. Palomar received the 2002–2003 Rosina Ribalta first prize for the Best Doctoral Thesis within the areas of information technologies and communications from the Epson Foundation. He also received the 2003 Prize for the Best Doctoral Thesis in advanced mobile communications from the Vodaphone Foundation and COIT. He has also received a Fulbright Research Fellowship.

Sergio Barbarossa (M'88) received the electrical engineering and the Ph.D. degrees in 1984 and 1988, respectively, both from the University of Rome "La Sapienza," Rome, Italy.

He started his research activity on synthetic aperture radar (SAR) in 1984, working at Selenia, Rome. In 1987, he was with the Environmental Research Institute of Michigan (ERIM), Ann Arbor. From 1988 to 1991, he was an Assistant Professor with the University of Perugia, Perugia, Italy. In November 1991, he joined the University of Rome "La Sapienza," where he is currently a full professor. He has held positions as visiting scientist and visiting professor at the University of Virginia, Charlotteville, in 1995 and 1997; the University of Minnesota, Minneapolis, in 1999; and the Polytechnic University of Catalonia, Barcelona, Spain, in 2001. He is the author of a research monograph entitled "Multiantenna Wireless Communication Systems." He has been responsible for the team from his University that is involved in the international projects *SATURN*, on space-time coding, and *ROMANTIK*, on multihop wireless networks, funded by the European Union. His current research interests lie in the areas of cooperative communications, random graphs, self-organizing networks, sensor networks, and space-time coding.

Dr. Barbarossa has been a member of the IEEE Signal Processing for Communications Technical Committee since 1997. He served as an Associate Editor for the IEEE TRANSACTIONS ON SIGNAL PROCESSING. He organized, as the general chairman, the Fourth IEEE Workshop on Signal Processing Advances in Wireless Communications, Rome, in June 2003. He received the 2000 IEEE Best Paper Award from the IEEE Transactions on Signal Processing Society in the area of Signal Processing for Communications as the co-author of a paper on optimal precoding and blind channel equalization. He is currently a member of the Editorial Board of the IEEE TRANSACTIONS ON SIGNAL PROCESSING, focusing on sensor networks.