Convex Primal Decomposition for Multicarrier Linear MIMO Transceivers

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***Abstract*—The design of linear transceivers for multiple-input–multiple-output (MIMO) communication systems with channel state information is particularly challenging for two main reasons. First, since several substreams are established through the MIMO channel, it is not even clear how the quality of the system should be measured. Second, once a cost function has been chosen to measure the quality, the optimization of the system according to such criterion is generally difficult due to the nonconvexity of the problem. Recent results have solved the problem for the wide family of Schur-concave/convex functions, resulting in simple closed-form solutions when the system is modeled as a single MIMO channel. However, with several MIMO channels (such as in multi-antenna multicarrier systems), the solution is generally more involved, leading in some cases to the need to employ general-purpose interior-point methods. This problem is specifically addressed in this paper by combining the closed-form solutions for single MIMO channels with a primal decomposition approach, resulting in a simple and efficient method for multiple MIMO channels. The extension to functions that are not Schur-concave/convex is also briefly considered, relating the present work with a recently proposed method to minimize the average bit error rate (BER) of the system.**

***Index Terms*—Convex optimization theory, linear precoder, majorization theory, multicarrier transceiver, multiple-input–multiple-output (MIMO) channel, primal decomposition technique, Schur convexity, waterfilling.**

I. Introduction

MULTIPLE-input–multiple-output (MIMO) channels provide a unified way to treat many different communication channels of diverse physical nature such as wireless communications with multiples antennas at both sides of the link [1]–[4] and digital subscriber line (DSL) systems [5]. MIMO channels can be conveniently and compactly represented by a channel matrix notation which is simple and powerful.

When channel state information (CSI) is available at both the transmitter and receiver, the system can adapt to each channel realization to improve the quality of the communication and/or the spectral efficiency. From an information-theoretic viewpoint, the best design in terms of capacity is well known and is given by the employment of ideal Gaussian codes [6], [1], [3], [7].

In practice, the ideal Gaussian codes are substituted with finite order constellations (such as QAM) and practical coding schemes. Furthermore, to simplify the design of such a system, it is customary to divide it into an inner uncoded part, which transmits symbols drawn from given constellations, and an outer coded part that adds redundancy in order to include error correction capabilities. Although the ultimate system performance depends on the combination of both parts (in fact, for some systems, such a division does not even apply), it is convenient, from the mathematical tractability point of view, to consider the uncoded and coded parts separately. This paper focuses on the uncoded part of the system and, specifically, on the employment of linear transceivers (commonly referred to as linear precoder at the transmitter and linear equalizer at the receiver).

The design of linear MIMO transceivers was initially considered in the 1970s by optimizing simple measures of quality of the system such as the sum of the mean square error (MSE) of all channel substreams or, equivalently, the trace of the MSE matrix [8]–[12] (see [13] for an extension to dispersive channels with arbitrary length, which allows to deal with overlapped block transmissions). In [14], the determinant of the MSE matrix was minimized instead. In [11], a maximum signal to interference-plus-noise ratio (SINR) criterion with a zero-forcing (ZF) constraint was also considered. The minimization of the bit error rate (BER) averaged over the channel substreams was treated in detail in [15], where a diagonal structure is imposed. Recently, the minimum BER design without the diagonal structure assumption was independently solved in [16] and [17] (for the case of equal constellations), obtaining an optimal nondiagonal structure. Similarly, the minimum BER design was considered in [18] for the class of orthogonal frequency division multiplexing (OFDM) transceivers with channel-independent unitary precoders, including a detailed comparison of the optimality of multicarrier and single-carrier approaches. The minimization of the BER was also considered in [19] for finite impulse response (FIR) MIMO channel and transceiver, where the transmitter and receiver were alternatively optimized in an iterative fashion. The joint design of the transceiver and constellations for a given probability of error was derived under a perfect reconstruction criterion in [20] and then extended to multiservice communications in [21], [22].

In [17], a general framework was developed to consider a wide range of different design criteria; in particular, the optimal design for Schur-concave and Schur-convex cost functions [23] was obtained (cf. Section III-A). Summarizing, when the performance of the system is measured by a Schur-concave/convex function and the system is modeled as a single channel matrix, then the solution admits a simple closed-form expression [17] that can be readily implemented. If the cost function is not Schur-concave/convex, the problem can still be optimally

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solved, although it is not as straightforward; an example is [24], where the minimum BER solution was obtained for the general case of different constellations. When the system is modeled as multiple MIMO channels (e.g., a multi-antenna multi-carrier system), the general solutions of [17] still apply to each of the MIMO channels but the problem of allocating the total power among the MIMO channels remains. The consequence is that in some cases, instead of having a closed-form solution, one has to resort to general-purpose iterative methods. For some specific cases, it is still possible to devise a tailored algorithm after a painstaking detailed analysis of the problem structure. Hence, having multiple MIMO channels, as in multicarrier systems, may constitute a barrier for a simple and practical development/implementation.

This paper proposes a primal decomposition approach [25]–[28] that allows the employment of the general solutions obtained in [17] for Schur-concave/convex functions to multiple MIMO channels in a simple and efficient way. The primal decomposition method is based on decomposing the original complicated problem into several simple subproblems controlled by a simple master problem. In a nutshell, a methodology is provided to extend the simple closed-form solutions for Schur-concave/convex functions in a single MIMO channel of [17] to multiple MIMO channels (e.g., multicarrier) in a straightforward way. The idea of using a primal decomposition to tackle an otherwise difficult problem was similarly used in [24] for a different purpose. To be more specific, [24] dealt with a non-Schur-concave/convex cost function via a primal decomposition using as a fundamental building block the subproblem characterized in [29], consisting on the minimization of the transmitted power subject to a set of quality-of-service (QoS) requirements.

The paper is structured as follows. Section II describes the system model and formulates the problem. The main contribution of the paper is given in Section III, where the subproblems and the master problem are properly characterized. The proposed approach is illustrated in Section IV with three interesting examples. Section V briefly describes the extension to cost functions that are not Schur-concave/convex (relating this paper with [24]). Then, a numerical assessment of the theory developed is given in Section VI with simulation results. Finally, Section VII concludes with a summary of the results of this paper.

A. Notation

Boldface upper-case letters denote matrices, boldface lowercase letters denote column vectors, and italics denote scalars. $\mathbb{R}$, $\mathbb{R}_{+}$, and $\mathbb{R}_{++}$ denote the set of real, nonnegative real, and positive real numbers, respectively. The super-scripts $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ denote transpose, complex conjugate, and Hermitian operations, respectively. $[\mathbf{X}]_{ij}$ denotes the (ith, jth) element of matrix $\mathbf{X}$ and, for vectors, x_i denotes the ith element of $\mathbf{x}$. $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix and $\mathrm{diag}(\{x_k\})$ is a diagonal matrix with diagonal elements given by the set $\{x_k\}$. The projection on the nonnegative orthant is denoted by $(x)^+ \triangleq \max(0,x)$. The derivative of function f is denoted by $f' \triangleq df/dx$, the partial derivative by $\partial f/\partial x$, the gradient by ∇f, and the subdifferential (set of subgradients) by ∂f.

II. System Model and Problem Formulation

A. System Model

This paper considers a general communication system composed of a set of N parallel and noninterfering MIMO channels each with n_T transmit and n_R receive dimensions:

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{s}_k + \mathbf{n}_k, \qquad 1 \le k \le N \tag{1}$$

where k is the channel index and, for each k, $\mathbf{s}_k$ is the transmitted $n_T \times 1$ vector, $\mathbf{H}_k$ is the $n_R \times n_T$ channel matrix, $\mathbf{y}_k$ is the received $n_R \times 1$ vector, and $\mathbf{n}_k$ is a zero-mean circularly symmetric complex Gaussian interference-plus-noise $n_R \times 1$ vector with arbitrary covariance matrix $\mathbf{R}_{n_k}$. All quantities are assumed complex and the noise among different MIMO channels is assumed independent. The most obvious example of the system model in (1) is a multicarrier MIMO system, where each carrier is modeled as a parallel channel. Typical examples are multiantenna multicarrier systems [2], [3], [1], [4] and wireline DSL systems [5].

Considering a linear processing approach, the transmitted vector can be written as (see Fig. 1)

$$\mathbf{s}_k = \mathbf{B}_k \mathbf{x}_k, \qquad 1 \le k \le N \tag{2}$$

where, for each k, $\mathbf{B}_k$ is the $n_T \times L_k$ transmit matrix (precoder) and $\mathbf{x}_k$ is the $L_k \times 1$ data vector that contains the L_k symbols to be transmitted (zero-mean,[1] normalized and uncorrelated, i.e., $\mathbb{E}\left[\mathbf{x}_k \mathbf{x}_k^H\right] = \mathbf{I}$) drawn from a set of constellations. For the sake of notation, it is assumed that $L_k \le \min(n_R, n_T)$. The total average transmitted power (in units of energy per transmission) is

$$P_T = \sum_{k=1}^{N} \mathbb{E}[\|\mathbf{s}_k\|^2] = \sum_{k=1}^{N} \mathrm{Tr}\left(\mathbf{B}_k \mathbf{B}_k^H\right). \tag{3}$$

Similarly, the estimated data vector at the receiver is (see Fig. 1)

$$\hat{\mathbf{x}}_k = \mathbf{A}_k^H \mathbf{y}_k \qquad 1 \le k \le N \tag{4}$$

where $\mathbf{A}_k^H$ is the $L_k \times n_R$ receive matrix (equalizer).

Focusing on the kth MIMO channel and ith substream, the signal model is

$$\hat{x}_{k,i} = \mathbf{a}_{k,i}^H \left(\mathbf{H}_k \mathbf{b}_{k,i} x_{k,i} + \mathbf{n}_{k,i}\right) \tag{5}$$

where $\mathbf{b}_{k,i}$ and $\mathbf{a}_{k,i}$ are the ith columns of $\mathbf{B}_k$ and $\mathbf{A}_k$, respectively, $\mathbf{n}_{k,i} = \sum_{j \ne i} \mathbf{H}_k \mathbf{b}_{k,j} x_{k,j} + \mathbf{n}_k$ is the equivalent noise seen by the (k,i)th substream, with covariance matrix $\mathbf{R}_{n_{k,i}} = \sum_{j \ne i} \mathbf{H}_k \mathbf{b}_{k,j} \mathbf{b}_{k,j}^H \mathbf{H}_k^H + \mathbf{R}_{n_k}$.

Measures of Quality: The quality of the (k,i)th established substream or link in (5) can be conveniently measured, among others, in terms of MSE, SINR, or BER, defined, respectively, as

$$\mathrm{MSE}_{k,i} \triangleq \mathbb{E}[|\hat{x}_{k,i} - x_{k,i}|^2] = \left|\mathbf{a}_{k,i}^H \mathbf{H}_k \mathbf{b}_{k,i} - 1\right|^2 + \mathbf{a}_{ki}^H \mathbf{R}_{n_{k,i}} \mathbf{a}_{k,i} \tag{6}$$

[1]If a constellation does not have zero mean, the receiver can always remove the mean and then proceed as if the mean was zero. Indeed, the mean of the signal does not carry any information and can always be set to zero saving power at the transmitter; otherwise, there is a loss of transmitted power.

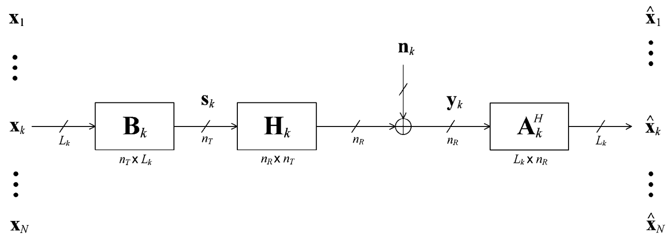

Fig. 1. Scheme of linear MIMO transceivers for multiple MIMO channels (muticarrier MIMO system).

$$\mathrm{SINR}_{k,i} \triangleq \frac{\text{desired component}}{\text{undesired component}} = \frac{\left|\mathbf{a}_{k,i}^H \mathbf{H}_k \mathbf{b}_{k,i}\right|^2}{\mathbf{a}_{k,i}^H \mathbf{R}_{n_{k,i}} \mathbf{a}_{k,i}} \tag{7}$$

$$\mathrm{BER}_{k,i} \triangleq \frac{\#\text{ bits in error}}{\#\text{ transmitted bits}} \approx \tilde{g}_{k,i}\left(\mathrm{SINR}_{k,i}\right) \tag{8}$$

where $\tilde{g}_{k,i}$ is a function that relates the SINR to the BER at the (k,i)th substream. For most types of modulations, the BER can indeed be analytically expressed as a function of the SINR when the interference-plus-noise term follows a Gaussian distribution [30], [31]; otherwise, it is an approximation (see [24] for a more detailed discussion).

It will be notationally convenient to define the kth MSE matrix as

$$\begin{aligned}\mathbf{E}_k &\triangleq \mathbb{E}[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^H] \\ &= \left(\mathbf{A}_k^H \mathbf{H}_k \mathbf{B}_k - \mathbf{I}\right)\left(\mathbf{B}_k^H \mathbf{H}_k^H \mathbf{A}_k - \mathbf{I}\right) + \mathbf{A}_k^H \mathbf{R}_{n_k} \mathbf{A}_k\end{aligned} \tag{9}$$

from which the MSE of the ith link is obtained as the ith diagonal element of $\mathbf{E}_k$, i.e., $\mathrm{MSE}_{k,i} = [\mathbf{E}_k]_{ii}$.

B. Problem Formulation

The problem considered in this paper is the design of the linear transceivers, i.e., of the linear transmitters $\mathbf{B}_k$ and receivers $\mathbf{A}_k$, to optimize some measure of quality of the system subject to a power constraint (or *vice versa*). A general formulation of problem can be adopted by expressing the global performance of the system with the arbitrary cost function $f_0(\alpha_1, \ldots, \alpha_N)$, where each α_k indicates the performance of each MIMO channel (or carrier) which, in turn, is measured by another arbitrary cost function $f_k\left(\{\mathrm{MSE}_{k,i}\}_{i=1}^{L_k}\right)$ that depends on the MSEs of that particular MIMO channel.[2]

The mathematical formulation of the problem is then

$$\begin{aligned}\min_{\{\mathbf{A}_k, \mathbf{B}_k, \alpha_k\}} \quad & f_0(\alpha_1, \ldots, \alpha_N) \\ \text{s.t.} \quad & \alpha_k = f_k\left(\{\mathrm{MSE}_{k,i}\}_{i=1}^{L_k}\right), \qquad 1 \le k \le N \\ & \sum_{k=1}^{N} \mathrm{Tr}\left(\mathbf{B}_k \mathbf{B}_k^H\right) \le P_0\end{aligned} \tag{10}$$

where P_0 is the maximum global power and the optimization variables are $\{\mathbf{A}_k, \mathbf{B}_k, \alpha_k\}$. The problem can be similarly formulated as the minimization of the power subject to a global quality (both formulations are in fact equivalent).

[2]Of course, it would be even more general to define the global cost function directly as a function of all the MSEs instead of using the intermediate functions f_k. The decomposition adopted, however, not only is very reasonable but also provides the problem with a rich structure that can be conveniently exploited.

A more general problem formulation could be adopted by allowing the cost functions to depend not just on the MSEs but also on the SINRs and BERs. However, as will be shortly justified, any function of the SINRs or BERs can be equivalently expressed as a function of the MSEs; hence, the formulation in (10) is without loss of generality in this sense.

By definition of "cost function," lower values are preferred (lower cost) and correspond to better systems. Also, if a link is evaluated in terms of MSE, it is clear (by definition) that lower values of the MSE are more desirable. As a consequence, it is completely reasonable to assume that all the cost functions f_0 and $f_1, \ldots, f_N$ are increasing in each argument.

C. Optimum Receiver

The receive matrices $\mathbf{A}_k$ can be independently and easily optimized for given and fixed transmit matrices $\mathbf{B}_k$. Interestingly, the minimization of each of the MSEs is decoupled and they can all be minimized simultaneously without any tradeoff. Therefore, the optimal receiver is independent of the particular choice of the cost functions (for more details, the reader is referred to [17], [29], and [32]). The simultaneous minimization of all the MSEs is achieved by the well-known linear MMSE receiver, also termed Wiener filter [33]. If the additional ZF constraint $\mathbf{A}_k^H \mathbf{H}_k \mathbf{B}_k = \mathbf{I}$ is imposed to avoid crosstalk among the substreams (which may happen with the MMSE receiver), then the well-known ZF receiver is obtained. Interestingly, the MMSE and ZF receivers are also optimum in the sense that they maximize simultaneously all SINRs and, consequently, minimize simultaneously all BERs (cf. [17] and [32]).

The MMSE and ZF receivers can be compactly written as

$$\mathbf{A}_k = \mathbf{R}_{n_k}^{-1} \mathbf{H}_k \mathbf{B}_k \left(\nu \mathbf{I} + \mathbf{B}_k^H \mathbf{H}_k^H \mathbf{R}_{n_k}^{-1} \mathbf{H}_k \mathbf{B}_k\right)^{-1} \tag{11}$$

where the parameter ν is 1 for the MMSE receiver and 0 for the ZF receiver. The MSE matrix (9) reduces then to the following concentrated expression:

$$\mathbf{E}_k = \left(\nu \mathbf{I} + \mathbf{B}_k^H \mathbf{R}_{H_k} \mathbf{B}_k\right)^{-1} \tag{12}$$

where $\mathbf{R}_{H_k} \triangleq \mathbf{H}_k^H \mathbf{R}_{n_k}^{-1} \mathbf{H}$ is the squared whitened channel matrix.

Relation Among Different Measures of Quality: It is convenient now to relate the different measures of quality, namely, MSE, SINR, and BER, to the concentrated MSE matrix in (12). By definition, the individual MSEs are given by the diagonal elements of the MSE matrix

$$\mathrm{MSE}_{k,i} = \left[\left(\nu \mathbf{I} + \mathbf{B}_k^H \mathbf{R}_{H_k} \mathbf{B}_k\right)^{-1}\right]_{ii}. \tag{13}$$

It turns out that the SINRs and the MSEs are trivially related when using the MMSE or ZF receivers as [31, Prob. 6.5], [17], [29], [32]

$$\mathrm{SINR}_{k,i} = \frac{1}{\mathrm{MSE}_{k,i}} - \nu. \tag{14}$$

Finally, the BERs can also be written as a function of the MSEs:

$$\mathrm{BER}_{k,i} = g_{k,i}\left(\mathrm{MSE}_{k,i}\right) \triangleq \tilde{g}_{k,i}\left(\mathrm{SINR}_{k,i} = \mathrm{MSE}_{k,i}^{-1} - \nu\right) \tag{15}$$

where $\tilde{g}_{k,i}$ was defined in (8). Observe that, although not explicitly indicated, $g_{k,i}$ depends on ν.

It is important to remark that the BER functions $g_{k,i}$ are convex increasing in the MSE for sufficiently small values of the argument (cf. [17], [32]). As a rule of thumb, the BER as a function of the MSE is convex for a BER less than 2×10^{-2}, which is a mild assumption; interestingly, for BPSK and QPSK constellations the BER function is always convex [17], [32].

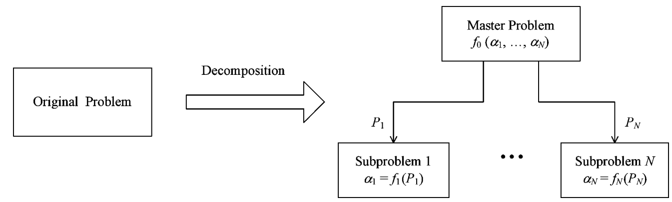

Fig. 2. Illustration of the decomposition of a large problem into several subproblems controlled by a master problem.

III. Primal Decomposition for Schur-Concave/Convex Functions

Many convex optimization problems stemming from real applications have a large number of variables and constraints. In principle, the existing general-purpose methods to solve convex problems, e.g., interior-point methods [34] or cutting-plane methods [28], are capable of handling such large problems. In many cases, however, problems have a very particular structure that allows simplification based on decomposing the original problem into smaller and simpler subproblems, which can be much more easily solved possibly in a parallel fashion [25]. For example, the problem may decouple into independent subproblems when some of the optimization variables are fixed. A master problem is then necessary to coordinate the subproblems by means of the coupling variables [25], [28], [27].

Most of the existing decomposition techniques can be classified into *primal decomposition* and *dual decomposition* methods. The former (also called decomposition by right-hand side allocation or decomposition with respect to variables) is based on decomposing the original primal problem, whereas the latter (also termed Lagrangian relaxation of the coupling constraints or decomposition with respect to constraints) is based on decomposing the dual of the problem [26], [28] (see [35] and [36] for two recent successful applications of dual decomposition). Primal decomposition methods have the interpretation that the master problem gives each subproblem an amount of resources that it can use; the role of the master problem is then to properly allocate the existing resources. Dual decomposition methods have the interpretation that the master problem sets the price for the resources to each subproblem, which has to decide the amount of resources to be used depending on the price; the role of the master problem is then to obtain the best pricing strategy.

This section gives the main result of the paper: a simple and efficient way to solve the problem (10) based on a primal decomposition approach (see Fig. 2). In particular, the proposed solution is based on the key fact that a problem can be optimized by first optimizing over some variables and then over the remaining ones [34, Sec. 4.1.3] (see also [28, Sec. 6.4.2])

$$\min_{\mathbf{x},\mathbf{y}} f(\mathbf{x},\mathbf{y}) = \min_{\mathbf{x}} \min_{\mathbf{y}} f(\mathbf{x},\mathbf{y}). \tag{16}$$

This is commonly called *concentration* in the literature of estimation theory [33].

To be more specific, the problem to be solved (10) assuming the employment of the optimal linear ZF/MMSE receiver (11) (i.e., using the MSEs in (13) from the concentrated MSE matrix in (12)) can be written as

$$\begin{aligned} \min_{\{\mathbf{B}_k,\alpha_k\}} \quad & f_0(\alpha_1,\ldots,\alpha_N) \\ \text{s.t.} \quad & f_k\left(\left\{\left[\left(\nu\mathbf{I}+\mathbf{B}_k^H\mathbf{R}_{H_k}\mathbf{B}_k\right)^{-1}\right]_{ii}\right\}_{i=1}^{L_k}\right) \le \alpha_k \quad 1\le k\le N \\ & \sum_{k=1}^{N} \mathrm{Tr}\left(\mathbf{B}_k\mathbf{B}_k^H\right) \le P_0 \end{aligned} \tag{17}$$

where the inequalities in the constraints $f_k(\cdot)\le\alpha_k$ are equivalent to equalities $f_k(\cdot)=\alpha_k$ since f_0 is increasing.

To successfully employ a primal decomposition approach, problem (17) has to be decomposed in the right way so that the subproblems and the master problem can be easily solved (see Fig. 2). The subproblems are fully characterized in Section III-A, which includes a closed-form expression of the solution [(21) and (22) in Theorem 2, and (27), (28)] and an analysis of the differentiability and convexity, including a closed-form expression of the subgradient (Propositions 1 and 2). The master problem is considered in Section III-B, where a simple solution is obtained (see (41) and Algorithms 1 and 2) because the problem was properly formulated so that the feasible set of the master problem is a simplex.

A. Characterization of the Subproblems

This section focuses on problem (17) for a single carrier k (the subindex k is therefore safely omitted). The optimal solution to the minimization of an arbitrary function subject to a power constraint was solved in [17] for the family of Schur-concave/convex functions. We now restate this result with an extension to other cost functions not necessary Schur-concave/convex.

Theorem 1: The following complicated nonconvex constrained optimization problem:

$$\begin{aligned} \min_{\mathbf{B}} \quad & f\left(\left\{\left[\left(\nu\mathbf{I}+\mathbf{B}^H\mathbf{R}_H\mathbf{B}\right)^{-1}\right]_{ii}\right\}\right) \\ \text{s.t.} \quad & \mathrm{Tr}\left(\mathbf{B}\mathbf{B}^H\right)\le P \end{aligned} \tag{18}$$

where $f:\mathbb{R}^L\to\mathbb{R}$ is an arbitrary increasing function (minimized when the arguments are sorted in decreasing order),[3] is

[3]In practice, most cost functions are minimized when the arguments are in a specific ordering (if not, one can always use instead the function $\tilde{f}_0(\mathbf{x}) = \min_{\mathbf{P}\in\mathcal{P}} f_0(\mathbf{P}\mathbf{x})$, where $\mathcal{P}$ is the set of all permutation matrices) and, hence, the decreasing ordering can be taken without loss of generality.

equivalent to the simple problem

$$\begin{aligned}\min_{\mathbf{p},\boldsymbol{\rho}} \quad & f(\rho_1,\ldots,\rho_L) \\ \text{s.t.} \quad & \sum_{j=i}^{L}\frac{1}{\nu+p_j\lambda_j} \le \sum_{j=i}^{L}\rho_j \qquad 1\le i\le L \\ & \rho_i \ge \rho_{i+1} \\ & \sum_{j=1}^{L} p_j \le P \\ & p_i \ge 0 \end{aligned} \tag{19}$$

where the λ_i's are L largest eigenvalues of $\mathbf{R}_H$ sorted in increasing order $\lambda_i \le \lambda_{i+1}$ and the ρ_i's are the MSEs achieved ($\rho_{L+1} \triangleq 0$). Furthermore, if f is a convex function, problem (19) is convex and the ordering constraints $\rho_i \ge \rho_{i+1}$ can be removed.

More specifically, the optimal solution to problem (18) is given by

$$\mathbf{B} = \mathbf{U}_{H,1}\boldsymbol{\Sigma}_B\mathbf{Q} \tag{20}$$

where $\mathbf{U}_{H,1}$ is an $n_T \times L$ (semi-)unitary matrix that has as columns the eigenvectors of $\mathbf{R}_H$ corresponding to the L largest eigenvalues in increasing order, $\boldsymbol{\Sigma}_B = \text{diag}\left(\left\{\sqrt{p_i}\right\}\right)$ is an $L \times L$ real diagonal matrix with the optimal power allocation $\{p_i\}$ obtained as the solution to problem (19), and $\mathbf{Q}$ is an $L \times L$ unitary matrix such that $\left[(\nu\mathbf{I}+\mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}\right]_{ii} = \rho_i$ for $1 \le i \le L$ (see [37, Sec. IV-A] and [32, Alg. 3.2] for a practical algorithm to obtain $\mathbf{Q}$).

Proof: The key simplification from (18) to (19) is based on an appropriate change of variable based on majorization theory [23]. A sketch of the proof is given in Appendix A (see [17] and [32] for details). ■

It is possible to further particularize the previous result to the case of Schur-concave/convex functions [23, 3.A.1]. In plain words, a function $f(\mathbf{x})$ is Schur-concave if it increases as the elements of the vector $\mathbf{x}$ become more uniform or less spread (for a given total sum of all its elements).[4] Similarly, a function $f(\mathbf{x})$ is Schur-convex if $-f(\mathbf{x})$ is Schur-concave, i.e., if the opposite happens. It is important to remark that most useful cost functions happen to fall within the family of Schur-concave/convex functions [17].

Theorem 2: The optimal solution (20) of Theorem 1 can be further characterized for two particular cases of cost functions (convexity is not required).

- *If f is Schur-concave, then an optimal solution is*

$$\mathbf{B} = \mathbf{U}_{H,1}\boldsymbol{\Sigma}_B. \tag{21}$$

- *If f is Schur-convex, then an optimal solution is*

$$\mathbf{B} = \mathbf{U}_{H,1}\boldsymbol{\Sigma}_B\mathbf{Q} \tag{22}$$

  where $\mathbf{Q}$ is a unitary matrix such that $(\mathbf{I}+\mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}$ has identical diagonal elements, i.e., the system has identical MSEs in all substreams. This "rotation" matrix $\mathbf{Q}$ can be obtained as any unitary matrix that satisfies $|[\mathbf{Q}]_{ik}| = |[\mathbf{Q}]_{il}|$, $\forall i,k,l$ such as the DFT matrix or the Hadamard matrix (when the dimensions are appropriate such as a power of two [31, p. 66]).

Proof: The proof is based on the definition of Schur-concavity/convexity within the context of the problem (see Appendix B). ■

In the following two subsections, the previous result is further studied for the class of Schur-concave/convex functions, which happen to be easily solved in practice as required for the primal decomposition approach. To be more precise, the minimum cost value of (18) as a function of the power P, denoted (with some abuse of notation) by $f(P)$, is characterized, as well as a corresponding gradient/subgradient.

1) Schur-Concave Functions: If $f(\boldsymbol{\rho})$ is Schur-concave [17] and increasing, Theorem 2 can be invoked to show that the MSEs are given (from (13) and (21)) by

$$\begin{aligned}\text{MSE}_i &= \left[\left(\nu\mathbf{I}+\boldsymbol{\Sigma}_B^H\mathbf{D}_H\boldsymbol{\Sigma}_B\right)^{-1}\right]_{ii} \\ &= \frac{1}{\nu+p_i\lambda_i} \qquad 1\le i\le L \end{aligned}\tag{23}$$

where $\mathbf{D}_H = \mathbf{U}_{H,1}^H\mathbf{R}_H\mathbf{U}_{H,1}$ is a diagonal matrix that contains the L largest eigenvalues of $\mathbf{R}_H$ in increasing order. If, in addition, $f(\boldsymbol{\rho})$ is convex, then $f(\mathbf{p})$ is also convex (since $\rho_i = 1/(\nu+p_i\lambda_i)$ is convex in p_i) [34, Sec. 3.2.4] and the problem (18) can be formulated in convex form as

$$\begin{aligned}\min_{\mathbf{p}} \quad & f\left(\left\{\frac{1}{\nu+p_i\lambda_i}\right\}\right) \\ \text{s.t.} \quad & \sum_{j=1}^{L} p_j \le P \\ & p_i \ge 0 \qquad 1\le i\le L. \end{aligned}\tag{24}$$

The optimal solution to problem (24) clearly depends on the particular choice of f and can be characterized from the Karush–Kuhn–Tucker (KKT) optimality conditions [28], [34] as follows (differentiability of f is assumed here for the sake of notation).

- For the ZF receiver ($\nu = 0$), the solution of (24) is the solution to the equations

$$\frac{1}{p_i^2\lambda_i}\left.\frac{\partial f}{\partial \rho_i}\right|_{\rho_i=(p_i\lambda_i)^{-1}} = \mu \qquad \forall i. \tag{25}$$

- For the MMSE receiver ($\nu = 1$), the solution of (24) is the solution to the equations

$$\begin{cases}\frac{\lambda_i}{(1+p_i\lambda_i)^2}\left.\frac{\partial f}{\partial \rho_i}\right|_{\rho_i=(1+p_i\lambda_i)^{-1}} = \mu & \text{if} \quad \mu < \lambda_i \left.\frac{\partial f}{\partial \rho_i}\right|_{\rho_i=1} \\ p_i = 0, & \text{otherwise}\end{cases} \quad \forall i. \tag{26}$$

In both cases, the optimal solution is parameterized by the Lagrange multiplier μ, commonly termed *waterlevel*, that has to be chosen such that $\sum_{i=1}^{L} p_i = P$.

For illustrative purposes, we now particularize the general expressions (25)–(26) to a couple of Schur-concave functions [17]

[4] A function f is Schur-concave if $\mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{x}) \ge f(\mathbf{y})$, where the majorization relation $\mathbf{x} \prec \mathbf{y}$ means that $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i$ for $1 \le k \le n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ (the elements of $\mathbf{x}$ and $\mathbf{y}$ are assumed in decreasing order) [23, 3.A.1].

(see [32] for other examples such as the minimization of the weighted sum/product of the MSEs and the maximization of the weighted sum/product of the SINRs).

- Sum of MSEs: $f(\boldsymbol{\rho}) = \sum_{i=1}^{L} \rho_i$. The solution is

$$p_i = \left(\mu^{-1/2}\lambda_i^{-1/2} - \nu\lambda_i^{-1}\right)^{+} \tag{27}$$

where the waterlevel is readily given in the ZF case by $\mu = \left(\sum_i \lambda_i^{-1/2}\right)^2 / P^2$ and (27) simplifies then to $p_i = \mu^{-1/2}\lambda_i^{-1/2}$.

- Product of MSEs: $f(\boldsymbol{\rho}) = \prod_{i=1}^{L} \rho_i$. The solution is

$$p_i = \left(\mu^{-1} f^{\star} - \nu\lambda_i^{-1}\right)^{+} \tag{28}$$

where the waterlevel is readily given in the ZF case by $\mu = f^{\star}L/P$ and $f^{\star}$ denotes the cost function evaluated at the optimal solution (28) (note that the optimal solution in the ZF case is simply a uniform power allocation $p_i = P/L$).

The previous particularized closed-form solutions (27)–(28) can be compared to those obtained in [17], [32]. When the solution is given by a waterfilling expression such as (27) and (28) (with the MMSE receiver), the waterlevel cannot be directly found and it is then necessary to use either an iterative approach or a hypothesis testing algorithm, which is still very simple in practice (cf. [38]).

Keeping in mind that problem (24) will constitute a subproblem in the primal decomposition approach, it is necessary to characterize its corresponding gradient ∇f or, if not differentiable, the subdifferential ∂f (set of subgradients) or simply one subgradient.

Proposition 1: Let $f(P)$ denote a function defined as the optimal cost value of the problem (24) when the power is constrained by P. Then, the following hold.

a) The function $f(P)$ is convex on $\mathbb{R}_+$ (recall that $f(\boldsymbol{\rho})$ is assumed convex).

b) A subgradient of $f(P)$ at P is given by $-\mu$, where μ is an optimal Lagrange multiplier of problem (24) associated to the power constraint $\sum_{i=1}^{L} p_i \leq P$ (which is implicitly obtained when $f(P)$ is evaluated or, equivalently, when (24) is solved).

c) The function $f(P)$ is differentiable on $\mathbb{R}_{++}$ if $f(\boldsymbol{\rho})$ is differentiable (and then $\nabla f(P) = -\mu$).

Proof: See Appendix C. ■

2) Schur-Convex Functions: If $f(\boldsymbol{\rho})$ is Schur-convex [17] and increasing, Theorem 2 can be invoked to show that the MSEs are equal and given [from (13) and (22)] by

$$\begin{aligned}\mathrm{MSE}_i &= \frac{1}{L}\mathrm{Tr}(\mathbf{E}) \\ &= \frac{1}{L}\mathrm{Tr}\left(\left(\nu\mathbf{I} + \boldsymbol{\Sigma}_B^H \mathbf{D}_H \boldsymbol{\Sigma}_B\right)^{-1}\right) \\ &= \frac{1}{L}\sum_{j=1}^{L}\frac{1}{\nu + p_j\lambda_j} \qquad 1 \leq i \leq L. \end{aligned} \tag{29}$$

If, in addition, $f(\boldsymbol{\rho})$ is convex, the problem can be formulated in convex form as

$$\begin{aligned}\min_{\mathbf{p}} \quad & f\left(\left\{\frac{1}{L}\sum_{j=1}^{L}\frac{1}{\nu + p_j\lambda_j}\right\}\right) \\ \text{s.t.} \quad & \sum_{j=1}^{L} p_j \leq P \\ & p_i \geq 0 \qquad 1 \leq i \leq L \end{aligned} \tag{30}$$

or, equivalently, as

$$\begin{aligned}\min_{\mathbf{p}} \quad & \sum_{j=1}^{L}\frac{1}{\nu + p_j\lambda_j} \\ \text{s.t.} \quad & \sum_{j=1}^{L} p_j \leq P \\ & p_i \geq 0 \qquad 1 \leq i \leq L \end{aligned} \tag{31}$$

which is convex even if $f(\boldsymbol{\rho})$ is nonconvex.

Surprisingly, this simplified convex problem for Schur-convex functions does not depend on the cost function f. The reason is that all the MSEs are equal and f is increasing in each argument; consequently, minimizing the cost function is equivalent to minimizing the equal arguments given by (29).

At this point, problem (31) is nothing else than a particular case of (24) with cost function $\sum_{i=1}^{L} \rho_i$ and the solution is given by (27).

The characterization of the gradient of the optimal cost value of (31) is just a particular case of Proposition 1. However, it is the optimal cost value of (30) that has to be characterized for the primal decomposition approach.

Proposition 2: Let $f(P)$ denote a function defined as the optimal cost value of the problem (30) when the power is constrained by P. Then, the following hold.

a) The function $f(P)$ is convex on $\mathbb{R}_+$ (recall that $f(\boldsymbol{\rho})$ is assumed convex).

b) A subgradient of $f(P)$ at P can be readily obtained from the optimal Lagrange multiplier μ of the simplified problem (31) associated to the power constraint $\sum_{i=1}^{L} p_i \leq P$ (which is implicitly obtained when $f(P)$ is evaluated or, equivalently, when (31) is solved). For example, if $f(\boldsymbol{\rho})$ is differentiable, then $\nabla f(P) = -\mu(1/L)\sum_{i=1}^{L} \partial f/\partial\rho_i$.

c) The function $f(P)$ is differentiable on $\mathbb{R}_{++}$ if $f(\boldsymbol{\rho})$ is differentiable.

Proof: Properties a) and c) are as in Proposition 1. Property b) follows from the chain rule $\partial f/\partial P = \sum_{i=1}^{L}(\partial f/\partial\rho_i)(\partial\rho_i/\partial P)$, where $\partial\rho_i/\partial P = -\mu/L$ by applying Proposition 1 to the simplified problem (31). ■

For illustrative purposes, we now consider some examples of Schur-convex functions [17] [with solution given by (27)] and obtain a subgradient [according to Proposition 2 (b)] recalling that, at an optimal point, $\rho_i = \rho$ for all i.

- Maximum of MSEs: $f(\boldsymbol{\rho}) = \max_i \{\rho_i\}$, which is not differentiable. Interestingly, the function $f(P) = \rho$ is differentiable with gradient
$$\nabla f(P) = -\frac{\mu}{L}. \tag{32}$$
- Harmonic mean of SINRs: The function to be minimized (inverse of the harmonic mean) is $f(\boldsymbol{\rho}) = \sum_i \left(\rho_i^{-1} - \nu\right)^{-1}$. The function $f(P) = L/\left(\rho^{-1} - \nu\right)$ is differentiable and the gradient is
$$\nabla f(P) = -\frac{\mu}{(1-\nu\rho)^2}. \tag{33}$$
- Average BER (assuming equal constellations, i.e., $g_i = g$ for all i): $f(\boldsymbol{\rho}) = (1/L)\sum_{i=1}^{L} g(\rho_i)$ where g is a convex (differentiable) function defined in (15) and (6). The function $f(P) = g(\rho)$ is differentiable with gradient
$$\nabla f(P) = -\frac{\mu}{L} g'(\rho). \tag{34}$$

B. Characterization of the Master Problem

Now that the subproblems have been properly characterized, we are ready to decompose problem (17) in a very convenient way (see Fig. 2).

Theorem 3: The power-constrained problem in (17), where $f_0 : \mathbb{R}^N \to \mathbb{R}$ and each $f_k : \mathbb{R}^{L_k} \to \mathbb{R}$ are assumed to be arbitrary convex and increasing functions, can be equivalently written in a decomposed and simplified way as the following convex (master) problem:
$$\begin{array}{ll} \min\limits_{\{P_k\}} & f_0(P_1,\ldots,P_N) \\ \text{s.t.} & \sum\limits_{k=1}^{N} P_k \le P_0 \\ & P_k \ge 0 \qquad 1 \le k \le N \end{array} \tag{35}$$
where $f_0(P_1,\ldots,P_N) \triangleq f_0(f_1(P_1),\ldots,f_N(P_N))$ and each $f_k(P_k)$ corresponds to the minimum cost value of the subproblem (as considered in Section III-A)
$$\begin{array}{ll} \min\limits_{\mathbf{B}_k} & f_k\left(\left\{\left[\left(\nu\mathbf{I} + \mathbf{B}_k^H \mathbf{R}_{H_k} \mathbf{B}_k\right)^{-1}\right]_{ii}\right\}_{i=1}^{L_k}\right) \\ \text{s.t.} & \operatorname{Tr}\left(\mathbf{B}_k \mathbf{B}_k^H\right) \le P_k. \end{array} \tag{36}$$

Proof: The proof follows easily by rewriting the original problem (17) with the additional variables $\{P_k\}$ and using (16):
$$\begin{array}{ll} \min\limits_{\{P_k\}} \min\limits_{\{\mathbf{B}_k,\alpha_k\}} & f_0(\alpha_1,\ldots,\alpha_N) \\ \text{s.t.} & f_k\left(\left\{\left[\left(\nu\mathbf{I}+\mathbf{B}_k^H \mathbf{R}_{H_k} \mathbf{B}_k\right)^{-1}\right]_{ii}\right\}_{i=1}^{L_k}\right) \le \alpha_k \\ & \quad 1 \le k \le N \\ & \operatorname{Tr}\left(\mathbf{B}_k \mathbf{B}_k^H\right) \le P_k \\ & \sum\limits_{k=1}^{N} P_k \le P_0 \\ & P_k \ge 0. \end{array} \tag{37}$$

It is then clear that for a given set of P_k's, the inner minimization can be done in a decoupled way for each k, which corresponds to the subproblem treated in detail in Section III-A and denoted here by $f_k(P_k)$. The simplified problem (35) follows then by realizing that $f_k(P_k) \le \alpha_k$ can be rewritten as an equality (since $f_0(\boldsymbol{\alpha})$ is increasing). The convexity of (35) follows from the convexity of $f_0(\boldsymbol{\alpha})$ and each $f_k(P_k)$, and from the increasingness of $f_0(\boldsymbol{\alpha})$ [34, Sec. 3.2.4]. ■

Theorem 3 says that the original problem (17) can be efficiently solved in practice by repeatedly evaluating $f_1(P_1), \ldots, f_N(P_N)$ and adjusting the power allocation $\{P_k\}$ according to the master problem (35) (see Fig. 2). As previously mentioned, decomposing a problem only makes sense when the subproblems and the master problem can be easily solved. In this case, the subproblems (36) are trivially solved for Schur-concave/convex functions (cf. Section III-A) and the master problem (35) can also be easily solved with a simple subgradient algorithm (as described in the following subsection) due to the simple structure of the feasible set in (35): A simplex (the reason is that the master algorithm requires a projection on the feasible set, which is straightforward for a simplex).

If instead the original problem (17) is formulated as the minimization of the total power subject to a global quality constraint, the resulting simplified problem is
$$\begin{array}{ll} \min\limits_{\{\alpha_k\}} & \sum\limits_{k=1}^{N} f_k^{-1}(\alpha_k) \\ \text{s.t.} & f_0(\alpha_1,\ldots,\alpha_N) \le \alpha_0 \end{array} \tag{38}$$
where the cost function is easily characterized as in Section III-A. This alternative formulation, however, may have an additional difficulty arising from the structure of the feasible set which is determined by $f_0(\alpha_1,\ldots,\alpha_N)$, as opposed to the simplex obtained in (35).

For the implementation of the master algorithm, it will be necessary to compute a subgradient of the cost function in the master problem (36). If each $f_k(P_k)$ is differentiable, the gradient of $f_0(P_1,\ldots,P_N)$ is easily obtained (since $\partial f_k/\partial P_l = 0$ for $k \ne l$) from
$$\frac{\partial f_0}{\partial P_k} = \frac{\partial f_0}{\partial \alpha_k}\frac{\partial f_k}{\partial P_k} \tag{39}$$
where each $\partial f_k/\partial P_k$ is readily obtained from Propositions 1 and 2 (e.g., (32)–(34)).

As a final comment, it is worth pointing out the possibility of including additional constraints in the form of spectral masks, i.e., constraining some carriers independently $P_k \le P_k^{\max}$ in addition to the global power constraint. This type of constraint can be readily included in the problem considered in Theorem 3 without affecting the simplicity of the decomposition and the solvability (cf. Section VI). Note that this constraint is not the same as the peak-power constraint per antenna considered in [39], [40], where each transmit dimension within a channel matrix is constrained as opposed to constraining the total power used on each channel matrix or carrier (cf. [32]).

1) Practical Algorithm for the Master Problem: The master problem obtained in the decomposition of Theorem 3, i.e., (35), is generally a nondifferentiable problem and, hence, many of the existing general-purpose methods to solve smooth optimization problems, such as interior-point methods [34] cannot be used. Cutting-plane [41], [28], ellipsoid [41], and subgradient [26], [28] methods arise then as excellent approaches to solve nondifferentiable problems: they simply require the value of the function $f_0(P_1, \ldots, P_N)$ and a subgradient (both available in this case). Subgradient methods are distinguished by their simplicity of implementation [26], [28], which is precisely the main interest in this paper. An extensive account of subgradient methods can be found in [26], [28].

Consider for the moment the following general convex minimization problem:

$$\begin{array}{ll} \min\limits_{\mathbf{x}} & f_0(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \mathcal{X}. \end{array} \tag{40}$$

The subgradient method generates a sequence of feasible points $\{\mathbf{x}_k\}$ (here k denotes the iteration) as [28, Sec. 6.3.1]

$$\mathbf{x}_{k+1} = [\mathbf{x}_k - \alpha_k \mathbf{s}_k]_{\mathcal{X}}^{+} \tag{41}$$

where $\mathbf{s}_k$ is a subgradient of $f_0(\mathbf{x})$ at $\mathbf{x}_k$, $[\cdot]_{\mathcal{X}}^{+}$ denotes projection on the feasible convex set $\mathcal{X}$, and α_k is a positive scalar stepsize. Such an iteration looks like a gradient projection method except that a subgradient is used instead of the gradient (which may not exist). However, there is a fundamental difference: each new iteration may not improve the objective value as happens with a gradient method. What makes the subgradient method work is that for sufficiently small stepsize α_k, the distance of the current solution $\mathbf{x}_k$ to the optimal solution $\mathbf{x}^\star$ decreases.

There are many results on convergence of the subgradient method [26], [28], [42]. For constant stepsize $\alpha_k = \alpha$, the subgradient algorithm is guaranteed to converge to within some range of the optimal value (assuming bounded subgradients) [42] in other words, the subgradient method finds an ϵ-suboptimal point within a finite number of steps. Note that for a differentiable function f_0, the gradient method with constant stepsize converges to the optimal value provided that the stepsize is sufficiently small (assuming that the gradient is Lipschitz [28]). For the diminishing step size rule

$$\alpha_k = \alpha_0 \frac{1+m}{k+m}, \tag{42}$$

where $\alpha_0 \in (0,1]$ is the initial stepsize and m is a fixed positive integer, the algorithm is guaranteed to converge to the optimal value [42]. The additional factor $\left|\hat{f}^\star - f_0(\mathbf{x}_k)\right| / \|\mathbf{s}_k\|^2$, where $\hat{f}^\star$ is an approximate value of the optimal value $f^\star \triangleq f_0(\mathbf{x}^\star)$, can be included in (42) [28, Sec. 6.3.1].

In order to successfully apply the subgradient method in (41) to solve the master problem (35), the projection on the feasible set must be easy. As the following result shows, a projection on a simplex is indeed trivial.

Lemma 1: Consider the projection of an N-dimensional complex point $\mathbf{x}_0$ on a simplex $\{\mathbf{x} \mid x_i \geq 0, \sum_i x_i \leq P\}$ described by the following convex optimization problem:

$$\begin{array}{lll} \min\limits_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}_0\|^2 & \\ \text{s.t.} & x_i \geq 0 & 1 \leq i \leq N \\ & \sum\limits_{i=1}^{N} x_i \leq P. & \end{array}$$

The optimal solution is unique and is given by

$$x_i = (x_{0,i} - \mu)^{+} \tag{43}$$

where μ is chosen as the minimum nonnegative value such that $\sum_i x_i \leq P$ (note that if $\mu > 0$, then $\sum_i x_i = P$). In addition, such a solution can be obtained very efficiently in practice with Algorithm 1.

Proof: The proof of the solution follows easily from the KKT optimality conditions of the convex problem. The proof that the algorithm indeed gives the optimal solution follows from [38]. ■

Algorithm 1 Practical algorithm to obtain the projection on a simplex as described in Lemma 1.

Input: Original point $\mathbf{x}_0$ and constraining value P.

Output: Projection $\mathbf{x}$.

1. First try $\mu = 0$: if $\sum_i (x_{0,i})^{+} \leq P$, then $x_i = (x_{0,i})^{+}$ and finish.
2. Obtain $\mu > 0$ such that $\sum_i x_i = P$ as follows.
   2.0 Reorder $x_{0,i}$ in decreasing order $x_{0,i} \geq x_{0,i+1}$ (define $x_{0,N+1} = -\infty$) and set $\tilde{N} = N$.
   2.1 If $x_{0,\tilde{N}} > x_{0,\tilde{N}+1}$ and $x_{0,\tilde{N}} > \left(\sum_{i=1}^{\tilde{N}} x_{0,i} - P\right) / \tilde{N}$ then accept hypothesis and go to step 2.2. Otherwise, reject hypothesis, form a new one by setting $\tilde{N} = \tilde{N} - 1$, and go to step 2.1.
   2.2 Set $\mu = \left(\sum_{i=1}^{\tilde{N}} x_{0,i} - P\right) / \tilde{N}$, obtain the optimal solution as $x_i = (\mathbf{x}_{0,i} - \mu)^{+}$, undo the reordering done at step 2.1, and finish.

If the cost function $f_0(\mathbf{x})$ is differentiable, it is possible to use a gradient method with better convergence properties than subgradient methods. In particular, if we consider feasible direction methods, the update in (41) becomes

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k \tag{44}$$

where $\mathbf{d}_k = \bar{\mathbf{x}}_k - \mathbf{x}_k$ is a feasible direction (assuming $\bar{\mathbf{x}}_k \in \mathcal{X}$) that satisfies $\nabla f_0(\mathbf{x}_k)^T \mathbf{d}_k < 0$ (provided that $\mathbf{x}_k$ is not stationary). A very simple way to obtain the feasible direction $\mathbf{d}_k$ or, equivalently, $\bar{\mathbf{x}}_k$ is by the conditional gradient method [28]

$$\bar{\mathbf{x}}_k = \arg\min_{\mathbf{x} \in \mathcal{X}} \nabla f_0(\mathbf{x}_k)^T (\mathbf{x} - \mathbf{x}_k) \tag{45}$$

which can be efficiently obtained in this case with Algorithm 2 (this was obtained in [24, Alg. 2] and is reproduced here for convenience). Interestingly, it is straightforward to define a termination criterion since we can easily compute the following bounds on the optimal value:

$$f_0(\mathbf{x}_k) \geq \min_{\mathbf{x} \in \mathcal{X}} f_0(\mathbf{x}) \geq f_0(\mathbf{x}_k) + \nabla f_0(\mathbf{x}_k)^T (\bar{\mathbf{x}}_k - \mathbf{x}_k). \tag{46}$$

Algorithm 2 Practical algorithm to obtain an optimal solution to the problem

$$\begin{array}{lll} \min_{\mathbf{x}} & \sum_{i=1}^{N} w_i x_i & \\ \text{s.t.} & 0 \leq x_i \leq x_i^{\max} \qquad 1 \leq i \leq N & \\ & \sum_{i=1}^{N} x_i \leq & P. \end{array}$$

Input: Bounds of the feasible set $\mathbf{x}^{\max}$, weights $\mathbf{w}$, and P.

Output: Optimal solution $\mathbf{x}$.

0. Set $\mathbf{x} = \mathbf{0}$.
1. If $\mathbf{1}^T \mathbf{x} = P$ or $w_{\min} \triangleq \min_{i:x_i \neq x_i^{\max}} \{w_i\} \geq 0$, then finish.
2. Set $x_i = \min\left(x_i + \left(P - \mathbf{1}^T \mathbf{x}\right)/|\mathcal{I}|, x_i^{\max}\right)$ for $i \in \mathcal{I} \triangleq \{i \mid x_i \neq x_i^{\max},\ w_i = w_{\min}\}$ and go to step 1.

IV. Some Relevant Examples

To illustrate the potential of the result obtained in Section III, we now consider three interesting examples and show how the general result in Theorem 3 particularizes.

A. Minimization of the Maximum MSE

Consider the minimization of $\max_{k,i}\{\mathrm{MSE}_{k,i}\}$. The cost function can be decomposed as required in Theorem 3, with $f_0(\boldsymbol{\alpha}) = \max\{\alpha_1, \ldots, \alpha_N\}$ and $f_k(\boldsymbol{\rho}_k) = \max_i\{\rho_{k,i}\}$, and the problem is then

$$\begin{array}{ll} \min_{\{\mathbf{B}_k, \alpha_k\}} & \max\{\alpha_1, \ldots, \alpha_N\} \\ \text{s.t.} & \max_i \left\{ \left[\left(\nu \mathbf{I} + \mathbf{B}_k^H \mathbf{R}_{H_k} \mathbf{B}_k \right)^{-1} \right]_{ii} \right\} \leq \alpha_k \\ & 1 \leq k \leq N \\ & \sum_{k=1}^{N} \mathrm{Tr}\left(\mathbf{B}_k \mathbf{B}_k^H \right) \leq P_0 \end{array} \tag{47}$$

which is usually referred to as a *finite minimax problem* [43].

Now, invoking Theorem 3, the problem reduces to

$$\begin{array}{ll} \min_{\{P_k\}} & \max\{f_1(P_1), \ldots, f_N(P_N)\} \\ \text{s.t.} & \sum_{k=1}^{N} P_k \leq P_0 \\ & P_k \geq 0 \qquad 1 \leq k \leq N. \end{array} \tag{48}$$

Since each $f_k(\boldsymbol{\rho}_k)$ is a Schur-convex function [17], the results of Section III-A-2 can be applied. To be more precise, each $f_k(P_k)$ can be evaluated by solving the problem (30) or the equivalent problem (31) with solution given by [cf. (27)]

$$p_{k,i} = \left(\mu_k^{-1/2} \lambda_{k,i}^{-1/2} - \nu \lambda_{k,i}^{-1} \right)^{+} \tag{49}$$

where the waterlevel μ_k chosen such that $\sum_{i=1}^{L_k} p_{k,i} = P_k$. In addition, the gradient is given by [cf. (32)]

$$\nabla f_k(P_k) = -\frac{\mu_k}{L_k}. \tag{50}$$

Thus, the evaluation of $f_0(P_1, \ldots, P_N) = \max\{f_1(P_1), \ldots, f_N(P_N)\}$ is straightforward after each $f_k(P_k)$ has been independently evaluated and its subdifferential is the convex hull of the active gradients [26], [43]

$$\boldsymbol{\partial} f_0(P_1, \ldots, P_N) = \mathrm{co}\left\{ -\frac{\mu_k}{L_k} \mathbf{e}_k : k \in \mathcal{K} \right\} \tag{51}$$

where co denotes convex hull, $\mathcal{K} \triangleq \{k \mid f_k(P_k) = f_0(P_1, \ldots, P_N)\}$ is the set of subproblems that achieve the maximum value (active subproblems), and $\mathbf{e}_k$ is the kth canonical vector (all-zero vector with a one in the kth element). One possible subgradient is then

$$-\frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \frac{\mu_k}{L_k} \mathbf{e}_k. \tag{52}$$

B. Maximization of the Harmonic Mean of the SINRs

The maximization of the harmonic mean of the SINRs can be formulated as the minimization of $\sum_{k,i} \mathrm{SINR}_{k,i}^{-1}$. The cost function can be decomposed as required by Theorem 3, with $f_0(\boldsymbol{\alpha}) = \sum_{k=1}^{N} \alpha_k$ and $f_k(\boldsymbol{\rho}_k) = \sum_{i=1}^{L_k} (\rho_{k,i}^{-1} - \nu)^{-1}$ [note that the f_k's are convex in the region of interest $\rho_{k,i} \in (0,1)$], and the problem reduces to

$$\begin{array}{ll} \min_{\{P_k\}} & \sum_{k=1}^{N} f_k(P_k) \\ \text{s.t.} & \sum_{k=1}^{N} P_k \leq P_0 \\ & P_k \geq 0 \qquad 1 \leq k \leq N. \end{array} \tag{53}$$

Since each $f_k(\boldsymbol{\rho}_k)$ is a Schur-convex function [17], the results of Section III-A-2 can be applied (see discussion in Section IV-A). Thus, the evaluation of $f_0(P_1, \ldots, P_N) = \sum_{k=1}^{N} f_k(P_k)$ is straightforward after each $f_k(P_k)$ has been independently evaluated and the gradient in this case is obtained [using (33) and (39)] from

$$\frac{\partial f_0}{\partial P_k} = -\frac{\mu_k}{(1 - \nu \rho_k)^2}.$$

C. Minimization of the Average BER

Consider the minimization of the average BER: $(1/N)\sum_k (1/L_k) \sum_i \mathrm{BER}_{k,i}$. The cost function can be decomposed as required by Theorem 3, with $f_0(\boldsymbol{\alpha}) = (1/N)\sum_{k=1}^{N} \alpha_k$ and

$f_k(\boldsymbol{\rho}_k) = (1/L_k)\sum_{i=1}^{L_k} g_{k,i}(\rho_{k,i})$, and the problem reduces to

$$\begin{aligned} \min_{\{P_k\}} \quad & \frac{1}{N}\sum_{k=1}^{N} f_k(P_k) \\ \text{s.t.} \quad & \sum_{k=1}^{N} P_k \le P_0 \\ & P_k \ge 0 \qquad 1 \le k \le N. \end{aligned} \tag{54}$$

Further assuming that equal constellations are used at each k ($g_{k,i} = g_k\ \forall i$), it follows that each $f_k(\boldsymbol{\rho}_k)$ is a Schur-convex function [17] and the results of Section III-A-2 can be applied (see discussion in Section IV-A). Thus, the evaluation of $f_0(P_1, \ldots, P_N) = (1/N)\sum_{k=1}^{N} f_k(P_k)$ is straightforward after each $f_k(P_k)$ has been independently evaluated and the gradient in this case is obtained [using (34) and (39)] from

$$\frac{\partial f_0}{\partial P_k} = -\frac{\mu_k}{N L_k} g_k'(\rho_k). \tag{55}$$

For completeness, an explicit approximate expression for $g(\rho)$ and $g'(\rho)$ is given below for M-ary QAM constellations (assuming that a Gray encoding is used to map the bits into the constellation points and that the interference-plus-noise can be approximated by a Gaussian term) [44], [30], [45]

$$\begin{aligned} g(\rho) &\simeq \frac{\alpha}{\log_2 M}\mathcal{Q}\left(\sqrt{\beta\left(\rho^{-1}-\nu\right)}\right) \\ g'(\rho) &\simeq \frac{\alpha}{\log_2 M}\sqrt{\frac{\beta}{8\pi}} e^{-\beta\left(\rho^{-1}-\nu\right)/2}\left(\rho^3-\nu\rho^4\right)^{-1/2}, \end{aligned}$$

where $\mathcal{Q}$ is defined as $\mathcal{Q}(x) \triangleq (1/\sqrt{(2\pi)})\int_x^\infty e^{-\lambda^2/2}d\lambda$ [44], [31],[5] $\alpha = 4(1-1/\sqrt{M})$ and $\beta = 3/(M-1)$ are parameters that depend on the constellation size.

V. Extension of Primal Decomposition to Non-Schur-Concave/Convex Functions

This section briefly considers the case of functions that are not Schur-concave/convex (as opposed to Section III), linking this work with [24]. In principle, Theorem 3 can still be applied and the simplified problem (35) is still obtained. However, the difference with respect to Section III is that the subproblems (36) are not easily solved anymore [the simplest known reformulation is as in (19)], as opposed to the simple reformulations for problems with Schur-concave and Schur-convex functions in (24) and (30) as discussed in Section III-A.

We now describe an alternative approach by decomposing the problem in a different way, yielding a different form for the subproblems which still can be easily solved. This approach was successfully adopted in [24] to minimize the average BER with arbitrary (not necessarily equal) constellations (such a cost function is Schur-convex only if the constellations are equal [17]).

[5]The $\mathcal{Q}$-function and the commonly used complementary error function "erfc" are related as $\operatorname{erfc}(x) = 2\,\mathcal{Q}(\sqrt{2}x)$ [31].

The new decomposition hinges on the problem solved in [29] (see also [24] for a more general formulation):

$$\begin{aligned} \min_{\mathbf{B}} \quad & \operatorname{Tr}\left(\mathbf{B}\mathbf{B}^H\right) \\ \text{s.t.} \quad & \left[\left(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B}\right)^{-1}\right]_{ii} \le \rho_i \qquad 1 \le i \le L \end{aligned} \tag{56}$$

which is the minimization of the power subject to a set of QoS constraints. This subproblem does not have a simple closed-form solution as the subproblems considered in Section III; however, it can still be easily solved in practice with the multilevel waterfilling algorithm given in [29] (see also [24]). We proceed similarly to Section III-A by defining $P(\boldsymbol{\rho})$ as the minimum cost value of (56) for the given set of QoS requirement $\boldsymbol{\rho}$, which happens to be a convex function with a subgradient implicitly obtained in the evaluation of $P(\boldsymbol{\rho})$ (cf. [24]).

The original problem (17) can be rewritten as

$$\begin{aligned} \min_{\{\boldsymbol{\rho}_k, \mathbf{B}_k, \alpha_k\}} \quad & f_0(\alpha_1, \ldots, \alpha_N) \\ \text{s.t.} \quad & f_k(\boldsymbol{\rho}_k) \le \alpha_k \qquad 1 \le k \le N,\ 1 \le i \le L_k \\ & \left[\left(\nu\mathbf{I} + \mathbf{B}_k^H\mathbf{R}_{H_k}\mathbf{B}_k\right)^{-1}\right]_{ii} \le \rho_{k,i} \\ & \sum_{k=1}^{N} \operatorname{Tr}\left(\mathbf{B}_k\mathbf{B}_k^H\right) \le P_0. \end{aligned} \tag{57}$$

It is now straightforward to recognize the structure of the subproblem (56) in (57), which can then be rewritten as the simple convex problem

$$\begin{aligned} \min_{\{\boldsymbol{\rho}_k\}} \quad & f_0(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N) \\ \text{s.t.} \quad & \sum_{k=1}^{N} P_k(\boldsymbol{\rho}_k) \le P_0 \end{aligned} \tag{58}$$

where $f_0(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N) \triangleq f_0(f_1(\boldsymbol{\rho}_1), \ldots, f_N(\boldsymbol{\rho}_N))$. Note that the problem could have been similarly formulated as the minimization of the total power subject to a global quality and a simplified problem like (58) would have been obtained but with the roles of the cost function and constraint function reversed (cf. [24]).

As explained in full detail in [24], to go from (57) to (58) it is necessary to assume that the functions f_k's are minimized when the arguments are in decreasing ordering and then include in (57) the constraints $\rho_{k,i} \ge \rho_{k,i+1}$, as required in (56). At that point, the problem can be relaxed by removing again these constraints. Realizing then that any solution to the relaxed problem will also satisfy these constraints, the equivalent (not relaxed) problem (58) is obtained.

With the new subproblem defined in (56), we have been able to decompose an arbitrary problem with not necessarily Schur-concave/convex functions. However, there is still one issue that should be addressed for the decomposition to make sense: the simplicity of solving the master problem. In Section III-B, the proposed subgradient method was indeed extremely simple due to the simple structure of the feasible set of the master problem (35), which is a simplex (the subgradient method requires a projection on the feasible set at each iteration). However, in the master problem (58), there is no such a simple structure. There are two possibilities to tackle this problem: i) to use another method for the master problem, and ii) to consider the opposite problem formulation (swapping the cost and constraint functions) whenever the set $f_0(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N) \leq \alpha_0$ is simple enough; this is indeed the case in [24], where the feasible set is a truncated simplex.

VI. Numerical Results

This section provides a numerical evaluation of the tools developed for the design of multicarrier linear MIMO transceivers, i.e., the subgradient/gradient method for the master problem (cf. Section III-B1) combined with the closed-form solutions for the subproblems (cf. Section III-A).

In particular, four different Schur-concave/convex cost functions have been chosen in the optimization process (with an MMSE receiver): The minimization of the sum of the MSEs (SUM-MSE) (cf. (27)), the minimization of the maximum of the MSEs (MAX-MSE) (Section IV-A), the maximization of the harmonic mean of the SINR's (HARM-SINR) (Section IV-B), and the minimization of the average BER (AVE-BER) (Section IV-C).

The SUM-MSE can be directly solved with a single water-filling, so the decomposition approach does not really make sense for this example. Both the MAX-MSE and the HARM-SINR approaches were studied in detail in [32] and specific exact algorithms were developed (see also [38]); for these two cases, the decomposition approach is a convenient and simple option. Regarding the AVE-BER criterion, no closed-form solution or simple implementation has been given so far (except by means of a general purpose interior-point method as in [17], [32]) and, hence, the simple decomposition approach proposed in this paper blossoms in all respects. For other methods, the decomposition approach is always an easy and simple way to obtain optimal solutions.

A simple model was used to randomly generate different realizations of the MIMO channel.[6] In particular, a system with $N = 16$ multiple MIMO channels (e.g., multicarrier) was considered where each channel matrix $\mathbf{H}_k$ was generated from a complex Gaussian distribution with i.i.d. zero-mean unit-variance elements, and the noise was modeled as white $\mathbf{R}_{n_k} = \sigma_n^2 \mathbf{I}$, where σ_n^2 is the noise power. The SNR is defined as $\mathrm{SNR} = P_0/\sigma_n^2$, which is essentially a measure of the transmitted power normalized with respect to the noise. An obvious initial point for the master problem is $P_k = P_0/N$ for all k, i.e., a uniform power allocation.

The convergence of the master problem is illustrated with one representative realization in Figs. 3 and 4 for the MAX-MSE method with a subgradient algorithm and for the HARM-SINR method with a gradient algorithm, respectively. In both cases, the algorithm has essentially converged after 20–40 iterations. A simple stopping criterion based on the derivative of the curve can be used (for the gradient method, one can also employ a stopping criterion based on the difference between the curve and the lower bound in (46), but that may be too conservative since the lower bound is not too tight). Interestingly, the initial point (uniform power allocation) seems to be reasonably close to the optimal and, hence, it may be a good candidate for a suboptimal solution (this behavior has been observed in all the realizations).

Fig. 5 shows steady-state results (i.e., after convergence) for the four methods previously described. The performance is plotted in terms of BER averaged over the substreams; to be

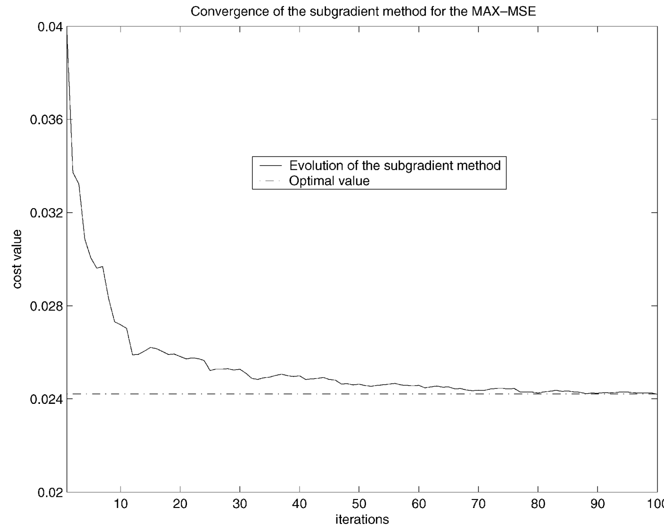

Fig. 3. Convergence of the subgradient method for the MAX-MSE design (along with the optimal value) in a 4×4 MIMO multicarrier channel ($N = 16$ carriers) with $L = 3$, $\mathrm{SNR} = 16$ dB/carrier and parameters of the gradient method: $\alpha_0 = 200$ (which also accounts for the additional term $|\hat{f}^\star - f_0(\mathbf{x}_k)| / \|\mathbf{s}_k\|^2$) and $m = 6$.

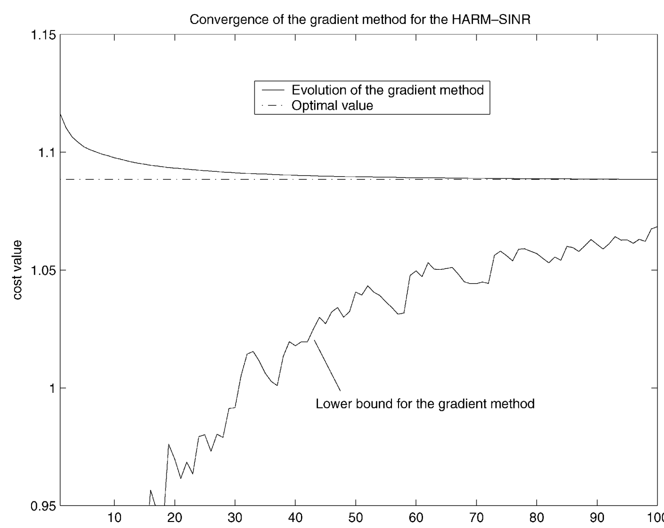

Fig. 4. Convergence of the gradient method for the HARM-SINR design (along with the lower bound in (46) and the optimal value) in a 4×4 MIMO multicarrier channel ($N = 16$ carriers) with $L = 3$, $\mathrm{SNR} = 16$ dB/carrier and parameters of the gradient method: $\alpha_0 = 0.01$ and $m = 10$.

[6]For the purposes of this paper, it is not necessary to resort to more realistic channel models (the interested reader is referred to [17], [29], and [32] for related simulations with more realistic wireless multiantenna channel models including spatial and frequency correlation).

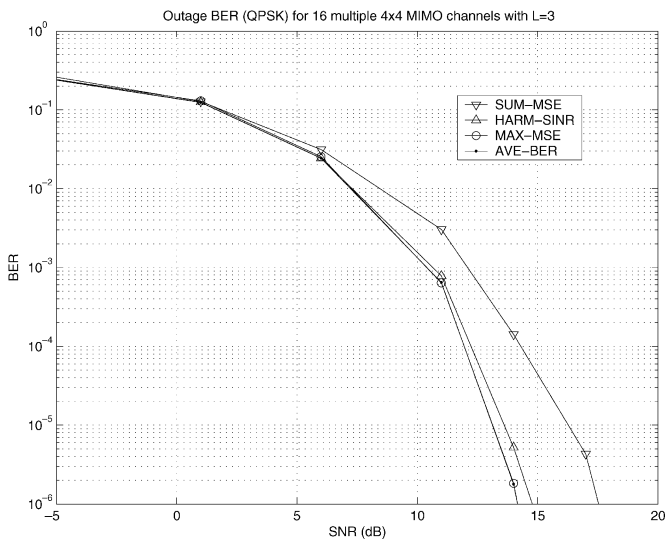

Fig. 5. Outage BER ($P_{\rm out} = 5\%$) versus the SNR/carrier in a 4×4 MIMO multicarrier channel ($N = 16$ carriers) with $L = 3$, for the methods: SUM-MSE, MAX-MSE, HARM-SINR, and AVE-BER.

more precise, the outage BER[7] (over 10^6 independent realizations of the channel) is considered since it is a more realistic measure than the average BER.[8] As previously mentioned, the novelty in these curves is in the way they have been obtained: With a simple subgradient/gradient method as opposed, for example, to an interior-point method. The comparison among the different methods and with other existing methods is out of the scope of the paper and the interested reader is referred to [17] and [32].

Finally, in Fig. 6, the design with additional spectral mask constraints of the form $P_k \leq P^{\max}$ has been considered. The upper plot gives the power allocation without peak constraints (only the average constraint), whereas the lower plot includes the same peak constraint in all carriers. It can be observed how the peak-power constrained design, as expected, satisfies the maximum power constraint. The resulting performance is $\text{BER} = 7.43 \times 10^{-5}$ without peak constraints and $\text{BER} = 1.11 \times 10^{-4}$ with peak constraints.

For numerical simulations corresponding to non-Schur-concave/convex functions, as treated in Section V, the reader is referred to [24], where the particular case of minimizing the average BER with arbitrary constellations is treated in full detail.

VII. Conclusion

This paper has extended the simple closed-form solutions for Schur-concave/convex cost functions in a single MIMO channel to the more general case of multiple MIMO channels such as in multi-antenna multicarrier systems. The extension is based on a primal decomposition approach that divides the original complicated problem into several simple subproblems controlled by a simple master problem. The subproblems are easily solved with closed-form solutions and the master problem with a subgradient/gradient algorithm. Hence, the problem can now be efficiently and optimally solved in practice with a very simple implementation. The method has also been extended to the case of functions that are not Schur-concave/convex. The approach has been illustrated by three examples of interest and numerical results have been provided to support and complement the mathematical development.

[7]The outage BER is the BER that is attained with some given probability (when it is not satisfied, an outage event is declared).

[8]The average BER only makes sense when the system does not have delay constraints and the duration of the transmission is sufficiently long such that the fading statistics of the channel can be averaged out.

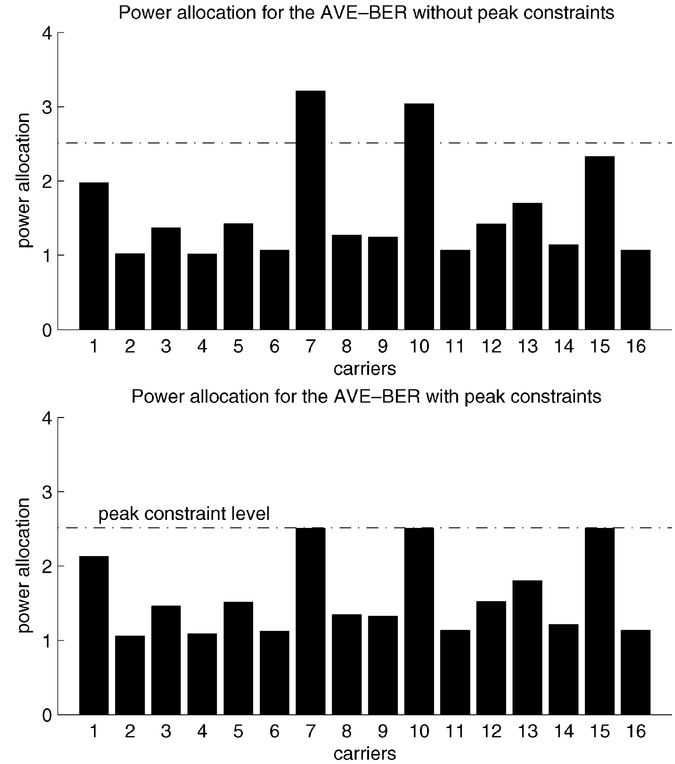

Fig. 6. Power distribution for the AVE-BER method in a 4×4 MIMO multicarrier channel ($N = 16$ carriers) with $L = 3$ (one realization), without and with peak power constraints [maximum average power 12 dB/carrier (total of 25.36) and maximum peak power 14 dB (2.51)].

Appendix I
Sketch of the Proof of Theorem 1

The proof hinges on majorization theory; the interested reader is referred to [23] for definitions and basic results on majorization theory (see also [32] for a brief overview) and to [32], [17], and [29] for details overlooked in this sketch of the proof.

To start with, the problem (18) can be written as

$$\begin{aligned} \min_{\mathbf{B},\boldsymbol{\rho}} \quad & f_0(\rho_1, \ldots, \rho_L) \\ \text{s.t.} \quad & \left[\left(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B}\right)^{-1}\right]_{ii} \leq \rho_i \qquad 1 \leq i \leq L \\ & \text{Tr}\left(\mathbf{B}\mathbf{B}^H\right) \leq P_0 \end{aligned} \tag{59}$$

which can always be done since f_0 is increasing in each argument. Also, since f_0 is minimized when $\rho_i \geq \rho_{i+1}$ and $\mathbf{B}$ can always include any desired permutation such that the diagonal elements of $\left(\nu\mathbf{I} + \mathbf{B}^H\mathbf{R}_H\mathbf{B}\right)^{-1}$ are in decreasing order, the constraint $\rho_i \geq \rho_{i+1}$ can be explicitly included without affecting the problem.

The first main simplification comes by rewriting the problem as [29, Th. 2]

$$\begin{array}{ll} \min\limits_{\tilde{\mathbf{B}},\boldsymbol{\rho}} & f_0(\rho_1,\ldots,\rho_L) \\ \text{s.t.} & \tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}} \quad \text{diagonal (increasing diag. elements)} \\ & \mathbf{d}\left(\left(\nu\mathbf{I}+\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}}\right)^{-1}\right) \succ^w \boldsymbol{\rho} \\ & \rho_i \geq \rho_{i+1} \\ & \mathrm{Tr}\left(\tilde{\mathbf{B}}\tilde{\mathbf{B}}^H\right) \leq P_0 \end{array} \tag{60}$$

where $\succ^w$ denotes the weakly majorization relation[9] [23] and $\mathbf{d}(\mathbf{X})$ denotes the diagonal elements of matrix $\mathbf{X}$ (similarly, $\boldsymbol{\lambda}(\mathbf{X})$ is used for the eigenvalues). The second constraint guarantees the existence of a unitary matrix $\mathbf{Q}$ such that $\mathbf{d}\left(\mathbf{Q}^H(\nu\mathbf{I}+\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}})^{-1}\mathbf{Q}\right) \leq \boldsymbol{\rho}$ [23, 9.B.2 and 5.A.9.a] or, in other words, such that $\left[(\nu\mathbf{I}+\mathbf{B}^H\mathbf{R}_H\mathbf{B})^{-1}\right]_{ii} \leq \rho_i$ with $\mathbf{B}=\tilde{\mathbf{B}}\mathbf{Q}$.

The second main simplification comes from the fact that $\tilde{\mathbf{B}}$ can be assumed without loss of optimality of the form $\tilde{\mathbf{B}}=\mathbf{U}_{H,1}\boldsymbol{\Sigma}_B$, as described in the theorem, since $\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}}$ is diagonal with diagonal elements in increasing order (cf. [17, Lem. 12], [32, Lem. 5.11], and [29, Lem. 7]).

Problem (19) follows then by plugging the expression of $\tilde{\mathbf{B}}$ into (60), denoting $p_i=|[\boldsymbol{\Sigma}_B]_{ii}|^2$ (which implies the need for the additional constraints $p_i \geq 0$), and by rewriting the weakly majorization constraint explicitly [23]. Note also that the constraints $p_i\lambda_i \leq p_{i+1}\lambda_{i+1}$ (to guarantee that the diagonal elements of $\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}}$ are in increasing order) are not necessary since any optimal solution must necessarily satisfy them (cf. [29, Lem. 7] and [32, Lem. 5.11]). If f_0 is convex, the constraints $\rho_i \geq \rho_{i+1}$ are not necessary since an optimal solution cannot have $\rho_i < \rho_{i+1}$ (because the problem would have a lower cost value by using instead $\tilde{\rho}_i=\tilde{\rho}_{i+1}=(\rho_i+\rho_{i+1})/2$, cf. [24]). ■

Appendix II
Proof of Theorem 2

To obtain the additional simplification for Schur-concave/convex cost functions, rewrite the MSE constraints of (59) (since they are satisfied with equality at an optimal point) as

$$\boldsymbol{\rho}=\mathbf{d}\left(\mathbf{Q}^H(\nu\mathbf{I}+\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}})^{-1}\mathbf{Q}\right). \tag{61}$$

Now it suffices to use the definition of Schur-concavity/convexity to obtain the desired result. In particular, if f_0 is Schur-concave, it follows from the definition of Schur-concavity [23] (the diagonal elements and eigenvalues are assumed here in decreasing order) that

$$f_0(\mathbf{d}(\mathbf{X})) \geq f_0(\boldsymbol{\lambda}(\mathbf{X})) \tag{62}$$

which means that $f_0(\boldsymbol{\rho})$ is minimum when $\mathbf{Q}=\mathbf{I}$ in (61) (since $(\nu\mathbf{I}+\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}})^{-1}$ is already diagonal with diagonal elements in decreasing order by definition). If f_0 is Schur-convex, the opposite happens

$$f_0(\mathbf{d}(\mathbf{X})) \geq f_0\left(\mathbf{1}\times\frac{\mathrm{Tr}(\mathbf{X})}{L}\right) \tag{63}$$

where $\mathbf{1}$ denotes the all-one vector. This means that $f_0(\boldsymbol{\rho})$ is minimum when $\mathbf{Q}$ is such that $\boldsymbol{\rho}$ has equal elements in (61), i.e., when $\mathbf{Q}^H(\nu\mathbf{I}+\tilde{\mathbf{B}}^H\mathbf{R}_H\tilde{\mathbf{B}})^{-1}\mathbf{Q}$ has equal diagonal elements. ■

[9]The weakly majorization relation $\mathbf{y} \succ^w \mathbf{x}$ is defined as $\sum_{j=i}^n y_j \leq \sum_{j=i}^n x_i$ for $1 \leq i \leq n$, where the elements of $\mathbf{y}$ and $\mathbf{x}$ are assumed in decreasing order [23].

Appendix III
Proof of Proposition 1

To show property a) it suffices to note that the function

$$f(P,\mathbf{p}) \triangleq \begin{cases} f\left(\left\{\frac{1}{\nu+p_i\lambda_i}\right\}\right), & \mathbf{p}\geq\mathbf{0},\, \mathbf{1}^T\mathbf{p}\leq P \\ +\infty, & \text{otherwise} \end{cases}$$

is jointly convex in P and $\mathbf{p}$. Then, since $f(P)=\inf_{\mathbf{p}} f(P,\mathbf{p})$, it follows that $f(P)$ is convex in P [34, Sec. 3.2.5], [28, Sec. 5.4.4]. Property (b) follows from a standard result [28, Sec. 5.4.4]. Property (c), i.e., the differentiability of $f(P)$ can be proved by showing that it has a unique subgradient at each P [28, Prop. B.24]. Since there is a one-to-one mapping between subgradients and optimal Lagrange multipliers [28, Sec. 5.4.4], it suffices to show the uniqueness of the Lagrange multiplier μ. This is indeed the case for $P>0$ [see (25) and (26)]. ■

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