Power Control By Geometric Programming

Mung Chiang, Chee Wei Tan, Daniel P. Palomar, Daniel O'Neill, and David Julian

***Abstract*— In wireless cellular or ad hoc networks where Quality of Service (QoS) is interference-limited, a variety of power control problems can be formulated as nonlinear optimization with a system-wide objective, *e.g.*, maximizing the total system throughput or the worst user throughput, subject to QoS constraints from individual users, *e.g.*, on data rate, delay, and outage probability. We show that in the high Signal-to-Interference Ratios (SIR) regime, these nonlinear and apparently difficult, nonconvex optimization problems can be transformed into convex optimization problems in the form of geometric programming; hence they can be very efficiently solved for global optimality even with a large number of users. In the medium to low SIR regime, some of these constrained nonlinear optimization of power control cannot be turned into tractable convex formulations, but a heuristic can be used to compute in most cases the optimal solution by solving a series of geometric programs through the approach of successive convex approximation. While efficient and robust algorithms have been extensively studied for centralized solutions of geometric programs, distributed algorithms have not been explored before. We present a systematic method of distributed algorithms for power control that is geometric-programming-based. These techniques for power control, together with their implications to admission control and pricing in wireless networks, are illustrated through several numerical examples.**

***Index Terms*— Convex optimization, CDMA power control, Distributed algorithms.**

I. Introduction

Due to the broadcast nature of radio transmission, data rates and other Quality of Service (QoS) in a wireless network are affected by interference. This is particularly important in Code Division Multiple Access (CDMA) systems where users transmit at the same time over the same frequency bands and their spreading codes are not perfectly orthogonal. Transmit power control is often used to tackle this problem of signal interference. In this paper, we study how to optimize over the transmit powers to create the optimal set of Signal-to-Interference Ratios (SIR) on wireless links. Optimality here can be with respect to a variety of objectives, such as maximizing a system-wide efficiency metric (*e.g.*, the total system throughput), or maximizing a Quality of Service (QoS) metric for a user in the highest QoS class, or maximizing a QoS metric for the user with the minimum QoS metric value (*i.e.*, a maxmin optimization).

While the objective represents a system-wide goal to be optimized, individual users' QoS requirements also need to be satisfied. Any power allocation must therefore be constrained by a feasible set formed by these minimum requirements from the users. Such a constrained optimization captures the tradeoff between user-centric constraints and some network-centric objective. Because a higher power level from one transmitter increases the interference levels at other receivers, there may not be any feasible power allocation to satisfy the requirements from all the users. Sometimes an existing set of requirements can be satisfied, but when a new user is admitted into the system, there exists no more feasible power control solutions, or the maximized objective is reduced due to the tightening of the constraint set, leading to the need for admission control and admission pricing, respectively.

Because many QoS metrics are nonlinear functions of SIR, which is in turn a nonlinear (and neither convex nor concave) function of transmit powers, in general power control optimization or feasibility problems are difficult nonlinear optimization problems that may appear to be NP-hard problems. This paper shows that, when SIR is much larger than 0dB, a class of nonlinear optimization called Geometric Programming (GP) can be used to efficiently compute the globally optimal power control in many of these problems, and efficiently determine the feasibility of user requirements by returning either a feasible (and indeed optimal) set of powers or a certificate of infeasibility. This also leads to an effective admission control and admission pricing method. The key observation is that despite the *apparent* nonconvexity, through logarithmic change of variable the GP technique turns these constrained optimization of power control into convex optimization, which is intrinsically tractable despite its nonlinearity in objective and constraints. When SIR is comparable to or below 0dB, the power control problems are *truly* nonconvex with no efficient and global solution methods. In this case, we present a heuristic that is provably convergent and empirically often computes the globally optimal power allocation by solving a sequence of GPs through the approach of successive convex approximations.

The GP approach reveals the hidden convexity structure, which implies efficient solution methods, in power control problems with nonlinear objective functions and specific SIR constraints. It also clearly differentiates the tractable formulations in high-SIR regime from the intractable ones in low-SIR regime. In contrast to the classical Foschini-Miljanic power control [8] and many of the related work, the power control problems in this paper have nonlinear objective functions in

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terms of system performance and are nonconvex optimization. In contrast to more recent work on optimal SIR assignment (e.g., [16]), the formulations here have hard QoS constraints, such as minimum SIR targets. Therefore, the range of power control problems that can be efficiently solved is widened.

Power control by GP is applicable to formulations in both cellular networks with single-hop transmission between mobile users and base stations, and ad hoc networks with mulithop transmission among the nodes, as illustrated through several numerical examples in this paper. Traditionally, GP is solved by centralized computation through the highly efficient interior point methods. In this paper we present a new result on how GP can be solved distributively with message passing, which has independent value to general maximization of coupled objective, and apply it to power control problems.

The rest of this paper is organized as follows. In section II, we provide a concise introduction to GP. Section III generalizes the results in [10] with several subsections, each discussing GP based power control with different representative formulations in cellular and multihop networks that can be transformed into convex problems. Then, generalizing the new results in [21], we present two extensions overcoming the two main limitations in [10]: solution method for nonconvex power control in low-SIR regime in section IV and distributed algorithm in section V.

II. Geometric Programming

GP is a class of nonlinear, nonconvex optimization problems with many useful theoretical and computational properties. Since a GP can be turned into a convex optimization problem[1], a local optimum is also a global optimum, Lagrange duality gap is zero under mild conditions, and a global optimum can always be computed very efficiently. Numerical efficiency holds both in theory and in practice: interior point methods applied to GP have provably polynomial time complexity [14], and are very fast in practice with high-quality software downloadable from the Internet (*e.g.*, the MOSEK package). Convexity and duality properties of GP are well understood, and large-scale, robust numerical solvers for GP are available. Furthermore, special structures in GP and its Lagrange dual problem lead to distributed algorithms, physical interpretations, and computational acceleration even beyond the generic results for convex optimization. A detailed tutorial of GP and comprehensive survey of its recent applications to communication systems can be found in [7]. This section contains a brief introduction of GP terminology for applications to be shown in the next three sections on power control problems.

There are two equivalent forms of GP: standard form and convex form. The first is a constrained optimization of a type of function called posynomial, and the second form is obtained from the first through a logarithmic change of variable.

We first define a monomial as a function $f : \mathbf{R}^n_{++} \rightarrow \mathbf{R}$:

$$f(\mathbf{x}) = d\, x_1^{a^{(1)}} x_2^{a^{(2)}} \ldots x_n^{a^{(n)}}$$

[1]Minimizing a convex objective function subject to upper bound inequality constraints on convex constraint functions and linear equality constraints is a convex optimization problem.

where the multiplicative constant $d \geq 0$ and the exponential constants $a^{(j)} \in \mathbf{R}, j = 1, 2, \ldots, n$. A sum of monomials, indexed by k below, is called a posynomial:

$$f(\mathbf{x}) = \sum_{k=1}^{K} d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \ldots x_n^{a_k^{(n)}}.$$

where $d_k \geq 0$, $k = 1, 2, \ldots, K$, and $a_k^{(j)} \in \mathbf{R}$, $j = 1, 2, \ldots, n, k = 1, 2, \ldots, K$. For example, $2x_1^{-\pi} x_2^{0.5} + 3x_1 x_3^{100}$ is a posynomial in $\mathbf{x}$, $x_1 - x_2$ is not a posynomial, and x_1/x_2 is a monomial, thus also a posynomial.

Minimizing a posynomial subject to posynomial upper bound inequality constraints and monomial equality constraints is called GP in *standard form*:

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 1, \quad i = 1, 2, \ldots, m, \\ & h_l(\mathbf{x}) = 1, \quad l = 1, 2, \ldots, M \end{array} \tag{1}$$

where $f_i, \quad i = 0, 1, \ldots, m$, are posynomials: $f_i(\mathbf{x}) = \sum_{k=1}^{K_i} d_{ik} x_1^{a_{ik}^{(1)}} x_2^{a_{ik}^{(2)}} \ldots x_n^{a_{ik}^{(n)}}$, and $h_l, \quad l = 1, 2, \ldots, M$ are monomials: $h_l(\mathbf{x}) = d_l x_1^{a_l^{(1)}} x_2^{a_l^{(2)}} \ldots x_n^{a_l^{(n)}}$.

GP in standard form is not a convex optimization problem, because posynomials are not convex functions. However, with a logarithmic change of the variables and multiplicative constants: $y_i = \log x_i, b_{ik} = \log d_{ik}, b_l = \log d_l$, and a logarithmic change of the functions' values, we can turn it into the following equivalent problem in $\mathbf{y}$:

$$\begin{array}{ll} \text{minimize} & p_0(\mathbf{y}) = \log \sum_{k=1}^{K_0} \exp(\mathbf{a}_{0k}^T \mathbf{y} + b_{0k}) \\ \text{subject to} & p_i(\mathbf{y}) = \log \sum_{k=1}^{K_i} \exp(\mathbf{a}_{ik}^T \mathbf{y} + b_{ik}) \leq 0, \quad \forall i, \\ & q_l(\mathbf{y}) = \mathbf{a}_l^T \mathbf{y} + b_l = 0, \quad l = 1, 2, \ldots, M. \end{array} \tag{2}$$

This is referred to as GP in *convex form*, which is a convex optimization problem since it can be verified that the log-sum-exp function is convex [5].

In summary, GP is a nonlinear, nonconvex optimization problem that can be transformed into a nonlinear, convex problem. GP in standard form can be used to formulate network resource allocation problems with nonlinear objectives under nonlinear QoS constraints. The basic idea is that resources are often allocated proportional to some parameters, and when resource allocations are optimized over these parameters, we are maximizing an inverted posynomial subject to lower bounds on other inverted posynomials, which are equivalent to GP in standard form.

III. Power Control by Geometric Programming: Convex Case

Various schemes for power control, centralized or distributed, have been extensively studied since 1990s based on different transmission models and application needs, *e.g.*, in [2], [8], [13], [19], [20], [23]. This section summarizes the new approach of formulating power control problems through GP. The key advantage is that globally optimal power allocations can be efficiently computed for a variety of *nonlinear* system-wide objectives and user QoS constraints, even when these nonlinear problems *appear* to be nonconvex optimization.

A. Basic model

Consider a wireless (cellular or multihop) network with n logical transmitter/receiver pairs. Transmit powers are denoted as $P_1, \ldots, P_n$. In the cellular uplink case, all logical receivers may reside in the same physical receiver, *i.e.*, the base station. In the multihop case, since the transmission environment can be different on the links comprising an end-to-end path, power control schemes must consider each link along a flow's path.

Under Rayleigh fading, the power received from transmitter j at receiver i is given by $G_{ij}F_{ij}P_j$ where $G_{ij} \geq 0$ represents the path gain (it may also encompass antenna gain and coding gain) that is often modeled as proportional to $d_{ij}^{-\gamma}$, where d_{ij} denotes distance, γ is the power fall-off factor, and F_{ij} models Rayleigh fading and are independent and exponentially distributed with unit mean. The distribution of the received power from transmitter j at receiver i is then exponential with mean value $\mathbf{E}\left[G_{ij}F_{ij}P_j\right] = G_{ij}P_j$. The SIR for the receiver on logical link i is

$$\mathsf{SIR}_i = \frac{P_i G_{ii} F_{ii}}{\sum_{j \neq i}^{N} P_j G_{ij} F_{ij} + n_i} \tag{3}$$

where n_i is the noise power for receiver i.

The constellation size M used by a link can be closely approximated for M-ary Quadrature Amplitude Modulation (MQAM) modulations as follows: $M = 1 + (-\phi_1 \mathsf{SIR})/(\ln(\phi_2 \mathsf{BER}))$, where BER is the bit error rate and ϕ_1, ϕ_2 are constants that depend on the modulation type [9]. Defining T as the symbol period and $K = (-\phi_1)/(\ln(\phi_2 \mathsf{BER}))$ leads to an expression of the data rate R_i on the ith link as a function of the SIR: $R_i = \frac{1}{T}\log_2(1 + K\mathsf{SIR}_i)$, which can be approximated as

$$R_i = \frac{1}{T}\log_2(K\mathsf{SIR}_i) \tag{4}$$

when $K\mathsf{SIR}$ is much larger than 1. This approximation is reasonable either when the signal level is much higher than the interference level or, in CDMA systems, when the spreading gain is large. For notational simplicity in the rest of this paper, we redefine G_{ii} as K times the original G_{ii}, thus absorbing constant K into the definition of SIR.

The aggregate data rate for the system can then be written as

$$R_{system} = \sum_i R_i = \frac{1}{T}\log_2\left[\prod_i \mathsf{SIR}_i\right].$$

So in the high SIR regime, aggregate data rate maximization is equivalent to maximizing a product of SIR. The system throughput is the aggregate data rate supportable by the system given a set of users with specified QoS requirements.

Outage probability is another important QoS parameter for reliable communication in wireless networks. A channel outage is declared and packets lost when the received SIR falls below a given threshold SIR_{th}, often computed from the BER requirement. Most systems are interference dominated and the thermal noise is relatively small, thus the ith link outage probability is

$$\begin{aligned} P_{o,i} &= \mathbf{Prob}\{\mathsf{SIR}_i \leq \mathsf{SIR}_{th}\} \\ &= \mathbf{Prob}\{G_{ii}F_{ii}P_i \leq \mathsf{SIR}_{th}\sum_{j\neq i} G_{ij}F_{ij}P_j\}. \end{aligned}$$

The outage probability can be expressed as $P_{o,i} = 1 - \prod_{j\neq i} 1/(1 + \frac{\mathsf{SIR}_{th}G_{ij}P_j}{G_{ii}P_i})$ [11], which means that the upper bound $P_{o,i} \leq P_{o,i,max}$ can be written as an upper bound on a posynomial in $\mathbf{P}$:

$$\prod_{j\neq i}\left(1 + \frac{\mathsf{SIR}_{th}G_{ij}P_j}{G_{ii}P_i}\right) \leq \frac{1}{1 - P_{o,i,max}}. \tag{5}$$

B. Cellular wireless networks

We first present how GP-based power control applies to cellular wireless networks with one-hop transmission from N users to a base station, extending the scope of power control by the classical solution in CDMA systems that equalizes SIRs, and those by the iterative algorithms (*e.g.*, in [2], [8], [13]) that minimize total power (a linear objective function) subject to SIR constraints.

We start the discussion on the suite of power control problem formulations with a simple objective function and simple constraints. The following constrained problem of maximizing the SIR of a particular user i^* is a GP:

$$\begin{array}{ll} \text{maximize} & R_{i^*}(\mathbf{P}) \\ \text{subject to} & R_i(\mathbf{P}) \geq R_{i,min}, \ \forall i, \\ & P_{i1}G_{i1} = P_{i2}G_{i2}, \\ & 0 \leq P_i \leq P_{i,max}, \ \forall i. \end{array}$$

The first constraint, equivalent to $\mathsf{SIR}_i \geq \mathsf{SIR}_{i,min}$, sets a floor on the SIR of other users and protects these users from user i^* increasing its transmit power excessively. The second constraint reflects the classical power control criterion in solving the near-far problem in CDMA systems: the expected received power from one transmitter $i1$ must equal that from another $i2$. The third constraint is regulatory or system limitations on transmit powers. All constraints can be verified to be inequality upper bounds on posynomials in transmit power vector $\mathbf{P}$.

Alternatively, we can use GP to maximize the minimum rate among all users. The maxmin fairness objective:

$$\text{maximize} \ \min_i \{R_i(\mathbf{P})\}$$

can also be accommodated in GP-based power control because it can be turned into equivalently maximizing an auxiliary variable t such that $\mathsf{SIR}_i(\mathbf{P}) \geq \exp(t), \ \forall i$, which has posynomial objective and constraints in $(\mathbf{P}, t)$.

Example 1. A simple system comprised of five users is used for a numerical example. The five users are spaced at distances d of $1, 5, 10, 15$, and 20 units from the base station. The power fall-off factor $\gamma = 4$. Each user has a maximum power constraint of $P_{max} = 0.5mW$. The noise power is 0.5μW for all users. The SIR of all users, other than the user we are optimizing for, must be greater than a common threshold SIR level β. In different experiments, β is varied to observe the effect on the optimized user's SIR. This is done independently for the near user at $d = 1$, a medium distance user at $d = 15$, and the far user at $d = 20$. The results are plotted in Figure 1.

Several interesting effects are illustrated. First, when the required threshold SIR in the constraints is sufficiently high, there are no feasible power control solutions. At moderate

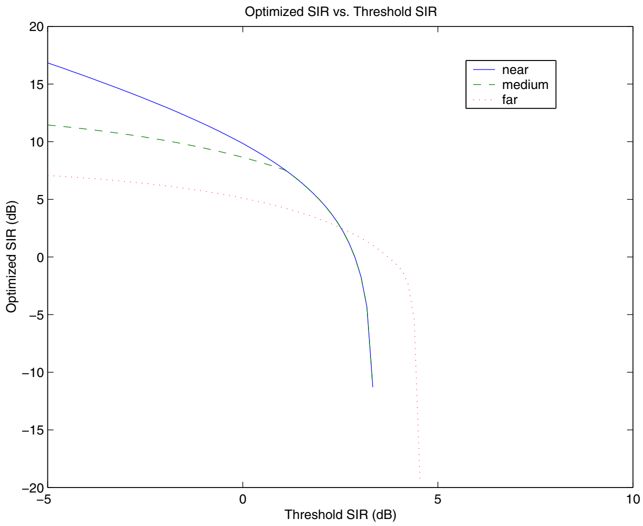

Fig. 1. Constrained optimization of power control in a cellular network (Example 1).

threshold SIR, as β is decreased, the optimized SIR initially increases rapidly. This is because it is allowed to increase its own power by the sum of the power reductions in the four other users, and the noise is relatively insignificant. At low threshold SIR, the noise becomes more significant and the power trade-off from the other users less significant, so the curve starts to bend over. Eventually, the optimized user reaches its upper bound on power and cannot utilize the excess power allowed by the lower threshold SIR for other users. This is exhibited by the transition from a sharp bend in the curve to a much shallower sloped curve.

We now proceed to show that GP can also be applied to the problem formulations with an overall system objective of total system throughput, under both user data rate constraints and outage probability constraints.

The following constrained problem of maximizing system throughput is a GP:

$$\begin{array}{ll} \text{maximize} & R_{system}(\mathbf{P}) \\ \text{subject to} & R_i(\mathbf{P}) \geq R_{i,min}, \ \ \forall i, \\ & P_{o,i}(\mathbf{P}) \leq P_{o,i,max}, \ \ \forall i, \\ & 0 \leq P_i \leq P_{i,max}, \ \ \forall i \end{array} \tag{6}$$

where the optimization variables are the transmit powers $\mathbf{P}$. The objective is equivalent to minimizing the posynomial $\prod_i \mathsf{ISR}_i$, where ISR is $1/\mathsf{SIR}$. Each ISR is a posynomial in $\mathbf{P}$ and the product of posynomials is again a posynomial. The first constraint is from the data rate demand $R_{i,min}$ by each user. The second constraint represents the outage probability upper bounds $P_{o,i,max}$. These inequality constraints put upper bounds on posynomials of $\mathbf{P}$, as can be readily verified through (4) and (5). Thus (6) is indeed a GP, and efficiently solvable for global optimality.

There are several obvious variations of problem (6) that can be solved by GP, *e.g.*, we can lower bound R_{system} as a constraint and maximize R_{i^*} for a particular user i^*, or have a total power $\sum_i P_i$ constraint or objective function. The objective function to be maximized can also be generalized to a weighted sum of data rates: $\sum_i w_i R_i$ where $\mathbf{w} \succeq 0$ is a given weight vector. This is still a GP because maximizing $\sum_i w_i \log \mathsf{SIR}_i$ is equivalent to maximizing $\log \prod_i \mathsf{SIR}_i^{w_i}$, which is in turn equivalent to minimizing $\prod_i \mathsf{ISR}_i^{w_i}$. Now use auxiliary variables $\{t_i\}$, and minimize $\prod_i t_i^{w_i}$ over the original constraints in (6) plus the additional constraints $\mathsf{ISR}_i. \leq t_i$ for all i. This is readily verified to be a GP, and is equivalent to the original problem.

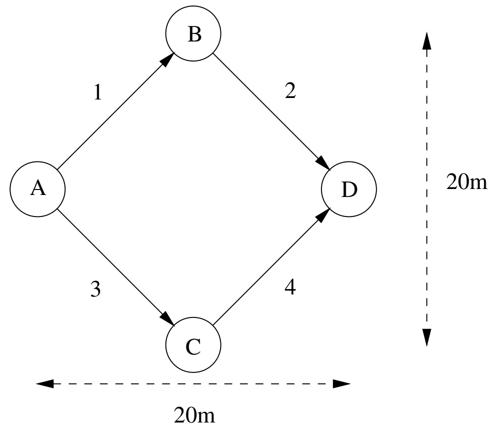

Fig. 2. A small wireless multihop network (Example 2).

Generalizing the above discussions and observing that high-SIR assumption is needed for GP formulation only when there are *sums* of $\log(1+\mathsf{SIR})$ in the optimization problem, we have the following summary.

Proposition 1: In the high-SIR regime, any combination of objectives (A)-(E) and constraints (a)-(e) in Table I is a power control problem that can be solved by GP, *i.e.*, can be transformed into a convex optimization with efficient algorithms to compute the globally optimal power vector. When objectives (C)-(D) and constraints (c)-(d) do not appear, the power control optimization problem can be solved by GP in any SIR regime.

In addition to efficient computation of the globally optimal power allocation with nonlinear objectives and constraints, GP can also be used for admission control based on feasibility study described in [7], and for determining which QoS constraint is a performance bottleneck, *i.e.*, met tightly at the optimal power allocation.[2]

C. Extensions: Multihop wireless networks

In wireless multihop networks, system throughput may be measured either by end-to-end transport layer utilities or by link layer aggregate throughput. GP application to the first approach has appeared in [6], and we focus on the second (and easier) approach in this section. Formulations in Table I can be readily extended to multihop case by indexing each logical link as i. An example is presented below for the total throughput maximization formulation (6).

Example 2. Consider a simple four node multihop network shown in Figure 2. There are two connections $A \to B \to D$

[2]This is because most GP solution algorithms solve both the primal GP and its Lagrange dual problem: by complementary slackness condition, a resource constraint is tight at optimal power allocation when the corresponding optimal dual variable is non-zero.

TABLE I
SUITE OF POWER CONTROL OPTIMIZATION SOLVABLE BY GP

Objective Function	Constraints
(A) Maximize R_{i^*} (specific user)	(a) $R_i \geq R_{i,min}$ (rate constraint)
(B) Maximize $\min_i R_i$ (worst-case user)	(b) $P_{i1}G_{i1} = P_{i2}G_{i2}$ (near-far constraint)
(C) Maximize $\sum_i R_i$ (total throughput)	(c) $\sum_i R_i \geq R_{system,min}$ (total throughput constraint)
(D) Maximize $\sum_i w_i R_i$ (weighted rate sum)	(d) $P_{o,i} \leq P_{o,i,max}$ (outage prob. constraint)
(E) Minimize $\sum_i P_i$ (total power)	(e) $0 \leq P_i \leq P_{i,max}$ (power constraint)

and $A \rightarrow C \rightarrow D$. Nodes A and D, as well as B and C, are separated by a distance of 20m. Path gain between a transmitter and a receiver has a common fall-off factor $\gamma = 4$. Each link has a maximum transmit power of 1mW. All nodes use MQAM modulation. The minimum data rate for each connection is 100bps, and the target BER is 10^{-3}. Assuming Rayleigh fading, we require outage probability be smaller than 0.1 on all links for an SIR threshold of 10dB. Spreading gain is 200. Using GP formulation (6), we find the maximized system throughput $R^* = 216.8$kbps, $R_i^* = 54.2$kbps for each link, $P_1^* = P_3^* = 0.709$mW and $P_2^* = P_4^* = 1$mW. The resulting optimized SIR is 21.7dB on each link. For this topology, we also consider an illustrative example of admission control and pricing. Three new users U_1, U_2, and U_3 are going to arrive to the network in order. U_1 and U_2 require 30kbps sent along the upper path $A \rightarrow B \rightarrow D$, while U_3 requires 10kbps sent from $A \rightarrow B$. All three users require the outage probability to be less than 0.1. When U_1 arrives at the system, its price is the baseline price. Next, U_2 arrives, and its QoS demands decrease the maximum system throughput from 216.82kbps to 116.63kbps, so its price is the baseline price plus an amount proportional to the reduction in system throughput. Finally, U_3 arrives, and its QoS demands produce no feasible power allocation solution, so she is not admitted to the system.

D. Extensions: Queuing models

We now turn to delay and buffer overflow properties to be included in constraints or objective function of power control optimization. The average delay a packet experiences traversing a network is an important design consideration in some applications. Queuing delay is often the primary source of delay, particularly for bursty data traffic in multihop networks. A node i first buffers the received packets in a queue and then transmits these packets at a rate R set by the SIR on the egress link, which is in turn determined by the transmit powers $\mathbf{P}$. A FIFO queuing discipline is used here for simplicity. Routing is assumed to be fixed, and is feed-forward with all packets visiting a node at most once.

Packet traffic entering the multihop network at the transmitter of link i is assumed to be Poisson with parameter λ_i and to have an exponentially distributed length with parameter Γ. Using the model of an $M/M/1$ queue as in [3], the probability of transmitter i having a backlog of $N_i = k$ packets to transmit is well-known to be $\mathbf{Prob}\{N_i = k\} = (1-\rho)\rho^k$ where $\rho = \lambda_i/\Gamma R_i(\mathbf{P})$, and the expected delay is $1/(\Gamma R_i(\mathbf{P}) - \lambda_i)$. Under the feed-forward routing and Poisson input assumptions, Burke's theorem in [3] can be applied. Thus the total packet arrival rate at node i is $\Lambda_i = \sum_{j \in I(i)} \lambda_j$ where $I(i)$ is the set of connections traversing node i. The expected delay $\bar{D}_i$ can be written as

$$\bar{D}_i = \frac{1}{\Gamma R_i(\mathbf{P}) - \Lambda_i}. \tag{7}$$

A bound $\bar{D}_{i,max}$ on $\bar{D}_i$ can thus be written as $1/(\Gamma \log_2(\mathsf{SIR}_i)/T - \Lambda_i) \leq \bar{D}_{i,max}$, or equivalently, $\mathsf{ISR}_i(\mathbf{P}) \leq 2^{-T(\bar{D}_{max}^{-1} + \Lambda_i)/\Gamma}$, an upper bound on a posynomial ISR of $\mathbf{P}$ that is allowed in GP formulations.

The probability P_{BO} of dropping a packet due to buffer overflow at a node is also important in several applications. It is again a function of $\mathbf{P}$ and can be written as $P_{BO,i} = \mathbf{Prob}\{N_i > B\} = \rho^{B+1}$ where B is the buffer size and $\rho = \Lambda_i/(\Gamma R_i(\mathbf{P}))$. Setting an upper bound $P_{BO,i,max}$ on the buffer overflow probability also gives a posynomial lower bound constraint in $\mathbf{P}$: $\mathsf{ISR}_i(\mathbf{P}) \leq 2^{-\Psi}$ where $\Psi = (T\Lambda_i)/(\Gamma (P_{BO,i,max})^{\frac{1}{B+1}})$, which is allowed in GP formulations. In summary, we have the following.

Proposition 2: The following nonlinear problem of optimizing powers to maximize system throughput in the high-SIR regime, subject to constraints on outage probability, expected delay, and the probability of buffer overflow, is a GP:

$$\begin{array}{ll} \text{maximize} & R_{system}(\mathbf{P}) \\ \text{subject to} & \bar{D}_i(\mathbf{P}) \leq \bar{D}_{i,max}, \ \forall i, \\ & P_{BO,i}(\mathbf{P}) \leq P_{BO,i,max}, \ \forall i, \\ & \text{Any combination of constraints (a)-(e)} \\ & \text{in Table (I)} \end{array} \tag{8}$$

where the optimization variables are the transmit powers $\mathbf{P}$.

Example 3. Consider a numerical example of the optimal tradeoff between maximizing the system throughput and bounding the expected delay for the network shown in Figure 3. There are six nodes, eight links, and five multihop connections. All sources are Poisson with intensity $\lambda_i = 200$ packets per second, and exponentially distributed packet lengths with an expectation of of 100 bits. The nodes use CDMA transmission scheme with a symbol rate of 10k symbols per second and the spreading gain is 200. Transmit powers are limited to 1mW and the target BER is 10^{-3}. The path loss matrix is calculated based on a power falloff of d^{-4} with the distance d, and a separation of 10m between any adjacent nodes along the perimeter of the network. Figure 4 shows the maximized system throughput for different upper bound numerical values in the expected delay constraints, obtained

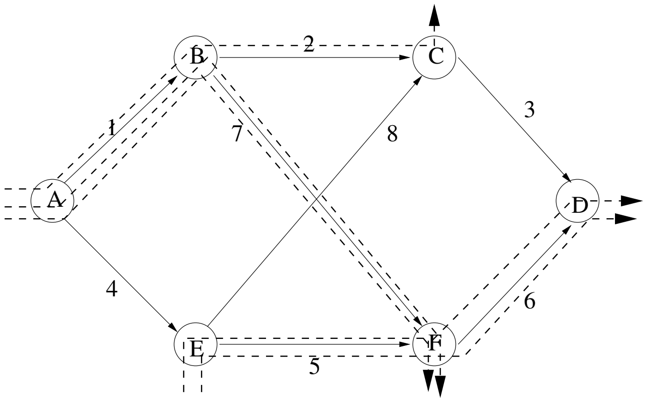

Fig. 3. Topology and flows in a multihop wireless network (Example 3).

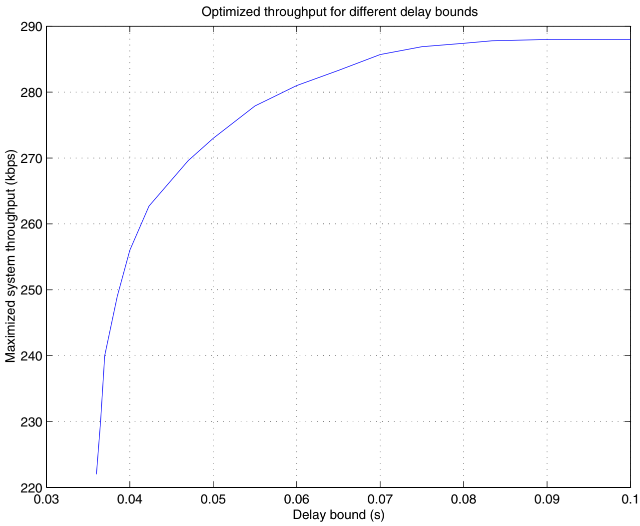

Fig. 4. Optimal tradeoff between maximized system throughput and average delay constraint (Example 3).

by solving a sequence of GPs, one for each point on the curve. There is no feasible power allocation to achieve delay smaller than 0.036s. As the delay bound is relaxed, the maximized system throughput increases sharply first, then more slowly until the delay constraints are no longer active. GP efficiently returns the globally Pareto-optimal tradeoff curve between system throughput and queuing delay.

There are two main limitations in the GP-based power control methods discussed so far: high-SIR assumption and centralized computation. Both can be overcome as discussed in the next two sections.

IV. Power Control by Geometric Programming: Non-convex Case

If we maximize the total throughput R_{system} in the medium to low SIR case, *i.e.*, when SIR is not much larger than 0dB, the approximation of $\log(1+\mathsf{SIR})$ as $\log \mathsf{SIR}$ does not hold. Unlike SIR, which is an inverted posynomial, $1+\mathsf{SIR}$ is not an inverted posynomial. Instead, $1/(1+\mathsf{SIR})$ is a ratio between two posynomials:

$$\frac{f(\mathbf{P})}{g(\mathbf{P})} = \frac{\sum_{j\neq i} G_{ij}P_j + n_i}{\sum_j G_{ij}P_j + n_i}. \quad (9)$$

Minimizing or upper bounding a ratio between two posynomials belongs to a truly nonconvex class of problems known as Complementary GP [1], [7] that is an intractable NP-hard problem. An equivalent generalization of GP is Signomial Programming (SP) [1], [7]: minimizing a signomial subject to upper bound inequality constraints on signomials, where a signomial $s(\mathbf{x})$ is a sum of monomials, possibly with *negative* multiplicative coefficients: $s(\mathbf{x}) = \sum_{i=1}^N c_i g_i(\mathbf{x})$ where $\mathbf{c} \in \mathbf{R}^N$ and $g_i(\mathbf{x})$ are monomials.[3]

A. Successive convex approximation method

Consider the following nonconvex problem:

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 1, \quad i = 1, 2, \ldots, m, \end{array} \quad (10)$$

where f_0 is convex without loss of generality[4], but the $f_i(\mathbf{x})$'s, $\forall i$ are nonconvex. Since directly solving this problem is difficult, we will solve it by a series of approximations $\tilde{f}_i(\mathbf{x}) \approx f_i(\mathbf{x}), \forall \mathbf{x}$, each of which can be optimally solved in an easy way. It turns out that if the approximations satisfy the following three properties, then the solutions of this series of approximations converge to a point satisfying the Karush-Kuhn-Tucker (KKT) conditions of the original problem [12]:

(1) $f_i(\mathbf{x}) \leq \tilde{f}_i(\mathbf{x})$ for all $\mathbf{x}$,

(2) $f_i(\mathbf{x}_0) = \tilde{f}_i(\mathbf{x}_0)$ where x_0 is the optimal solution of the approximated problem in the previous iteration,

(3) $\nabla f_i(\mathbf{x}_0) = \nabla \tilde{f}_i(\mathbf{x}_0)$.

Condition (1) guarantees that the approximation $\tilde{f}_i(\mathbf{x})$ is tightening the constraints in (10), and any solution of the approximated problem will be a feasible point of the original problem in (10). Condition (2) guarantees that the solution of each approximated problem will decrease the cost function: $f_0(\mathbf{x}^{(k)}) \leq f_0(\mathbf{x}^{(k-1)})$, where $x^{(k)}$ is the solution to the k-th approximated problem. Condition (3) guarantees that the KKT conditions of the original problem in (10) will be satisfied after the series of approximations converges.

The following algorithm describes the generic successive approximation approach.

Algorithm Successive approximation to a nonconvex problem

Input Method to approximate $f_i(\mathbf{x})$ with $\tilde{f}_i(\mathbf{x})$, $\forall i$, around some point of interest $\mathbf{x}_0$.

Output A vector that satisfies the KKT conditions of the original problem.

0) Choose an initial feasible point $\mathbf{x}^{(0)}$ and set $k = 1$.

1) Form the k-th approximated problem of (10) based on approximating $f_i(\mathbf{x})$ with $\tilde{f}_i(\mathbf{x})$ around the previous point $\mathbf{x}^{(k-1)}$.

2) Solve the k-th approximated problem to obtain $\mathbf{x}^{(k)}$.

3) Increment k and go to step 2 until convergence to a stationary point.

[3]An SP can always be converted into a Complementary GP, because an inequality in SP, which can be written as $f_{i1}(\mathbf{x}) - f_{i2}(\mathbf{x}) \leq 1$, where f_{i1}, f_{i2} are posynomials, is equivalent to an inequality $f_{i1}(\mathbf{x})/(1+f_{i2}(\mathbf{x})) \leq 1$ in Complementary GP. Similarly, an upper bound on a ratio of posynomials in Complementary GP can be rewritten as an SP inequality constraint.

[4]If f_0 is nonconvex, we can move the objective function to the constraints by introducing auxiliary scalar variable t and writing minimize t subject to the additional constraint $f_0(\mathbf{x}) - t \leq 0$.

B. Examples of successive convex approximation

1) Logarithmic approximation for GP: In [17], [18], a nonconvex problem involving the function $\log(1+\mathsf{SIR})$ is approximated by $a + b\log(\mathsf{SIR})$ for some a and b that satisfy the above three conditions.

2) Single condensation method for GP: Complementary GPs involve upper bounds on the ratio of posynomials as in (9); they can be turned into GPs by approximating the denominator of the ratio of posynomials, $g(\mathbf{x})$, with a monomial $\tilde{g}(\mathbf{x})$, but leaving the numerator $f(\mathbf{x})$ as a posynomial.

Lemma 1: Let $g(\mathbf{x}) = \sum_i u_i(\mathbf{x})$ be a posynomial. Then

$$g(\mathbf{x}) \geq \tilde{g}(\mathbf{x}) = \prod_i \left(\frac{u_i(\mathbf{x})}{\alpha_i}\right)^{\alpha_i}. \tag{11}$$

If, in addition, $\alpha_i = u_i(\mathbf{x}_0)/g(\mathbf{x}_0), \ \forall i$, for any fixed positive $\mathbf{x}_0$, then $\tilde{g}(\mathbf{x}_0) = g(\mathbf{x}_0)$, and $\tilde{g}(\mathbf{x}_0)$ is the best local monomial approximation to $g(\mathbf{x}_0)$ near $\mathbf{x}_0$ in the sense of first order Taylor approximation.

Proof: The arithmetic-geometric mean inequality states that $\sum_i \alpha_i v_i \geq \prod_i v_i^{\alpha_i}$, where $\mathbf{v} \succ 0$ and $\boldsymbol{\alpha} \succeq 0, \ \mathbf{1}^T\boldsymbol{\alpha} = 1$. Letting $u_i = \alpha_i v_i$, we can write this basic inequality as $\sum_i u_i \geq \prod_i (u_i/\alpha_i)^{\alpha_i}$. The inequality becomes an equality if we let $\alpha_i = u_i/\sum_i u_i, \ \forall i$, which satisfies the condition that $\boldsymbol{\alpha} \succeq 0$ and $\mathbf{1}^T\boldsymbol{\alpha} = 1$. The best local monomial approximation $\tilde{g}(\mathbf{x}_0)$ of $g(\mathbf{x}_0)$ near $\mathbf{x}_0$ can be easily verified [4]. ∎

Proposition 3: The approximation of a ratio of posynomials $f(\mathbf{x})/g(\mathbf{x})$ with $f(\mathbf{x})/\tilde{g}(\mathbf{x})$, where $\tilde{g}(\mathbf{x})$ is the monomial approximation of $g(\mathbf{x})$ using the arithmetic-geometric mean approximation of Lemma 1, satisfies the three conditions for the convergence of the successive approximation method.

Proof: Conditions (1) and (2) are clearly satisfied since $g(\mathbf{x}) \geq \tilde{g}(\mathbf{x})$ and $\tilde{g}(\mathbf{x}_0) = g(\mathbf{x}_0)$ (Lemma 1). Condition (3) is easily verified by taking derivatives of $g(\mathbf{x})$ and $\tilde{g}(\mathbf{x})$. ∎

Suppose we want to minimize $f_0(\mathbf{x})$ subject to an equality constraint on a ratio of posynomials $f(\mathbf{x})/g(\mathbf{x}) = 1$. Then, we can rewrite this as minimizing $f_0(\mathbf{x}) + \phi t$ subject to $f(\mathbf{x})/g(\mathbf{x}) \leq 1$ and $f(\mathbf{x})/g(\mathbf{x}) \geq 1 - t$ where ϕ is a sufficiently large number to guarantee that the optimum solution will have $t \approx 0$.[5] The second constraint can be rewritten as $g(\mathbf{x})/(f(\mathbf{x}) + tg(\mathbf{x})) \leq 1$ where we can apply the single condensation method to the denominators of both constraints.

3) Double condensation method for GP[1]: Another choice of approximation is to make a double monomial approximation for both the denominator and numerator in (9). We can still use the arithmetic-geometric mean approximation of Lemma 1 as a monomial approximation for the denominator. But, Lemma 1 cannot be used as a monomial approximation for the numerator. To satisfy the three conditions for the convergence of the successive approximation method, a monomial approximation for the numerator $f(\mathbf{x})$ should satisfy $f(\mathbf{x}) \leq \tilde{f}(\mathbf{x})$.

C. Applications to power control

Figure 5 shows a block diagram of the approach of GP-based power control for general SIR regime. In the high SIR regime, we need to solve only *one* GP. In the medium to low SIR regimes, we solve truly nonconvex power control problems that cannot be turned into convex formulation through a *series* of GPs.

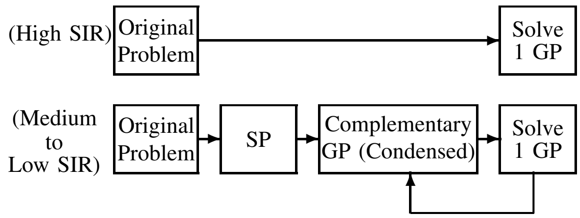

Fig. 5. GP-based power control for general SIR regime.

GP-based power control problems in the medium to low SIR regimes become SP (or, equivalently, Complementary GP), which can be solved by the single or double condensation method. We focus on the single condensation method here. Consider a representative problem formulation of maximizing total system throughput in a cellular wireless network subject to user rate and outage probability constraints in problem (6), which can be explicitly written out as

$$\begin{array}{ll} \text{minimize} & \prod_{i=1}^{N} \frac{1}{1+\mathsf{SIR}_i} \\ \text{subject to} & (2^{TR_{i,min}} - 1)\frac{1}{\mathsf{SIR}_i} \leq 1, \ \forall i, \\ & (\mathsf{SIR}_{th})^{N-1}(1 - P_{o,i,max}) \prod_{j\neq i}^{N} \frac{G_{ij}P_j}{G_{ii}P_i} \leq 1, \ \forall i, \\ & P_i(P_{i,max})^{-1} \leq 1, \ \forall i. \end{array} \tag{12}$$

All the constraints are posynomials. However, the objective is not a posynomial, but a ratio between two posynomials as in (9). This power control problem can be solved by the condensation method by solving a series of GPs. Specifically, we have the following single-condensation algorithm:

Algorithm Single condensation GP power control

Input An initial feasible power vector $\mathbf{P}$.

Output A power allocation that satisfies the KKT conditions.

1) Evaluate the denominator posynomial of the objective function in (12) with the given $\mathbf{P}$.

2) Compute for each term i in this posynomial,

$$\alpha_i = \frac{\text{value of } i\text{th term in posynomial}}{\text{value of posynomial}}.$$

3) Condense the denominator posynomial of the (12) objective function into a monomial using (11) with weights α_i.

4) Solve the resulting GP using an interior point method.

5) Go to step 1 using $\mathbf{P}$ of step 4.

6) Terminate the kth loop if $\parallel \mathbf{P}^{(k)} - \mathbf{P}^{(k-1)} \parallel \leq \epsilon$ where ϵ is the error tolerance for exit condition.

As condensing the objective in the above problem gives us an underestimate of the objective value, each GP in the condensation iteration loop tries to improve the accuracy of the approximation to a particular minimum in the original feasible region. All three conditions for convergence in subsection IV.A are satisfied, and the algorithm is provably convergent. Empirically through extensive numerical experiments, we observe that it often compute the globally optimal power allocation.

Example 4. We consider a wireless cellular network with 3 users. Let $T = 10^{-6}$s, $G_{ii} = 1.5$, and generate $G_{ij}, i \neq j$, as

[5] In our numerical analysis, we use $\phi = 1 + k$ at the k-th iteration.

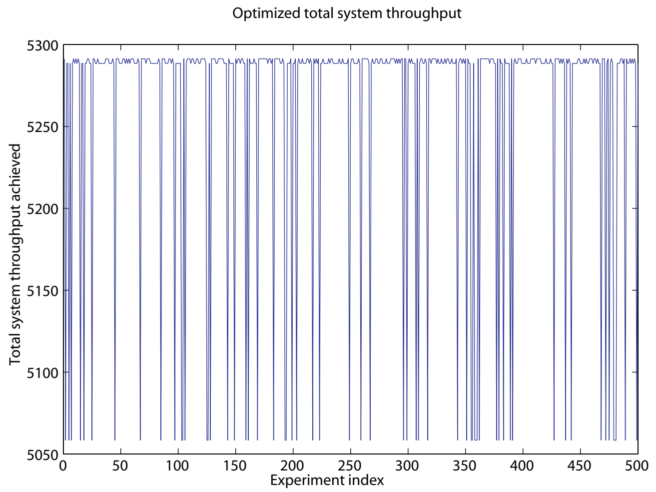

Fig. 6. Maximized total system throughput achieved by (single) condensation method for 500 different initial feasible vectors (Example 4). Each point represents a different experiment with a different initial power vector.

independent random variables uniformly distributed between 0 and 0.3. Threshold SIR is $\mathsf{SIR}_{th} = -10$dB, and minimal data rate requirements are 100 kbps, 600 kbps and 1000 kbps for logical links 1, 2 and 3 respectively. Maximal outage probabilities are 0.01 for all links, and maximal transmit powers are 3mW, 4mW and 5mW for link 1, 2 and 3, respectively. For each instance of SP power control (12), we pick a random initial feasible power vector $\mathbf{P}$ uniformly between 0 and $\mathbf{P}_{max}$. Figure 6 compares the maximized total network throughput achieved over five hundred sets of experiments with different initial vectors. With the (single) condensation method, SP converges to different optima over the entire set of experiments, achieving (or coming very close to) the global optimum at 5290 bps 96% of the time and a local optimum at 5060 bps 4% of the time, thus is very likely to converge to or very close to the global optimum. The average number of GP iterations required by the condensation method over the same set of experiments is 15 if an extremely tight exit condition is picked for SP condensation iteration: $\epsilon = 1 \times 10^{-10}$. This average can be substantially reduced by using a larger ϵ, *e.g.*, increasing ϵ to 1×10^{-2} requires on average only 4 GPs.

We have thus far discussed a power control problem (12) where the objective function needs to be condensed. The method is also applicable if some constraint functions are signomials and need to be condensed. For example, consider the case of differentiated services where a user expects to obtain a predicted QoS relatively better than the other users. We may have a proportional delay differentiation model where a user who pays more tariff obtains a delay proportionally lower as compared to users who pay less. Then for a particular ratio between any user i and j, σ_{ij}, we have

$$\frac{\overline{D}_i}{\overline{D}_j} = \sigma_{ij}, \qquad (13)$$

which, by (7), is equivalent to

$$\frac{1+\mathsf{SIR}_j}{(1+\mathsf{SIR}_i)^{\sigma_{ij}}} = 2^{(\lambda_j - \sigma_{ij}\lambda_i)T/\Gamma}. \qquad (14)$$

TABLE II

EXAMPLES 4 AND 5. ROW 2 SHOWS EXAMPLE 4 USING THE SINGLE CONDENSATION ON SYSTEM THROUGHPUT MAXIMIZATION ONLY, AND ROW 1 SHOWS THE OPTIMAL SOLUTIONS FOUND BY EXHAUSTIVE SEARCH. ROW 4 SHOWS EXAMPLE 5 WITH AN ADDITIONAL DIFFSERV CONSTRAINT $D_1/D_3 = 1$, WHICH ALSO RECOVERS THE NEW OPTIMAL SOLUTION OF 6624 KBPS AS VERIFIED BY EXHAUSTIVE SEARCH, AND ROW 3 SHOWS THE OPTIMAL SOLUTIONS FOUND BY EXHAUSTIVE SEARCH.

Method	System throughput	P_1^*	P_2^*	P_3^*
Exhaustive search	6626 kbps	0.65	0.77	0.79
Single condensation	6626 kbps by solving 17 GPs	0.65	0.77	0.78
Exhaustive search	6624 kbps	0.68	0.75	0.68
With DiffServ constraint	6624 kbps by solving 17 GPs	0.68	0.75	0.68

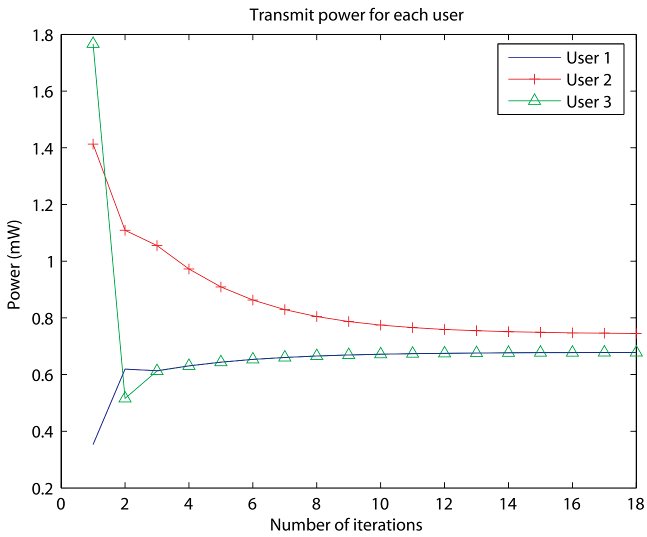

Fig. 7. Convergence of power variables (Example 5).

The denominator on the left hand side is a ratio between posynomials raised to a positive power. Therefore, the single condensation method can be readily used to solve the proportional delay differentiation problem because the function on the left hand side can be condensed using successive approximation on an equality constraint of a posynomial in Section IV-B.

Example 5. We consider the wireless cellular network in Example 4 with an additional constraint $D_1/D_3 = 1$. The arrival rates of each user at base station is measured and input as network parameters into (14). Figures 7 and 8 show the convergence towards satisfying all the QoS constraints including the DiffServ constraint. As shown on the figures, the convergence is fast, with the power allocations very close to the optimal power allocation by the 8th GP iteration. Also, Table II summarizes the optimizers for Examples 4 and 5 using $\epsilon = 1 \times 10^{-10}$.

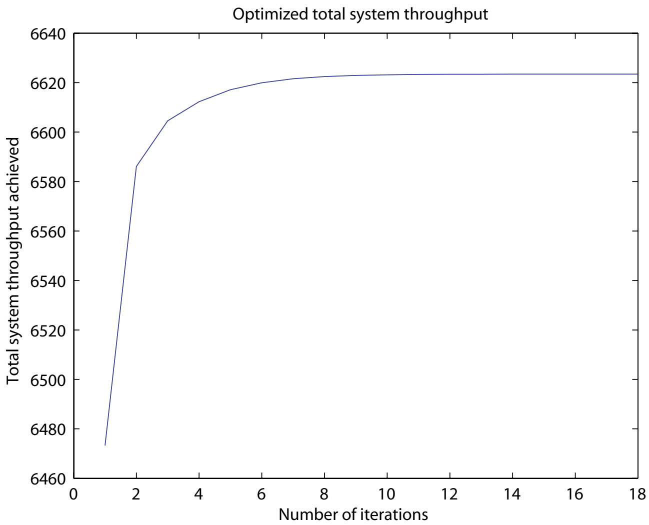

Fig. 8. Convergence of total system throughput (Example 5).

V. Distributed Implementation

A limitation for GP based power control in ad hoc networks without base stations is the need for centralized computation (*e.g.*, by interior point methods). The GP formulations of power control problems can also be solved by a new method of distributed algorithm for GP. The basic idea is that each user solves its own local optimization problem and the coupling among users is taken care of by message passing among the users. Interestingly, the special structure of coupling for the problem at hand (essentially, all coupling among the logical links can be lumped together using interference terms) allows one to further reduce the amount of message passing among the users. Specifically, we use a dual decomposition method to decompose a GP into smaller subproblems whose solutions are jointly and iteratively coordinated by the use of dual variables. The key step is to introduce auxiliary variables and to add extra equality constraints, thus transferring the coupling in the objective to coupling in the constraints, which can be solved by introducing 'consistency pricing'. We illustrate this idea through an unconstrained GP followed by an application of the technique to power control.

A. Distributed algorithm for GP

Suppose we have the following unconstrained standard form GP in $\mathbf{x} \succ 0$:

$$\text{minimize} \quad \textstyle\sum_i f_i(x_i, \{x_j\}_{j \in I(i)}) \tag{15}$$

where $I(i)$ is the set of users coupled with the ith user, and x_i denotes the local variable of the ith user, $\{x_j\}_{j \in I(i)}$ denote the coupled variables from other users, and f_i is either a monomial or posynomial. Making a change of variable $y_i = \log x_i, \forall i$, in the original problem, we obtain

$$\text{minimize} \quad \textstyle\sum_i f_i(e^{y_i}, \{e^{y_j}\}_{j \in I(i)}).$$

We now rewrite the problem by introducing auxiliary variables y_{ij} for the coupled arguments and additional equality constraints to enforce consistency:

$$\begin{array}{ll} \text{minimize} & \sum_i f_i(e^{y_i}, \{e^{y_{ij}}\}_{j \in I(i)}) \\ \text{subject to} & y_{ij} = y_j, \forall j \in I(i), \forall i. \end{array} \tag{16}$$

Each ith user controls the local variables $(y_i, \{y_{ij}\}_{j \in I(i)})$. Next, the Lagrangian of (16) is formed as

$$\begin{aligned} & L(\{y_i\}, \{y_{ij}\}; \{\gamma_{ij}\}) \\ = & \sum_i f_i(e^{y_i}, \{e^{y_{ij}}\}_{j \in I(i)}) + \sum_i \sum_{j \in I(i)} \gamma_{ij}(y_j - y_{ij}) \\ = & \sum_i L_i(y_i, \{y_{ij}\}; \{\gamma_{ij}\}) \end{aligned}$$

where

$$\begin{aligned} & L_i(y_i, \{y_{ij}\}; \{\gamma_{ij}\}) \\ = \; & f_i(e^{y_i}, \{e^{y_{ij}}\}_{j \in I(i)}) + \left(\sum_{j: i \in I(j)} \gamma_{ji} \right) y_i - \sum_{j \in I(i)} \gamma_{ij} y_{ij}. \end{aligned} \tag{17}$$

The minimization of the Lagrangian with respect to the primal variables $(\{y_i\}, \{y_{ij}\})$ can be done simultaneously and distributively by each user in parallel. In the more general case where the original problem (15) is constrained, the additional constraints can be included in the minimization at each L_i.

In addition, the following master Lagrange dual problem has to be solved to obtain the optimal dual variables or consistency prices $\{\gamma_{ij}\}$:

$$\max_{\{\gamma_{ij}\}} \quad g(\{\gamma_{ij}\}) \tag{18}$$

where

$$g(\{\gamma_{ij}\}) = \sum_i \min_{y_i, \{y_{ij}\}} L_i(y_i, \{y_{ij}\}; \{\gamma_{ij}\}).$$

Note that the transformed primal problem (16) is convex, and the duality gap is zero under mild conditions; hence the Lagrange dual problem (18) indeed solves the original standard GP problem (15). A simple way to solve the maximization in (18) is with the following subgradient update for the consistency prices:

$$\gamma_{ij}(t+1) = \gamma_{ij}(t) + \delta(t)(y_j(t) - y_{ij}(t)). \tag{19}$$

Appropriate choice of the stepsize $\delta(t) > 0$, *e.g.*, $\delta(t) = \delta_0/t$ for some constant $\delta_0 > 0$, leads to convergence of the dual algorithm [7].

Summarizing, the ith user has to: i) minimize the Lagrangian L_i in (18) involving only *local* variables upon receiving the updated dual variables $\{\gamma_{ji}, j : i \in I(j)\}$ (note that $\{\gamma_{ij}, j \in I(i)\}$ are local dual variables), and ii) update the local consistency prices $\{\gamma_{ij}, j \in I(i)\}$ with (19), and broadcast the updated prices to the coupled users.

B. Applications to power control

As an illustrative example, we maximize the total system throughput in the high SIR regime with constraints local to each user. If we directly applied the distributed approach described in the last subsection, the resulting algorithm would not be very practical since it would require knowledge by each user of the interfering channels and interfering transmit powers, which would translate into a large amount of message passing. To obtain a practical distributed solution, we can leverage the structures of power control problems at hand, and instead keep a local copy of each of the *effective received powers* $P_{ij}^R = G_{ij}P_j$. Again using problem (6) as an example formulation and assuming high SIR, we can write the problem as following (after the logarithmic change of variable $\tilde{x} = \log x$):

$$\begin{array}{ll} \text{minimize} & \sum_i \log\left(G_{ii}^{-1}\exp(-\tilde{P}_i)\left(\sum_{j\neq i}\exp(\tilde{P}_{ij}^R)+\sigma^2\right)\right) \\ \text{subject to} & \tilde{P}_{ij}^R = \tilde{G}_{ij} + \tilde{P}_j, \\ & \text{Constraints local to each user, i.e., (a),(d) and (e)} \\ & \text{in Table (I).} \end{array} \quad (20)$$

The partial Lagrangian is

$$\begin{aligned} L &= \sum_i \log\left(G_{ii}^{-1}\exp(-\tilde{P}_i)\left(\sum_{j\neq i}\exp(\tilde{P}_{ij}^R)+\sigma^2\right)\right) + \\ &\quad \sum_i\sum_{j\neq i}\gamma_{ij}\left(\tilde{P}_{ij}^R - \left(\tilde{G}_{ij}+\tilde{P}_j\right)\right), \end{aligned} \quad (21)$$

and the separable property of the Lagrangian in (21) can be exploited to determine the optimal power allocation $\mathbf{P}^*$ distributively.[6] The distributed power control algorithm is summarized as follows.

Algorithm Distributed power allocation update to maximize R_{system}.

Input Each ith user sets $\gamma_{ij}(0) = 0, \forall j$ and agrees among all users on a constant $\delta_0 > 0$ for the stepsize.

Output Optimal power allocation $\mathbf{P}^*$ for all users.

At each iteration t:

1) Each ith user receives the term $\left(\sum_{j\neq i}\gamma_{ji}(t)\right)$ involving the dual variables from the interfering users by message passing and minimizes the following local Lagrangian with respect to $\tilde{P}_i(t), \left\{\tilde{P}_{ij}^R(t)\right\}_j$ subject to the local constraints:

$$\begin{aligned} & L_i\left(\tilde{P}_i(t), \left\{\tilde{P}_{ij}^R(t)\right\}_j; \{\gamma_{ij}(t)\}_j\right) \\ &= \log\left(G_{ii}^{-1}\exp(-\tilde{P}_i(t))\left(\sum_{j\neq i}\exp(\tilde{P}_{ij}^R(t))+\sigma^2\right)\right) \\ &+ \sum_{j\neq i}\gamma_{ij}\tilde{P}_{ij}^R(t) - \left(\sum_{j\neq i}\gamma_{ji}(t)\right)\tilde{P}_i(t). \end{aligned} \quad (22)$$

2) Each ith user estimates the effective received power from each of the interfering users $P_{ij}^R(t) = G_{ij}P_j(t)$ for $j \neq i$, updates the dual variable as

$$\gamma_{ij}(t+1) = \gamma_{ij}(t) + (\delta_0/t)\left(\tilde{P}_{ij}^R(t) - \log G_{ij}P_j(t)\right), \quad (23)$$

and then broadcast them by message passing to all interfering users in the system.

The amount of message passing can be further reduced in practice by ignoring the messages from links that are physically far apart, leading to suboptimal distributed heuristics. If the system can be divided into clusters so that interference happens primarily within the clusters but not across clusters, the signaling can be significantly reduced by tailoring the distributed approach on a cluster-basis [22].

[6] A small quadratic term in $\tilde{P}_i$ for each i is added to the partial Lagrangian in (21) to enforce strict convexity of (21) in $\tilde{\mathbf{P}}$.

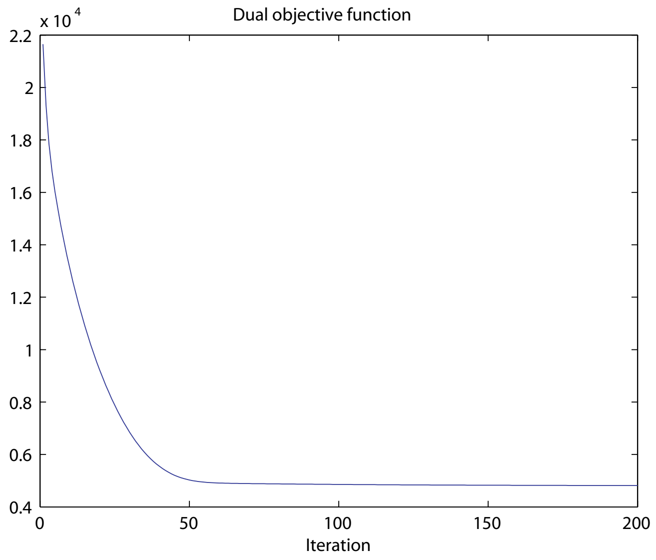

Fig. 9. Convergence of the dual objective function through distributed algorithm (Example 6).

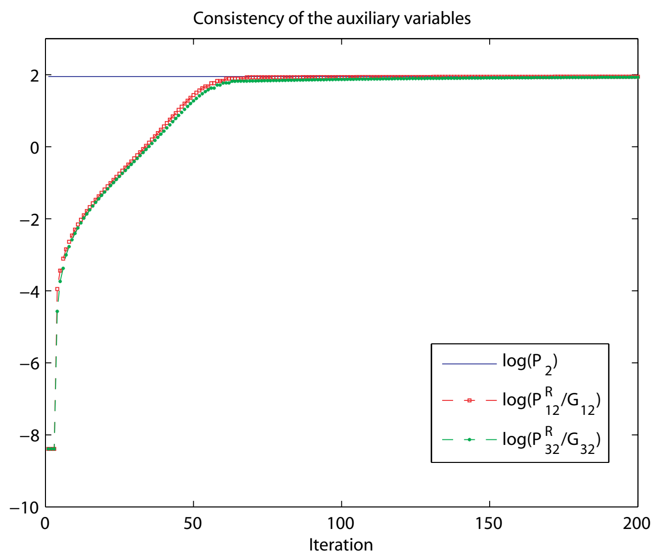

Fig. 10. Convergence of the consistency constraints through distributed algorithm (Example 6).

Example 6. We apply the distributed algorithm to solve the above power control problem for three logical links with $G_{ij} = 0.2, i \neq j$, $G_{ii} = 1, \forall i$, maximal transmit powers of 6mW, 7mW and 7mW for link 1, 2 and 3 respectively. Figure 9 shows the convergence of the dual objective function towards the globally optimal total throughput of the network. Figure 10 shows the convergence of the two auxiliary variables in link 1 and 3 towards the optimal solutions.

VI. Conclusions

Power control problems with nonlinear objective and constraints may seem to be difficult, NP-hard problems to solve for global optimality. However, when SIR is much larger than

0dB, GP can be used to turn these problems, with a variety of possible combinations of objective and constraint functions involving data rate, delay, and outage probability, into tractable, convex formulations. Then interior point algorithms can efficiently compute the globally optimal power allocation even for a large network. Feasibility analysis of GP naturally lead to admission control and pricing schemes. When the high SIR approximation cannot be made, these power control problems become SP and may be solved by the heuristic of condensation method through a series of GPs. Distributed optimal algorithms for GP-based power control in multihop networks can also be carried out through message passing of 'consistency prices'.

Several interesting research issues remain to be further explored: reduction of SP solution complexity by using high-SIR approximation to obtain the initial power vector and by solving the series of GPs only approximately (except the last GP), combination of SP solution and distributed algorithm for distributed power control in low SIR regime, inclusion of fading and mobility to the framework, and application to optimal spectrum management in DSL broadband access systems with interference-limited performance across the tones and among competing users sharing a cable binder.

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REFERENCES

[1] M. Avriel, Ed., *Advances in Geometric Programming*. New York: Plenum Press, 1980.

[2] N. Bambos, "Toward power-sensitive network architectures in wireless communications: Concepts, issues, and design aspects." *IEEE Personal Commun. Mag.*, vol. 5, no. 3, pp. 50-59, 1998.

[3] D. P. Bertsekas and R. G. Gallager, *Data Networks*. Prentice Hall, 1987.

[4] S. Boyd, S. J. Kim, L. Vandenberghe, and A. Hassibi, "A tutorial on geometric programming", to appear in *Optimization and Engineering*, 2005.

[5] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[6] M. Chiang, "Balancing transport and physical layers in wireless multihop networks: jointly optimal congestion control and power control," *IEEE J. Sel. Area Commun.,* vol. 23, no. 1, pp. 104-116, Jan. 2005.

[7] M. Chiang, "Geometric programming for communication systems," *Foundations and Trends of Communications and Information Theory*, vol. 2, no. 1-2, pp. 1-156, Aug. 2005.

[8] G. Foschini and Z. Miljanic, "A simple distributed autonomous power control algorithm and its convergence", *IEEE Trans. Veh. Tech.*, vol. 42, no. 4, 1993.

[9] A. Goldsmith, *Wireless Communications*. Cambridge University Press, 2004.

[10] D. Julian, M. Chiang, D. O'Neill, and S. Boyd, "QoS and fairness constrained convex optimization of resource allocation for wireless cellular and ad hoc networks," in *Proc. IEEE Infocom*, New York, NY, June 2002.

[11] S. Kandukuri and S. Boyd, "Optimal power control in interference limited fading wireless channels with outage probability specifications," *IEEE Trans. Wireless Commun.*, vol. 1, no. 1, pp. 46-55, Jan. 2002.

[12] B. R. Marks and G. P. Wright, "A general inner approximation algorithm for nonconvex mathematical programs," *Operations Research*, vol. 26, no. 4, pp. 681-683, 1978.

[13] D. Mitra, "An asynchronous distributed algorithm for power control in cellular radio systems," in *Proc. WINLAB Workshop* Rutgers University, NJ, 1993.

[14] Y. Nesterov and A. Nemirovsky, *Interior Point Polynomial Methods in Convex Programming*. SIAM Press, 1994.

[15] D. P. Palomar and M. Chiang, "Alternative decompositions for distributed maximization of network utility: framework and applications," *IEEE Trans. Automatic Control*, to appear.

[16] P. Hande, S. Rangan, and M. Chiang, "Distributed algorithms for optimal QoS assignment in wireless cellular data networks," in *Proc. IEEE Infocom*, Barcelona, Spain, April 2006.

[17] J. Papandriopoulos, S. Dey, and J. Evans, "Distributed cross-layer optimization of MANETs in composite fading," in *Proc. IEEE ICC*, Istanbul, Turkey, June 2006.

[18] J. Papandriopoulos and J. Evans, "Low-complexity distributed algorithms for spectrum balancing in multi-user DSL networks," in *Proc. IEEE ICC*, Istanbul, Turkey, June 2006.

[19] C. Saraydar, N. Mandayam, and D. Goodman, "Pricing and power control in a multicell wireless data network," *IEEE J. Sel. Areas Commun.*, vol. 19, no. 10, pp. 1883-1892, 2001.

[20] C. Sung and W. Wong, "Power control and rate management for wireless multimedia CDMA systems," *IEEE Trans. Commun.* vol. 49, no. 7, pp. 1215-1226, 2001.

[21] C. W. Tan, D. P. Palomar, and M. Chiang, "Solving non-convex power control problems in wireless networks: low SIR regime and distributed algorithms," in *Proc. IEEE Globecom*, St. Louis, MO, Nov. 2005.

[22] C. W. Tan, D. P. Palomar, and M. Chiang, "Distributed optimization of coupled systems with applications to network utility maximization," in *Proc. IEEE ICASSP 2006*, Toulouse, France, May 2006.

[23] R. Yates, "A framework for uplink power control in cellular radio systems," *IEEE J. Sel. Areas Commun.*, vol. 13, no. 7, pp. 1341-1347, 1995.

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