Ordered Eigenvalues of a General Class of Hermitian Random Matrices With Application to the Performance Analysis of MIMO Systems

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***Abstract*—In this paper, we present a general formulation that unifies the probabilistic characterization of Hermitian random matrices with a specific structure. Based on a general expression for the joint pdf of the ordered eigenvalues, we obtain i) the joint cdf; ii) the marginal cdfs; and iii) the marginal pdfs of the ordered eigenvalues, where ii) and iii) follow as simple particularizations of i). Our formulation is shown to include the distribution of some common MIMO channel models such as the uncorrelated, semicorrelated, and double-correlated Rayleigh MIMO fading channel and the uncorrelated Rician MIMO fading channel, although it is not restricted only to these. Hence, the proposed formulation and derived results provide a solid framework for the simultaneous analytical performance analysis of MIMO systems under different channel models. As an illustrative application, we obtain the exact outage probability of a spatial multiplexing MIMO system transmitting through the strongest channel eigenmodes.**

***Index Terms*—Channel eigenmodes, Hermitian random matrices, linear MIMO transceivers, ordered eigenvalues, Pseudo-Wishart distribution, Wishart distribution.**

I. Introduction

MULTIPLE-INPUT multiple-output (MIMO) channels are an abstract and general way to model many different communication systems of diverse physical nature; ranging from wireless multi-antenna channels [2]–[5], to wireline digital subscriber line (DSL) systems [6], and to single-antenna frequency-selective channels [7]. In particular, wireless MIMO channels have been recently attracting a great interest since they provide significant improvements in terms of spectral efficiency and reliability with respect to single-input single-output (SISO) channels [4], [5].

Assuming that the communication link has n_T transmit and n_R receive dimensions, the MIMO channel is mathematically described by an $n_\mathsf{R} \times n_\mathsf{T}$ channel matrix $\mathbf{H}$, whose (i,j)th entry characterizes the path between the jth transmit and the ith receive antenna. In particular, when communicating over MIMO fading channels, $\mathbf{H}$ is a random matrix that depends on the particular system architecture and the particular propagation conditions. Hence, $\mathbf{H}$ is assumed to be drawn from a certain probability distribution, which characterizes the system and scenario of interest and is known as channel model. The system behavior is then evaluated on the average or outage sense, taking into account all possible channel states.

The performance of a MIMO system is usually related to the eigenstructure of $\mathbf{H}$ (channel eigenmodes) or, more exactly, to the nonzero eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ (or $\mathbf{H}^\dagger\mathbf{H}$). Consequently, the probabilistic characterization of these eigenvalues for the adopted channel model is necessary in order to derive analytical expressions for the average and outage performance measures.

In MIMO wireless communications $\mathbf{H}$ is commonly modeled with Gaussian distributed entries, leading to the MIMO generalization of the well-known SISO Rayleigh or Rician fading channels (depending on whether the entries are zero mean or not). Some important particular cases of the MIMO Rayleigh and Rician channel models result in $\mathbf{H}\mathbf{H}^\dagger$ (or $\mathbf{H}^\dagger\mathbf{H}$) being a Wishart random matrix. The Wishart distribution and some closely related distributions have been widely studied during the sixties and seventies in the mathematical literature (see, e.g.,[1] [8]–[13]), due to its importance in various areas of research such as the analysis of time series [14] or nuclear physics [15], [16]. More recently, the statistical properties of the eigenvalues of Wishart matrices have been investigated and effectively applied to analyze the information theoretical limits of MIMO channels [4], [17]–[31] as well as the performance of practical MIMO systems [32]–[45]. Some other interesting cases entail the study of complex Pseudo-Wishart distributed matrices. However, this distribution and its eigenvalues have been only marginally considered in the MIMO literature [17], [19], [23], [30], [37], [38], [46]. This is also the case of the more general complex Quadratic form distributions [30], [47]–[53], which include the Wishart and Pseudo-Wishart distributions as particular cases.

Most of these works deal with the joint pdf of the ordered eigenvalues [17]–[24], [30], [39], [46], [49]–[52], the marginal distribution of an unordered eigenvalue [4], [25]–[29], or the distribution of the smallest eigenvalue [32], [45] to evaluate the

[1] We only include here some relevant references that focus on the *complex* Wishart distribution.

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system performance for the uninformed transmitter case. In contrast, when perfect channel state information is available at the transmitter, the weakest channel eigenmodes can be discarded, and the marginal statistics of the ordered eigenvalues become necessary. In this context, useful closed-form expressions for the distribution of the largest eigenvalue have been derived in [17], [33]–[38], [45], [54], and [55] to analyze the performance of the beamforming scheme (also referred as maximum-ratio transmission [56]). Nevertheless, an exhaustive analysis of the marginals of all ordered eigenvalues was still missing.[2] Some initial contributions in this direction are [40]–[42]. In particular, the first order Taylor expansion of the marginal pdfs of all the ordered eigenvalues was given in [40], [41] for the uncorrelated central Wishart distribution to characterize the high-SNR performance of the individual MIMO channel eigenmodes and of linear MIMO transceivers [57]. With the same purpose, [42] derived the exact marginal cdfs and the first order Taylor expansion of the marginal pdfs of the ordered eigenvalues for the uncorrelated noncentral Wishart distribution.

In this paper, we present a general formulation that unifies the probabilistic characterization of Hermitian random matrices with a specific structure. Based on a general expression for the joint pdf, we obtain i) the joint cdf; ii) the marginal cdfs; and iii) the marginal pdfs of the ordered eigenvalues, where ii) and iii) follow as simple particularizations of i). Then, in order to illustrate the utility of our unified approach, we particularize these results for uncorrelated and correlated central Wishart, correlated central Pseudo-Wishart, and uncorrelated noncentral Wishart matrices. To the best of the authors' knowledge, the joint cdf was unknown for all these distributions and the marginal cdfs and pdfs were only available for the uncorrelated central and noncentral Wishart distributions. Recently, other unified treatments have been proposed in [28] and [36], including, however, only uncorrelated and correlated central Wishart and uncorrelated noncentral Wishart matrices. Furthermore, only the distribution of the largest and the smallest eigenvalue was derived in [36] and the distribution of an unordered eigenvalue in both [28], [36]. Specifically, in the context of MIMO performance analysis, our results can be applied to investigate the spatial multiplexing system that results from transmitting independent substreams through the strongest eigenmodes when perfect channel state information is available at both sides of the link (also termed as MIMO SVD systems). The motivation behind the analysis of this particular communication scheme is that it was proven to be optimal in the design of linear MIMO transceivers under a wide rage of different optimization criteria [57]. The most common performance measures such as the average BER or outage probability of the established substreams have been (partially or approximately) obtained in [40]–[44] under an specific channel model. Here, we exploit the proposed unified formulation and results to characterize the exact outage probability of the channel eigenmodes and the global outage probability of the system for different channel models simultaneously.

[2]Simultaneously to the publication of this work in [2], the marginal cdfs of all the ordered eigenvalues were obtained in [45] following the unified approach by the same authors in [36] that includes uncorrelated and correlated central Wishart and uncorrelated noncentral Wishart matrices.

The rest of the paper is outlined as follows. Section II is devoted to introducing the Rayleigh and Rician MIMO channel models and the corresponding joint pdfs of its ordered eigenvalues. Section III contains the main contribution of this paper, i.e., the derivations of the joint cdf and both the marginal cdfs and pdfs of the ordered eigenvalues of a general class of Hermitian random matrices. In Section IV we establish the matching between this class and the Rayleigh and Rician MIMO channel models in Section II. More exactly, we provide the expressions for the parameters describing the general joint pdf of the eigenvalues considered in Section III as well as the expressions needed to particularize the derived distributions. As a straightforward example of application of the proposed characterization, the exact outage performance of a MIMO spatial multiplexing scheme is obtained in Section V. The paper is finally summarized in Section VI.

II. MIMO Channel Model

In this section, we introduce the Rayleigh and Rician flat-fading[3] MIMO channel models used in the analytical derivations and performance analysis of Sections IV and V.

A. Rayleigh and Rician MIMO Channels

Recall from the introduction that a MIMO channel with n_{T} transmit and n_{R} receive dimensions can be modeled as an $n_{\mathsf{R}} \times n_{\mathsf{T}}$ random matrix $\mathbf{H}$. Usually, since there are a large number of scatters in the channel that contribute to the signal at the receiver, application of the central limit theorem results in $\mathbf{H}$ having zero-mean Gaussian distributed coefficients. Analogously to the single antenna case, this model is referred to as Rayleigh MIMO fading channel [5].

In realistic environments, the SISO channels connecting each pair of transmit and receive antenna elements are not independent due, for instance, to insufficient spacing between antenna elements or insufficient scattering. In such cases, a convenient approach is to construct a correlation model that can provide a reasonable description of the propagation environment and physical setup for the wireless application of interest (see [59] for a review on MIMO channel models). The most common correlation model assumes that antenna correlation at the transmitter side and at the receiver side are caused by independent phenomena and is known as Kronecker model [60]–[64]. Thus, correlation can be separated and the correlated MIMO Rayleigh channel can be modeled as

$$\mathbf{H}^{(\mathrm{Rayleigh})} = \mathbf{\Sigma}_{\mathsf{R}}^{1/2}\mathbf{H}_{\mathsf{w}}\mathbf{\Sigma}_{\mathsf{T}}^{1/2} \quad (1)$$

where $\mathbf{\Sigma}_{\mathsf{T}} = \left(\mathbf{\Sigma}_{\mathsf{T}}^{1/2}\right)\left(\mathbf{\Sigma}_{\mathsf{T}}^{1/2}\right)^{\dagger}$ is the transmit correlation matrix, $\mathbf{\Sigma}_{\mathsf{R}} = \left(\mathbf{\Sigma}_{\mathsf{R}}^{1/2}\right)\left(\mathbf{\Sigma}_{\mathsf{R}}^{1/2}\right)^{\dagger}$ is the receive correlation matrix, and $\mathbf{H}_{\mathsf{w}}$ is the random channel matrix with i.i.d. zero-mean unit-variance circulary symmetric Gaussian entries, i.e., $[\mathbf{H}_{\mathsf{w}}]_{ij} \sim \mathcal{CN}(0,1)$. Although this simple correlation model is not completely general (see, e.g., [65] and [66] for environments where it does not apply), it has been validated experimentally in [67]–[69] as well

[3]Observe that in wideband MIMO systems a multicarrier approach is usually applied and the flat-fading assumption holds then for the channel seen by each subcarrier (cf. [58]). Henceforth, we use the term "fading" instead of "flat-fading," although a flat-fading channel is implicitly considered.

as using ray-tracing simulations in [62]. Hence, it is widely accepted as an accurate representation of the fade correlation seen in actual cellular systems.

In addition, for scenarios where a line-of-sight or specular component is present, the channel matrix is modeled as having nonzero mean [63], [64]

$$\mathbf{H}^{(\mathrm{Ricean})} = \sqrt{\frac{K_{\mathsf{c}}}{K_{\mathsf{c}}+1}}\overline{\mathbf{H}} + \sqrt{\frac{1}{K_{\mathsf{c}}+1}}\mathbf{\Sigma}_{\mathsf{R}}^{1/2}\mathbf{H}_{\mathsf{w}}\mathbf{\Sigma}_{\mathsf{T}}^{1/2} \tag{2}$$

where $K_{\mathsf{c}} \in [0,\infty)$ is power normalization factor known as the Rician K_{c}-factor and $\overline{\mathbf{H}}$ is a deterministic $n_{\mathsf{R}} \times n_{\mathsf{T}}$ matrix containing the line-of-sight components of the channel. Analogously to the single antenna case, this model is referred to as MIMO Ricean fading channel [70].

Observe that the MIMO Ricean fading model in (2) includes channels ranging from a fully random Rayleigh channel when $K_{\mathsf{c}} = 0$ to a fully deterministic channel when $K_{\mathsf{c}} \to \infty$. For a fair comparison of the different cases, the total average received power is assumed to be constant, i.e.,

$$\begin{aligned}\mathrm{E}\left\{\|\mathbf{H}\|^2\right\} &= \frac{K_{\mathsf{c}}}{K_{\mathsf{c}}+1}\mathrm{tr}\left(\overline{\mathbf{H}}\overline{\mathbf{H}}^{\dagger}\right) \\ &\quad + \frac{1}{K_{\mathsf{c}}+1}\mathrm{E}\left\{\mathrm{tr}\left(\mathbf{\Sigma}_{\mathsf{R}}\mathbf{H}_{\mathsf{w}}\mathbf{\Sigma}_{\mathsf{T}}\mathbf{H}_{\mathsf{w}}^{\dagger}\right)\right\} &(3)\\ &= \frac{K_{\mathsf{c}}}{K_{\mathsf{c}}+1}\mathrm{tr}\left(\overline{\mathbf{H}}\overline{\mathbf{H}}^{\dagger}\right) \\ &\quad + \frac{1}{K_{\mathsf{c}}+1}\mathrm{tr}\left(\mathbf{\Sigma}_{\mathsf{R}}\right)\mathrm{tr}\left(\mathbf{\Sigma}_{\mathsf{T}}\right) = n_{\mathsf{R}}n_{\mathsf{T}} &(4)\end{aligned}$$

and, hence, we can impose without loss of generality that

$$\mathrm{tr}\left(\mathbf{\Sigma}_{\mathsf{T}}\right) = n_{\mathsf{T}}, \quad \mathrm{tr}\left(\mathbf{\Sigma}_{\mathsf{R}}\right) = n_{\mathsf{R}} \tag{5a}$$

and

$$\mathrm{tr}\left(\overline{\mathbf{H}}\overline{\mathbf{H}}^{\dagger}\right) = n_{\mathsf{R}}n_{\mathsf{T}} \tag{5b}$$

B. Particular Cases of Rayleigh and Rician MIMO Channels

We now present the different distributions of $\mathbf{H}\mathbf{H}^{\dagger}$ or $\mathbf{H}^{\dagger}\mathbf{H}$ and the joint pdf of its ordered eigenvalues which result when $\mathbf{H}$ follows some important particular cases of the Rayleigh and Rician MIMO channel models introduced in the previous section (see physical justifications of these channels in [71]).

1) Uncorrelated Rayleigh Fading MIMO Channel:

Definition 2.1: The uncorrelated Rayleigh MIMO fading channel model is defined as

$$\mathbf{H} = \mathbf{H}_{\mathsf{w}} \tag{6}$$

where $\mathbf{H}_{\mathsf{w}}$ is an $n_{\mathsf{R}} \times n_{\mathsf{T}}$ random channel matrix with i.i.d. zero-mean unit-variance complex Gaussian entries.

Consider an uncorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2.1, then the random Hermitian matrix $\mathbf{W}$ $(n \times n)$ defined as

$$\mathbf{W} = \begin{cases}\mathbf{H}\mathbf{H}^{\dagger} & n_{\mathsf{R}} \le n_{\mathsf{T}} \\ \mathbf{H}^{\dagger}\mathbf{H} & n_{\mathsf{R}} > n_{\mathsf{T}}\end{cases} \tag{7}$$

follows a complex uncorrelated central Wishart distribution [8]–[10], [13], denoted as $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$, where $n = \min(n_{\mathsf{T}}, n_{\mathsf{R}})$ and $m = \max(n_{\mathsf{T}}, n_{\mathsf{R}})$. The joint pdf of the ordered eigenvalues, $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$ is given by [9, eq. (95)], [10, eq. (7.1.7)], [18, eq. (10)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n}\frac{1}{(m-i)!(n-i)!}|\mathbf{V}(\boldsymbol{\lambda})|^2\prod_{i=1}^{n}e^{-\lambda_i}\lambda_i^{m-n} \tag{8}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix A.1).

2) Min-Semicorrelated Rayleigh Fading MIMO Channel:

Definition 2.2: The semicorrelated Rayleigh fading MIMO channel model with correlation at the side with minimum number of antennas is defined as

$$\mathbf{H} = \begin{cases}\mathbf{\Sigma}^{1/2}\mathbf{H}_{\mathsf{w}} & n_{\mathsf{R}} \le n_{\mathsf{T}} \\ \mathbf{H}_{\mathsf{w}}\mathbf{\Sigma}^{1/2} & n_{\mathsf{R}} > n_{\mathsf{T}}\end{cases} \tag{9}$$

where $\mathbf{\Sigma} = \left(\mathbf{\Sigma}^{1/2}\right)\left(\mathbf{\Sigma}^{1/2}\right)^{\dagger}$ is the $n \times n$ positive definite correlation matrix with $n = \min(n_{\mathsf{T}}, n_{\mathsf{R}})$ and $\mathbf{H}_{\mathsf{w}}$ is an $n_{\mathsf{R}} \times n_{\mathsf{T}}$ random channel matrix with i.i.d. zero-mean unit-variance complex Gaussian entries.

Consider a min-semicorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2.2, then the random Hermitian matrix $\mathbf{W}$ $(n \times n)$ in (7) follows a complex correlated central Wishart distribution [9], [10], [13], denoted as $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$, where $n = \min(n_{\mathsf{T}}, n_{\mathsf{R}})$, $m = \max(n_{\mathsf{T}}, n_{\mathsf{R}})$, and $\mathbf{\Sigma}$ is the $n \times n$ positive definite correlation matrix. The joint pdf of the ordered eigenvalues, $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$ is given by [9, eq. (95)], [18, eq. (17)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n}\frac{1}{\sigma_i^m(m-i)!}\prod_{i<j}^{n}\frac{\sigma_i\sigma_j}{\sigma_j-\sigma_i}|\mathbf{E}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})|\prod_{i=1}^{n}\lambda_i^{m-n} \tag{10}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix A.1) and $\mathbf{E}(\boldsymbol{\lambda})$ is defined as

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = e^{-\lambda_v/\sigma_u} \qquad \text{for } u,v = 1,\ldots,n \tag{11}$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ are the eigenvalues of $\mathbf{\Sigma}$ ordered such that[4] $(\sigma_1 > \cdots > \sigma_n > 0)$.

3) Max-Semicorrelated Rayleigh Fading MIMO Channel:

Definition 2.3: The semicorrelated Rayleigh fading MIMO channel model with correlation at the side with maximum number of antennas is defined as

$$\mathbf{H} = \begin{cases}\mathbf{H}_{\mathsf{w}}\mathbf{\Sigma}^{1/2} & n_{\mathsf{R}} \le n_{\mathsf{T}} \\ \mathbf{\Sigma}^{1/2}\mathbf{H}_{\mathsf{w}} & n_{\mathsf{R}} > n_{\mathsf{T}}\end{cases} \tag{12}$$

where $\mathbf{\Sigma} = \left(\mathbf{\Sigma}^{1/2}\right)\left(\mathbf{\Sigma}^{1/2}\right)^{\dagger}$ is the $m \times m$ positive definite correlation matrix with $m = \max(n_{\mathsf{T}}, n_{\mathsf{R}})$ and $\mathbf{H}_{\mathsf{w}}$ is an $n_{\mathsf{R}} \times n_{\mathsf{T}}$

[4]If some σ_i's are equal the result is obtained by taking the limiting case of (10) (see [30, Sec. IV]). However, the most common practical scenarios which give rise to eigenvalue multiplicities are when all the eigenvalues are the same, i.e., $\mathbf{\Sigma} = \mathbf{I}_n$[53], and, hence, $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$ as in Section II-B-1). This observation also holds for the max-semicorrelated Rayleigh fading MIMO channel in Section II-B-3).

random channel matrix with i.i.d. zero-mean unit-variance complex Gaussian entries.

Consider a max-semicorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2.3, then the random Hermitian matrix $\mathbf{W}$ $(m \times m)$ defined as

$$\mathbf{W} = \begin{cases} \mathbf{H}^\dagger\mathbf{H} & n_\mathsf{R} \leq n_\mathsf{T} \\ \mathbf{H}\mathbf{H}^\dagger & n_\mathsf{R} > n_\mathsf{T} \end{cases} \tag{13}$$

is not full-rank and follows a complex correlated central Pseudo-Wishart distribution[5] [46], [73], [74], denoted as $\mathbf{W} \sim \mathcal{PW}_m(n, \mathbf{0}_m, \mathbf{\Sigma})$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$, $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and $\mathbf{\Sigma}$ is the $m \times m$ positive definite correlation matrix. The joint pdf of the ordered nonzero eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$,[6] of $\mathbf{W} \sim \mathcal{PW}_m(n, \mathbf{0}_m, \mathbf{\Sigma})$ is given by [19, eq. (25)], [30, eq. (43)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \frac{1}{(n-i)!} \prod_{i<j}^{m} \frac{1}{(\sigma_j - \sigma_i)} |\mathbf{E}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \tag{14}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix A.1) and $\mathbf{E}(\boldsymbol{\lambda})$ is defined as

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = \begin{cases} \sigma_u^{v-1} & 1 \leq v \leq m-n \\ \sigma_u^{m-n-1} e^{-\lambda_{v-m+n}/\sigma_u} & m-n < v \leq m \end{cases} \tag{15}$$

for $u, v = 1, \ldots, m$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$ are the eigenvalues of $\mathbf{\Sigma}$ ordered such that $(\sigma_1 > \cdots > \sigma_m > 0)$. Performing the Laplace expansion (see, e.g., [75, Sec. 33]) over the first $m-n$ columns of $\mathbf{E}(\boldsymbol{\lambda})$, it follows that

$$|\mathbf{E}(\boldsymbol{\lambda})| = \sum_{\boldsymbol{\iota} \in \mathcal{I}} (-1)^{\sum_{i=1}^{m-n}(\iota_i + i)} |\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})||\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})| \tag{16}$$

where the summation over $\boldsymbol{\iota} = (\iota_1, \ldots, \iota_m)$ is for all permutation of integers $(1, \ldots, m)$ such that $(\iota_1 < \cdots < \iota_{m-n})$ and $(\iota_{m-n+1} < \cdots < \iota_m)$ and the matrices $\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})$ $((m-n) \times (m-n))$ and $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ $(n \times n)$ are defined as

$$[\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})]_{u,v} = \sigma_{\iota_u}^{v-1} \quad \text{for } u, v = 1, \ldots, m-n \tag{17}$$

$$[\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) = \sigma_{\iota_{m-n+u}}^{m-n-1} e^{-\lambda_v/\sigma_{\iota_{m-n+u}}} \quad \text{for } u, v = 1, \ldots, n. \tag{18}$$

Observing that $\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})$ is a Vandermonde matrix (see Appendix A.1), we can finally rewrite the joint pdf as

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \frac{1}{(n-i)!} \prod_{i<j}^{m} \frac{1}{(\sigma_j - \sigma_i)} \times \sum_{\boldsymbol{\iota} \in \mathcal{I}} (-1)^{\sum_{i=1}^{m-n}(\iota_i + i)} \prod_{i<j}^{m-n} (\sigma_{\iota_j} - \sigma_{\iota_i}) |\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})|. \tag{19}$$

[5]The Pseudo-Wishart distribution is also referred to as singular Wishart [72] or anti-Wishart [73] distribution.

[6]There will be also $m-n$ additional zero eigenvalues.

4) *Uncorrelated Rician Fading MIMO Channel:*

Definition 2.4: The uncorrelated Ricean fading MIMO channel model is defined as

$$\mathbf{H} = \sqrt{\frac{K_\mathsf{c}}{K_\mathsf{c}+1}}\overline{\mathbf{H}} + \sqrt{\frac{1}{K_\mathsf{c}+1}}\mathbf{H}_\mathsf{w} \tag{20}$$

where $K_\mathsf{c} \in (0, \infty)$, $\overline{\mathbf{H}}$ is an $n_\mathsf{R} \times n_\mathsf{T}$ deterministic matrix, and $\mathbf{H}_\mathsf{w}$ is an $n_\mathsf{R} \times n_\mathsf{T}$ random channel matrix with i.i.d. zero-mean unit-variance complex Gaussian entries.

Consider a uncorrelated Rician fading MIMO channel $\mathbf{H}$ as given in Definition 2.4, then the random Hermitian matrix $\widetilde{\mathbf{W}} = (K_\mathsf{c}+1)\mathbf{W}$ $(n \times n)$, where $\mathbf{W}$ is given in (7), follows a complex uncorrelated noncentral Wishart distribution [9], [10], [13], denoted as $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$ and $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and the non-centrality parameter $\mathbf{\Omega}$ is defined as

$$\mathbf{\Omega} = \begin{cases} K_\mathsf{c}\overline{\mathbf{H}}\overline{\mathbf{H}}^\dagger & n_\mathsf{R} \leq n_\mathsf{T} \\ K_\mathsf{c}\overline{\mathbf{H}}^\dagger\overline{\mathbf{H}} & n_\mathsf{R} > n_\mathsf{T}. \end{cases} \tag{21}$$

Note that in this case the nonzero channel eigenvalues, i.e., the eigenvalues of $\mathbf{W}$, are a scaled version of the eigenvalues of the complex uncorrelated central Wishart distributed matrix $\widetilde{\mathbf{W}}$. The joint pdf of the ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ of $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$ is given by [9, eq. (102)], [34, eq. (45)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \frac{e^{-\sum_{i=1}^{n}\omega_i}}{((m-n)!)^n} \prod_{i<j}^{n} \frac{1}{(\omega_j - \omega_i)} \times |\mathbf{E}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \prod_{i=1}^{n} e^{-\lambda_i} \lambda_i^{m-n} \tag{22}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix A.1) and $\mathbf{E}(\boldsymbol{\lambda})$ is defined as

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = {}_0F_1(m-n+1; \omega_u\lambda_v) \text{ for } u, v = 1, \ldots, n \tag{23}$$

where ${}_0F_1(\cdot;\cdot)$ is a generalized hypergeometric function (see [76, eq. (9.14.1)]) and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ are the eigenvalues of $\mathbf{\Omega}$ ordered such that[7] $(\omega_1 > \cdots > \omega_n > 0)$.

III. Ordered Eigenvalues of a General Class of Random Matrices

Observe that the joint pdf of the ordered eigenvalues of the distributions associated with the channel models presented in Section II-B follow a very similar structure [cf. (8), (10), (19), and (22)]. This enables to perform the probabilistic characterization of the ordered eigenvalues simultaneously for these distributions by assuming this particular structure for the joint pdf of the ordered eigenvalues in all derivations. Indeed, in this section we derive the joint cdf and both the marginal cdfs and pdfs of the ordered eigenvalues of a general class of Hermitian random matrices (formalized next in Assumption 3.1). This general class is shown in Section IV to include $\mathbf{H}\mathbf{H}^\dagger$ or $\mathbf{H}^\dagger\mathbf{H}$ when $\mathbf{H}$ is drawn from the particular cases of the general Rayleigh and Rician

[7]If some ω_i's are equal or zero the result is obtained by taking the limiting case of (22) (see [34, App. B]).

MIMO channel models introduced in Section II-B. Hence, the results obtained in this section can be applied to analyze the performance of MIMO systems as we will illustrate in Section V.

Assumption 3.1: We consider the class of Hermitian random matrices, for which the joint pdf of its n nonzero ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ can be expressed as

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = f_{\boldsymbol{\lambda}}(\lambda_1, \ldots, \lambda_n) = \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} |\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \prod_{t=1}^{n} \varphi(\lambda_t) \tag{24}$$

where $\boldsymbol{\iota}$ is a vector of indices and the summation is for all vectors $\boldsymbol{\iota}$ in the set $\mathcal{I}$, $\mathbf{V}(\boldsymbol{\lambda})$ $(n \times n)$ is a Vandermonde matrix (see Appendix A.1) and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ $(n \times n)$ satisfies

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) \qquad \text{for } u, v = 1, \ldots, n. \tag{25}$$

The dimension of $\boldsymbol{\iota}$, the set $\mathcal{I}$, the constant $K^{(\boldsymbol{\iota})}$, and the functions $\zeta_u^{(\boldsymbol{\iota})}(\lambda)$ and $\varphi(\lambda)$ depend on the particular distribution of the random matrix.

The adoption of Assumption 3.1 for the joint pdf of the eigenvalues is not only supported by the identification of the common structure for the distributions in Section II-B. In addition, it can be also motivated by investigating the form of the most common Hermitian matrix distributions as done in Appendix B.1.

A. Joint CDF of the Ordered Eigenvalues

This section presents the main contribution of this paper, since all the other results follow as straightforward particularizations of the next theorem.

Theorem 3.1: The joint cdf of the ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ of a random Hermitian matrix satisfying Assumption 3.1 is given by

$$F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \Pr(\lambda_1 \leq \eta_1, \ldots, \lambda_n \leq \eta_n) = \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\boldsymbol{i} \in \mathcal{S}} \frac{1}{\tau(\boldsymbol{i})} \mathcal{T}\{\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i}; \boldsymbol{\eta})\} \tag{26}$$

where $(\eta_1 \geq \cdots \geq \eta_n > 0)$,[8] the summation over $\boldsymbol{i} = (i_1, \ldots, i_n)$ is for all $\boldsymbol{i}$ in the set $\mathcal{S}$ defined as[9]

$$\mathcal{S} = \{\boldsymbol{i} \in \mathbb{N}^n | \max(i_{s-1}, s) \leq i_s \leq n,\ i_s \neq r \text{ if } \eta_r = \eta_{r+1}\} \tag{27}$$

and

$$\tau(\boldsymbol{i}) = \prod_{u=1}^{n} \left((1 - \delta_{i_u, i_{u+1}}) \sum_{v=1}^{u} \delta_{i_u, i_v} \right)! \tag{28}$$

where $\delta_{u,v}$ denotes the Kronecker delta. The operator $\mathcal{T}\{\cdot\}$ is defined in Appendix A.1 and the tensor $\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i}; \boldsymbol{\eta})$ $(n \times n \times n)$ is defined as

$$[\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i}; \boldsymbol{\eta})]_{u,v,t} = \int_{\eta_{i_t+1}}^{\eta_{i_t}} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda \tag{29}$$

for $u, v, t = 1, \ldots, n$, where $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$ (see Assumption 3.1).

Proof: See Appendix B.2.

B. Marginal CDF and PDF of the kth Largest Ordered Eigenvalue

In this section we particularize the joint cdf of the ordered eigenvalues given in Theorem 3.1 to derive the marginal cdf and the marginal pdf of the kth largest eigenvalue.

Theorem 3.2: The marginal cdf of the kth largest eigenvalue, λ_k, of a random Hermitian matrix satisfying Assumption 3.1 is given by

$$F_{\lambda_k}(\eta) = \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{i=1}^{k} \sum_{\boldsymbol{\mu} \in \mathcal{P}(i)} |\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)| \tag{30}$$

where $\mathcal{P}(i)$ is the set of all permutations $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ of the integers $(1, \ldots, n)$ such that $(\mu_1 < \cdots < \mu_{i-1})$ and $(\mu_i < \cdots < \mu_n)$, matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)$ $(n \times n)$ is defined as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)]_{u,v} = \begin{cases} \int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda & 1 \leq \mu_v < i \\ \int_{0}^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda & i \leq \mu_v \leq n \end{cases} \tag{31}$$

for $u, v = 1, \ldots, n$, and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$ (see Assumption 3.1).

Proof: See Appendix B.3.

In the following, we particularize Theorem 3.2 to obtain a simplified expression for the marginal cdf of the largest and smallest eigenvalues.

Corollary 3.1: The marginal cdf of the largest eigenvalue, λ_1, of a random Hermitian matrix satisfying Assumption 3.1 is given by

$$F_{\lambda_1}(\eta) = \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} |\mathbf{F}^{(\boldsymbol{\iota})}(\eta)| \tag{32}$$

where matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\eta)$ $(n \times n)$ is defined as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\eta)]_{u,v} = \int_{0}^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda \qquad \text{for } u, v = 1, \ldots, n \tag{33}$$

and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$ (see Assumption 3.1).

Proof: See Appendix B.4.

Corollary 3.2: The marginal cdf of the smallest nonzero eigenvalue, λ_n, of a random Hermitian matrix satisfying Assumption 3.1 is given by

$$F_{\lambda_n}(\eta) = 1 - \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} |\mathbf{F}^{(\boldsymbol{\iota})}(\eta)| \tag{34}$$

where matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\eta)$ $(n \times n)$ is defined as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\eta)]_{u,v} = \int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda \qquad \text{for } u, v = 1, \ldots, n \tag{35}$$

and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$ (see Assumption 3.1).

Proof: See Appendix B.5.

[8] If $\eta_{k-1} < \eta_k$ then $F_{\boldsymbol{\lambda}}(\eta_1, \ldots, \eta_{k-1}, \eta_k, \ldots, \eta_n) = F_{\boldsymbol{\lambda}}(\eta_1, \ldots, \eta_{k-1}, \eta_{k-1}, \ldots, \eta_n)$ and if some $\eta_k = 0$, then $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = 0$.

[9] Note that $i_n = n$ and by definition $i_0 = 0$, $i_{n+1} = n + 1$ and $\eta_{n+1} = 0$.

Similarly, the marginal pdf of the kth largest eigenvalue can be easily derived from Theorem 3.2 as we illustrate in the following corollary.

Corollary 3.3: The marginal pdf of the kth largest eigenvalue, λ_k, of a random Hermitian matrix satisfying Assumption 3.1 is given by

$$f_{\lambda_k}(\eta) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{i=1}^{k} \sum_{\boldsymbol{\mu}\in\mathcal{P}(i)} \sum_{t=1}^{n} |\mathbf{D}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i, t; \eta)| \quad (36)$$

where $\mathcal{P}(i)$ is the set of all permutations $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ of the integers $(1, \ldots, n)$ such that $(\mu_1 < \cdots < \mu_{i-1})$ and $(\mu_i < \cdots < \mu_n)$, matrix $\mathbf{D}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i, t; \eta)$ $(n \times n)$ is defined as

$$[\mathbf{D}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i, t; \eta)]_{u,v} = \begin{cases} \int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda & 1 \le \mu_v < i, v \ne t \\ -\xi_{u,v}^{(\boldsymbol{\iota})}(\eta) & 1 \le \mu_v < i, v = t \\ \int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda & i \le \mu_v \le n, v \ne t \\ \xi_{u,v}^{(\boldsymbol{\iota})}(\eta) & i \le \mu_v \le n, v = t \end{cases} \quad (37)$$

for $u, v = 1, \ldots, n$, and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$ (see Assumption 3.1).

Proof: See Appendix B.6.

Observe that the simplifications used in the proofs of Corollaries 3.1 and 3.2 can be straightforwardly applied to obtain simple expressions for the marginal pdf of the largest and smallest eigenvalues using Corollary 3.3.

Remark 3.1: This unified approach allows also the derivation of the marginal cdf (the marginal pdf could be analogously obtained) of an unordered eigenvalue λ chosen from the set of the $1 \le \kappa \le n$ largest eigenvalues, $\{\lambda_1, \ldots, \lambda_\kappa\}$, observing that

$$F_\lambda(\eta) = \frac{1}{\kappa} \sum_{k=1}^{\kappa} F_{\lambda_k}(\eta). \quad (38)$$

This expression is more general than the marginal cdf of an unordered eigenvalue chosen from the set of all nonzero ordered eigenvalues, $\{\lambda_1, \ldots, \lambda_n\}$, available in the literature (see, e.g., [28], [36]), which can be obtained by setting $\kappa = n$ in (38).

IV. Ordered Eigenvalues of Rayleigh and Rician MIMO Channels

In this section, we provide the expressions for the parameters describing the general joint pdf of the eigenvalues in Assumption 3.1, as well as the expressions needed to particularize the distributions given in Section III when $\mathbf{H}$ follows the Rayleigh and Rician fading MIMO channel models described in Section II-B. Although we restrict here to simple cases of the Rayleigh and Rician fading MIMO channels (for which closed-form expressions can be obtained), other interesting cases are also included in our unified formulation, for instance, the double-correlated Rayleigh fading MIMO channel considered in [53], since the joint pdf of the ordered eigenvalues in [53, eq. (57)] can be expressed as the joint pdf in Assumption 3.1 (see Appendix C).

A. Uncorrelated Rayleigh Fading MIMO Channel

Consider an uncorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2.1, then the random Hermitian matrix $\mathbf{W}$ $(n \times n)$ in (7) follows a complex uncorrelated central Wishart distribution, i.e., $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$ and $m = \max(n_\mathsf{T}, n_\mathsf{R})$. Since the nonzero eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ and $\mathbf{H}^\dagger\mathbf{H}$ coincide, we can derive without loss of generality the statistical properties of the nonzero channel eigenvalues by analyzing the eigenvalues of $\mathbf{W}$.

Joint pdf: Identifying terms, the joint pdf of the ordered eigenvalues of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$ in (8) coincides with the general pdf given in Assumption 3.1 if we let $\mathcal{I}$ be a singleton (the superindex $(\boldsymbol{\iota})$ can then be dropped), define $K^{(\boldsymbol{\iota})}$ as

$$K^{(\boldsymbol{\iota})} = \prod_{i=1}^{n} \frac{1}{(m-i)!(n-i)!} \quad (39)$$

the function $\varphi(\lambda)$ as

$$\varphi(\lambda) = e^{-\lambda}\lambda^{m-n} \quad (40)$$

and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ $(n \times n)$, equal to $\mathbf{V}(\boldsymbol{\lambda})$, with entries given by

$$[\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) = \lambda_v^{u-1} \text{ for } u, v = 1, \ldots, n. \quad (41)$$

Hence, it follows that $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)$ is

$$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1} = e^{-\lambda}\lambda^{d(u+v-1)} \quad (42)$$

where we have introduced the function $d(v) = m - n + v - 1$.

Marginal distributions: In order to derive the marginal cdf and pdf of the kth largest eigenvalue using the results presented in Section III-B, we only have to particularize

$$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \Gamma(d(u+v), \eta) \quad (43)$$

$$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \gamma(d(u+v), \eta) \quad (44)$$

where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the upper and lower incomplete gamma functions defined in Appendix A.2.

The marginal cdf of the largest eigenvalue of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$ was initially derived in [73, Thm. 2] and extended to the marginal cdf and pdf of the kth largest eigenvalue in [78, eq. (16)]. Recently, the cdfs of the largest and smallest eigenvalue were obtained in [33, eq. (18)], [34, Cor. 2], [35, eq. (6)], [54, Thm. 5], [36, eq. (5)] and [32, eq. (38)], respectively. In addition, the pdfs of the largest and smallest eigenvalue were provided in [33, eq. (22)], [34, Cor. 3], [35, eq. (7)], [36, eq. (23)] and [36, eq. (24)], respectively.

B. Min-Semicorrelated Rayleigh Fading MIMO Channel

Consider a min-semicorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2.2, then the random Hermitian matrix $\mathbf{W}$ $(n \times n)$ in (7) follows a complex correlated central Wishart distribution, i.e., $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$, $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and $\mathbf{\Sigma}$ is the

$n \times n$ positive definite correlation matrix with eigenvalues $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ ordered such that $(\sigma_1 > \cdots \sigma_n > 0)$.

Joint pdf: Identifying terms, the joint pdf of the ordered eigenvalues of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$ in (10) coincides with the general pdf given in Assumption 3.1 if we let $\mathcal{I}$ be a singleton (the superindex $(\boldsymbol{\iota})$ can then be dropped), define $K^{(\boldsymbol{\iota})}$ as

$$K^{(\boldsymbol{\iota})} = \prod_{i=1}^{n} \frac{1}{\sigma_i^m (m-i)!} \prod_{i<j}^{n} \frac{\sigma_i \sigma_j}{\sigma_j - \sigma_i} \tag{45}$$

the function $\varphi(\lambda)$ as

$$\varphi(\lambda) = \lambda^{m-n} \tag{46}$$

and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ with entries as given in (11), i.e.

$$[\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) = e^{-\lambda_v/\sigma_u} \text{ for } u,v = 1,\ldots,n. \tag{47}$$

Hence, it follows that

$$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1} = e^{-\lambda/\sigma_u}\lambda^{d(v)} \tag{48}$$

where we have introduced the function $d(v) = m - n + v - 1$.

Marginal distributions: In order to derive the marginal cdf and pdf of the kth largest eigenvalue using the results presented in Section III-B, we only have to particularize

$$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_u^{d(v+1)} \Gamma\left(d(v+1), \frac{\eta}{\sigma_u}\right) \tag{49}$$

$$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_u^{d(v+1)} \gamma\left(d(v+1), \frac{\eta}{\sigma_u}\right) \tag{50}$$

where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the upper and lower incomplete gamma functions defined in Appendix A.2.

The marginal cdfs of the largest and smallest eigenvalue of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$ were recently derived in [17, Thm. 4 (1)], [36, eq. (7)], [37, eq. (9)], [38, eq. (18)] and in [36, eq. (9)], [37, eq. (13)], [38, eq. (22)], respectively. The corresponding marginal pdfs were obtained in [36, eq. (26)], [37, eq. (17)], [38, eq. (24)] and in [37, eq. (18)], [38, eq. (25)]. To the best of authors' knowledge, the marginal cdf and pdf of the kth largest eigenvalue were not available in the literature.

C. Max-Semicorrelated Rayleigh Fading MIMO Channel

Consider a max-semicorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2.3, then the random Hermitian matrix $\mathbf{W}$ $(m \times m)$ in (13) follows a complex correlated central Pseudo-Wishart distribution, i.e., $\mathbf{W} \sim \mathcal{PW}_m(n, \mathbf{0}_m, \mathbf{\Sigma})$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$, $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and $\mathbf{\Sigma}$ is the $m \times m$ positive definite correlation matrix with eigenvalues $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$ ordered such that $(\sigma_1 > \cdots \sigma_m > 0)$.

Joint pdf: Identifying terms, the joint pdf of the ordered nonzero eigenvalues in (19) coincides with the general pdf given in Assumption 3.1 by defining the set $\mathcal{I}$ as

$$\mathcal{I} = \{(\iota_1, \ldots, \iota_m) = \pi(1, \ldots, m) | (\iota_1 < \cdots < \iota_{m-n}) \text{ and } (\iota_{m-n+1} < \cdots < \iota_n)\} \tag{51}$$

where $\pi(\cdot)$ denotes permutation, the constant $K^{(\boldsymbol{\iota})}$ as

$$K^{(\boldsymbol{\iota})} = \frac{(-1)^{\sum_{i=1}^{m-n}(\iota_i + i)}}{\prod_{i=1}^{n}(n-i)!} \frac{\prod_{i<j}^{m-n}(\sigma_{\iota_j} - \sigma_{\iota_i})}{\prod_{i<j}^{m}(\sigma_j - \sigma_i)} \tag{52}$$

the function $\varphi(\lambda) = 1$, and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ with entries as given in (18). Hence, it follows that

$$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \sigma_{\iota_{d(u+1)}}^{m-n-1} e^{-\lambda/\sigma_{\iota_{d(u+1)}}} \lambda^{v-1} \tag{53}$$

where we have introduced the function $d(u) = m - n + u - 1$.

Marginal distributions: In order to derive the marginal cdf and pdf of the kth largest eigenvalue using the results presented in Section III-B, we only have to particularize

$$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_{\iota_{d(u+1)}}^{d(v)} \Gamma\left(\frac{v, \eta}{\sigma_{\iota_{d(u+1)}}}\right) \tag{54}$$

$$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_{\iota_{d(u+1)}}^{d(v)} \gamma\left(\frac{v, \eta}{\sigma_{\iota_{d(u+1)}}}\right) \tag{55}$$

where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the upper and lower incomplete gamma functions defined in Appendix A.2.

The marginal cdf of the largest eigenvalue and smallest eigenvalue of $\mathbf{W} \sim \mathcal{PW}_m(n, \mathbf{0}_m, \mathbf{\Sigma})$ was recently derived in [17, Th. 4 (2)], [37, eq. (21)], [38, eq. (40)] and [38, eq. (41)], respectively, and the marginal pdfs of the largest and smallest eigenvalue were calculated in [37, eq. (22)], [38, eq. (42)] and in [37, eq. (25)], [38, eq. (43)]. To the best of authors' knowledge, the marginal cdf and pdf of the kth largest eigenvalue were not available in the literature.

D. Uncorrelated Rician Fading MIMO Channel

Consider a uncorrelated Rician fading MIMO channel as given in Definition 2.4, then the random Hermitian matrix $\widetilde{\mathbf{W}} = (K_\mathrm{c} + 1)\mathbf{W}$ $(m \times m)$, where $\mathbf{W}$ is given in (7), follows a complex uncorrelated noncentral Wishart distribution, i.e., $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$ and $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and the non-centrality parameter $\mathbf{\Omega}$ is defined in (21) with eigenvalues $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ ordered such that $(\omega_1 > \cdots \omega_n > 0)$.

Joint pdf: Identifying terms, the joint pdf of the ordered eigenvalues in (22) coincides with the general pdf given in Assumption 3.1 if we let $\mathcal{I}$ be a singleton (the superindex $(\boldsymbol{\iota})$ can then be dropped), define $K^{(\boldsymbol{\iota})}$ as

$$K^{(\boldsymbol{\iota})} = \frac{e^{-\sum_{i=1}^{n}\omega_i}}{((m-n)!)^n} \prod_{i<j}^{n} \frac{1}{(\omega_j - \omega_i)} \tag{56}$$

the function $\varphi(\lambda)$ as

$$\varphi(\lambda) = e^{-\lambda}\lambda^{m-n} \tag{57}$$

and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ with entries as given in (23), i.e.,

$$[\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) = {}_0F_1(m-n+1; \omega_u\lambda_v) \tag{58}$$

for $u, v = 1, \ldots, n$. Hence, it follows that

$$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = {}_0F_1(m-n+1; \omega_u\lambda)e^{-\lambda}\lambda^{d(v)} \tag{59}$$

where we have introduced the function $d(v) = m - n + v - 1$.

Marginal distributions: In order to derive the marginal cdf and pdf of the kth largest eigenvalue using the results presented in Section III-B, we only have to particularize

$$\int_0^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda = \int_0^{\eta} {}_0F_1(m-n+1;\omega_u\lambda)e^{-\lambda}\lambda^{d(v)}d\lambda \quad (60)$$

$$\int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda = \int_{\eta}^{\infty} {}_0F_1(m-n+1;\omega_u\lambda)e^{-\lambda}\lambda^{d(v)}d\lambda. \quad (61)$$

Using [79, eq. (9.6.47)], it holds that

$$\begin{aligned}\int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda &= \frac{\Gamma(m-n+1)}{\left(\sqrt{\omega_u}\right)^{m-n}}\int_{\eta}^{\infty} e^{-\lambda}\lambda^{(m-n)/2+v-1} \\ &\quad \times I_{m-n}\left(2\sqrt{\omega_u\lambda}\right)d\lambda \quad (62)\\ &= \frac{e^{\omega_u}2^{1-v}(m-n)!}{\left(\sqrt{2\omega_u}\right)^{m-n}}Q_{d(2v),m-n}\left(\sqrt{2\omega_u},\sqrt{2\eta}\right) \quad (63)\end{aligned}$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of integer order n [79, eq. (9.6.10)] and $Q_{m,n}(\cdot,\cdot)$ is the Nuttall Q-function defined in Appendix A.2. Similarly, using [76, eq. (6.643.2)] and [8, eq. (9.220.2)], it follows that

$$\begin{aligned}\int_0^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda &= \frac{\Gamma(m-n+1)}{\left(\sqrt{\omega_u}\right)^{m-n}}\int_0^{\infty} e^{-\lambda}\lambda^{(m-n)/2+v-1} \\ &\quad \times I_{m-n}(2\sqrt{\omega_u\lambda})d\lambda \quad (64)\\ &= \Gamma(d(v+1))\,{}_1F_1(d(v+1);m-n+1;\omega_u) \quad (65)\end{aligned}$$

and we have that

$$\begin{aligned}\int_0^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda &= \int_0^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda - \int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda \quad (66)\\ &= d(v)!\,{}_1F_1(d(v+1);m-n+1;\omega_u) \\ &\quad - \frac{e^{\omega_u}2^{1-v}(m-n)!}{\left(\sqrt{2\omega_u}\right)^{m-n}}Q_{d(2v),m-n}\left(\sqrt{2\omega_u},\sqrt{2\eta}\right). \quad (67)\end{aligned}$$

Observe that the sum of the two indices of the Nuttall Q-functions in (63) and (67) is always odd and, hence, Remark A.2 in Appendix A.2 holds.

The marginal cdf of the kth largest eigenvalue of $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m,\boldsymbol{\Omega},\mathbf{I}_n)$ was initially derived in [80, eq. (9)] in terms of an infinite series of determinants. Recently, the marginal cdf the kth largest eigenvalue was obtained in terms of a finite sum of determinants in [42, Th. 3] and the particular cases of the largest and smallest eigenvalue in [34, Thm. 1], [42, Th. 2] and in [42, Th. 1], respectively. In addition, the marginal pdf of the maximum eigenvalue was given in [34, Cor. 3] and the case of $\boldsymbol{\Omega}$ being rank 1 was considered in [34, Cor. 1 and Cor. 3].

V. Outage Probability of Spatial Multiplexing MIMO Systems With CSI

As an illustrative application for the joint and marginal cdfs of the ordered eigenvalues given in Section III, we analyze in this section the outage probability of a spatial multiplexing MIMO system with perfect channel state information (CSI) at both sides of the link, and which transmits independent substreams through the channel eigenmodes [57]. Assuming the channel models presented in Definitions 2.1–2.4, we first obtain the outage probability of each individual substream, i.e., each channel eigenmode, and, then, we derive different global outage probabilities taking into account all established substreams, i.e., all used channel eigenmodes.

A. Signal Model

The signal model corresponding to a transmission through a general MIMO channel with n_{T} transmit and n_{R} receive dimensions is

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{w} \quad (68)$$

where $\mathbf{s} \in \mathbb{C}^{n_{\mathsf{T}}\times 1}$ is the transmitted vector, $\mathbf{H} \in \mathbb{C}^{n_{\mathsf{R}}\times n_{\mathsf{T}}}$ is the channel matrix as given in Definitions 2.1–2.4, $\mathbf{y} \in \mathbb{C}^{n_{\mathsf{R}}\times 1}$ is the received vector, and $\mathbf{w} \in \mathbb{C}^{n_{\mathsf{R}}\times 1}$ is a spatially white zero-mean circularly symmetric complex Gaussian noise vector normalized so that $\mathbb{E}\{\mathbf{w}\mathbf{w}^\dagger\} = \mathbf{I}_{n_{\mathsf{R}}}$.

Following the singular value decomposition (SVD), the channel matrix $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{U}\sqrt{\boldsymbol{\Lambda}}\mathbf{V}^\dagger \quad (69)$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, and $\sqrt{\boldsymbol{\Lambda}}$ is a diagonal matrix containing the singular values of $\mathbf{H}$ sorted in descending order. This way, the channel matrix is effectively decomposed into $\mathrm{rank}\,(\mathbf{H}) = \min\{n_{\mathsf{T}}, n_{\mathsf{R}}\}$ independent orthogonal modes of excitation, which are referred to as channel eigenmodes [3]–[5].

Assuming that perfect CSI is available at the transmitter and that $\kappa \le \min\{n_{\mathsf{T}}, n_{\mathsf{R}}\}$ independent data symbols per channel use have to be communicated, the transmitted vector can be written as

$$\mathbf{s} = \mathbf{V}_\kappa\sqrt{\mathbf{P}}\mathbf{x} \quad (70)$$

where $\mathbf{x}$ gathers the κ data symbols (zero mean, unit energy and uncorrelated, i.e., $\mathbb{E}\{\mathbf{x}\mathbf{x}^\dagger\} = \mathbf{I}_\kappa$), $\mathbf{V}_\kappa$ is formed with the κ columns of $\mathbf{V}$ associated with the κ strongest channel eigenmodes and $\mathbf{P} = \mathrm{diag}\left(\{p_k\}_{k=1,\ldots,\kappa}\right)$ is a diagonal matrix containing the power allocated to each established substream. The transmitted power is constrained such that

$$\mathbb{E}\left\{\|\mathsf{s}\|^2\right\} = \sum_{k=1}^{\kappa} p_k \le \mathsf{snr} \quad (71)$$

where snr is the SNR per receive antenna. Assuming perfect channel knowledge also at the receiver, the symbols transmitted

TABLE I
PARAMETERS OF THE UNCORRELATED, MIN-SEMICORRELATED, AND MAX-SEMICORRELATED RAYLEIGH FADING MIMO CHANNELS (DEFINITIONS 2.1, 2.2, AND 2.3)

	$\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$	$\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$	$\mathbf{W} \sim \mathcal{PW}_m(n, \mathbf{0}_m, \mathbf{\Sigma})$
$\mathcal{I}$	$\{1\}$	$\{1\}$	$\{\boldsymbol{\iota} = \text{perm}(1, \ldots, m) \mid (\iota_1 < \cdots < \iota_{m-n})$ and $(\iota_{m-n+1} < \cdots < \iota_n)\}$
$K^{(\boldsymbol{\iota})}$	$\prod_{i=1}^{n} \frac{1}{(m-i)!(n-i)!}$	$\prod_{i=1}^{n} \frac{1}{\sigma_i^m (m-i)!} \prod_{i<j}^{n} \frac{\sigma_i \sigma_j}{\sigma_j - \sigma_i}$	$\frac{(-1)^{\sum_{i=1}^{m-n}(\iota_i + i)}}{\prod_{i=1}^{n}(n-i)!} \frac{\prod_{i<j}^{m-n}(\sigma_{\iota_j} - \sigma_{\iota_i})}{\prod_{i<j}^{m}(\sigma_j - \sigma_i)}$
$\varphi(\lambda)$	$e^{-\lambda}\lambda^{m-n}$	λ^{m-n}	1
$\zeta_u^{(\boldsymbol{\iota})}(\lambda)$	λ^{u-1}	$e^{-\lambda/\sigma_u}$	$\sigma_{\iota_{d(u+1)}}^{m-n-1} e^{-\lambda_v/\sigma_{\iota_{d(u+1)}}}$
$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)$	$e^{-\lambda}\lambda^{d(u+v-1)}$	$e^{-\lambda/\sigma_u}\lambda^{d(v)}$	$\sigma_{\iota_{d(u+1)}}^{m-n-1} e^{-\lambda_v/\sigma_{\iota_{d(u+1)}}} \lambda^{v-1}$
$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$\Gamma(d(u+v), \eta)$	$\sigma_u^{d(v+1)} \Gamma(d(v+1), \eta/\sigma_u)$	$\sigma_{\iota_{d(u+1)}}^{d(v)} \Gamma(v, \eta/\sigma_{\iota_{d(u+1)}})$
$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$\gamma(d(u+v), \eta)$	$\sigma_u^{d(v+1)} \gamma(d(v+1), \eta/\sigma_u)$	$\sigma_{\iota_{d(u+1)}}^{d(v)} \gamma(v, \eta/\sigma_{\iota_{d(u+1)}})$
$d(v)$	$m-n+v-1$	$m-n+v-1$	$m-n+v-1$

TABLE II
PARAMETERS OF THE UNCORRELATED RICEAN FADING MIMO CHANNEL (DEFINITION 2.4)

	$\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$
$\mathcal{I}$	$\{1\}$
$K^{(\boldsymbol{\iota})}$	$\frac{e^{-\sum_{i=1}^{n}\omega_i}}{((m-n)!)^n} \prod_{i<j}^{n} \frac{1}{(\omega_j - \omega_i)}$
$\varphi(\lambda)$	$e^{-\lambda}\lambda^{m-n}$
$\zeta_u^{(\boldsymbol{\iota})}(\lambda)$	${}_0F_1(m-n+1; \omega_u \lambda)$
$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)$	${}_0F_1(m-n+1; \omega_u \lambda) e^{-\lambda} \lambda^{d(v)}$
$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$d(v)!\,{}_1F_1(d(v+1); m-n+1; \omega_u) - \frac{e^{\omega_u} 2^{1-v} (m-n)!}{\left(\sqrt{2\omega_u}\right)^{m-n}} Q_{d(2v),m-n}\left(\sqrt{2\omega_u}, \sqrt{2\eta}\right)$
$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$\frac{e^{\omega_u} 2^{1-v} (m-n)!}{\left(\sqrt{2\omega_u}\right)^{m-n}} Q_{d(2v),m-n}\left(\sqrt{2\omega_u}, \sqrt{2\eta}\right)$
$d(v)$	$m-n+v-1$

through the channel eigenmodes are recovered from the received signal $\mathbf{y}$ with matrix $\mathbf{U}_\kappa$, similarly defined to $\mathbf{V}_\kappa$, as

$$\hat{\mathbf{x}} = \mathbf{U}_\kappa^\dagger \left(\mathbf{H} \mathbf{V}_\kappa \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{w} \right) = \sqrt{\mathbf{\Lambda}_\kappa} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{n} \tag{72}$$

where $\sqrt{\mathbf{\Lambda}_\kappa}$ is a diagonal matrix that contains the κ largest singular values in descending order, and the noise vector $\mathbf{n} = \mathbf{U}_\kappa^\dagger \mathbf{w}$ has the same statistical properties as $\mathbf{w}$, possibly with a reduced dimension. Each substream experiences then an instantaneous SNR given by

$$\rho_k = \lambda_k p_k \qquad \text{for } k = 1, \ldots, \kappa \tag{73}$$

where λ_k denotes the kth ordered eigenvalue and p_k defines the power allocation policy. In the following, we focus only on fixed (channel non-dependent) power allocation policies, i.e.,

$$p_k = \phi_k \mathsf{snr} \qquad \text{for } k = 1, \ldots, \kappa \tag{74}$$

where ϕ_k is a positive constant independent of the channel with $\sum_{i=1}^{\kappa} \phi_k = 1$. Although linear MIMO transceivers optimized under the most common design criteria imply channel-dependent power allocation policies [57], this simple case serves as a starting point in the analysis of these systems [40]–[43]; channel-dependent power allocation strategies can be studied as in [41]. In addition, if the power constraint in (71) is substituted by a peak power constraint:

$$\lambda_{\max} \left\{ \mathbb{E}\left\{ \mathbf{s}\mathbf{s}^\dagger \right\} \right\} \leq \frac{\mathsf{snr}}{\kappa} \tag{75}$$

the optimum power allocation coincides with the fixed power allocation in (74) for $\phi_k = 1/\kappa$, i.e., a uniform power allocation [81].

B. Outage Probability

The outage probability is defined as the probability that the instantaneous SNR, denoted by ρ, falls below a certain threshold $\bar{\rho}$ [82, eq. (1.4)]

$$\mathsf{P}_{\mathsf{out}}(\bar{\rho}) \triangleq \mathsf{Pr}(\rho \leq \bar{\rho}). \tag{76}$$

Assuming that the instantaneous bit error rate (BER) can be approximated as[10] [82, Sec. 8.1.1]

$$\mathsf{BER}(\rho) = \frac{\alpha}{\log_2 M} \mathcal{Q}\left(\sqrt{\beta\rho}\right) \tag{77}$$

where $\mathcal{Q}(\cdot)$ is the Gaussian Q-function defined as [82, eq. (4.1)]

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt \tag{78}$$

and the parameters $\alpha \triangleq \alpha(M)$, and $\beta \triangleq \beta(M)$ depend on the M-ary constellation[11] used to map the data bits to symbols. Observe that if $\bar{\rho}$ is chosen as

$$\bar{\rho} = \frac{1}{\beta} \mathcal{Q}^{-2} \left(\frac{\log_2 M}{\alpha} \overline{\mathsf{BER}}\right) \tag{79}$$

where $\mathcal{Q}^{-1}(\cdot)$ is the inverse of the Gaussian Q-function in (78), the outage probability is the probability that the instantaneous BER overcomes certain target BER denoted by $\overline{\mathsf{BER}}$, i.e., $\Pr(\mathsf{BER}(\rho) > \overline{\mathsf{BER}})$.

1) Individual Outage Probability: Let us consider the diagonal spatial multiplexing MIMO system in (72) with the fixed power allocation policy in (74). Then, the instantaneous SNR of the channel eigenmodes in (73) can be rewritten as

$$\rho_k = \lambda_k \phi_k \mathsf{snr} \qquad \text{for } k = 1, \ldots, \kappa \tag{80}$$

and the individual outage probability as defined in (76) of the substream transmitted through the kth strongest channel eigenmode is given by

$$\mathsf{P}_{\mathsf{out}}^{(k)}(\mathsf{snr}) \triangleq \Pr(\rho_k \leq \bar{\rho}) = F_{\lambda_k}\left(\frac{\bar{\rho}}{\phi_k \mathsf{snr}}\right) \tag{81}$$

where $F_{\lambda_k}(\cdot)$ denotes the marginal cdf of the kth channel eigenvalue. Under the MIMO channel models presented in Definitions 2.1–2.4, $F_{\lambda_k}(\cdot)$ can be easily obtained by particularizing Theorem 3.2 with the corresponding expressions given in Tables I and II.

In Fig. 1 we provide the individual outage probability defined in (81) of the substream transmitted through the third $(k = 3)$ channel eigenmode in a spatial multiplexing MIMO system with $n_{\mathsf{T}} = 6$ and $n_{\mathsf{R}} = 4$ antennas when $\kappa = 4$ substreams are established, a QPSK modulation is used on each substream, and the power is uniformly allocated $(\phi_k = 1/4)$. The performance threshold has been chosen using (79) to guarantee a target BER of $\overline{\mathsf{BER}} = 10^{-3}$ and we have defined the correlation matrix as $[\mathbf{\Sigma}]_{ij} = r^{|i-j|}$ with $r = 0.9$ for the min- and max-semicorrelated Rayleigh, and the rician factor $K_{\mathsf{c}} = 10$ for the uncorrelated Rician fading MIMO channel.

The individual outage probability when transmitting through the strongest eigenmode, i.e., for $\kappa = k = 1$ in (81) and $p_1 = \mathsf{snr}$, has been widely analyzed in the literature, since it corresponds to the outage probability of the beamforming scheme (or maximum ratio transmission [56]). In particular, the outage probability under uncorrelated Rayleigh fading was obtained in [33, Sec. IV], [34, Sec. III], [54, Sec. II], [55, Sec. IV], under semicorrelated Rayleigh fading in [17, Sec. IV], and under uncorrelated Rician fading in [34, Sec. III]. Additionally, the case of double-correlated Rayleigh fading MIMO channels (not considered in detail in this paper) has been recently addressed in [53, Sec. IV].

[10]The BER approximation in (77) implicitly assumes that a Gray encoding mapping and coherent detection is used.

[11]The value of these parameters for M-QAM and M-PSK can be found in [82, eq. (8.15)] and [82, eq. (8.33)] respectively.

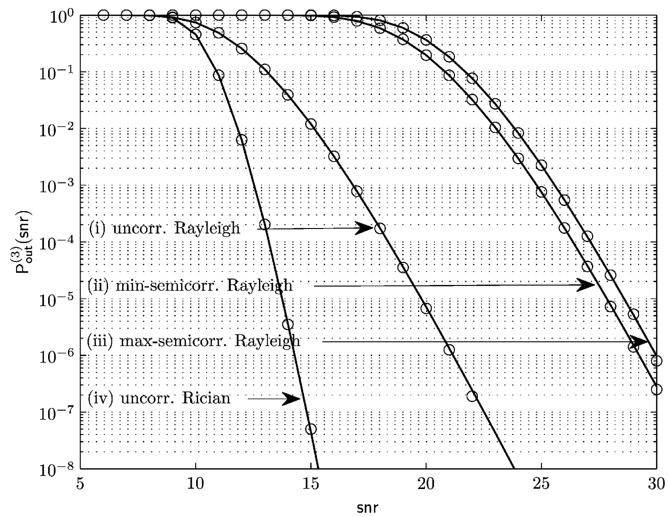


Fig. 1. Exact $(-)$ and simulated $(\circ)$ individual outage probability for the substream transmitted through the third $(k = 3)$ channel eigenmode of a 6×4 MIMO system with $\kappa = 4$, $\phi_k = 1/4$, and using a QPSK modulation.

2) Global Outage Probability: In this section we analyze the global outage probability of the spatial multiplexing MIMO system described in Section V-A. Consider, for instance, that κ services or substreams with possibly different performance constraints are multiplexed by accommodating each service in a different channel eigenmode. The global outage probability can be defined in many different ways depending on how the application of interest takes into account the individual outages of the established substreams. In the following we provide two illustrative examples.

All-Outage Probability: Assume that the communication process is considered to be successful if at least one of the substreams achieves the desired performance. Then, a global outage event is declared only when all used channel eigenmodes fail to offer their corresponding target performance and, hence, the global outage probability is defined as

$$\begin{aligned}
&\mathsf{P}_{\mathsf{out}}^{(\mathrm{all})}(\mathsf{snr}) \\
&\quad \triangleq \Pr(\rho_1 \leq \bar{\rho}_1, \ldots, \rho_\kappa \leq \bar{\rho}_\kappa) \\
&\quad = F_{\boldsymbol{\lambda}}\left(\frac{\bar{\rho}_1}{\phi_1 \mathsf{snr}}, \ldots, \frac{\bar{\rho}_\kappa}{\phi_\kappa \mathsf{snr}}, \infty, \ldots, \infty\right) && (82) \\
&\quad = F_{\boldsymbol{\lambda}}\left(\frac{\bar{\rho}_1}{\phi_1 \mathsf{snr}}, \ldots, \frac{\bar{\rho}_\kappa}{\phi_\kappa \mathsf{snr}}, \frac{\bar{\rho}_\kappa}{\phi_\kappa \mathsf{snr}}, \ldots, \frac{\bar{\rho}_\kappa}{\phi_\kappa \mathsf{snr}}\right) && (83)
\end{aligned}$$

where $\{\bar{\rho}_k\}_{k=1,\ldots,\kappa}$ are the target performances and $F_{\boldsymbol{\lambda}}(\cdot)$ denotes the joint cdf of the ordered channel eigenvalues. Under the MIMO channel models presented in Definitions 2.1–2.4, $F_{\boldsymbol{\lambda}}(\cdot)$ can be obtained particularizing Theorem 3.1 with the expression of the corresponding parameters in Tables I and II.

When equal target performances are imposed on all κ established substreams, i.e., $\bar{\rho}_k = \bar{\rho}$, and a uniform power allocation

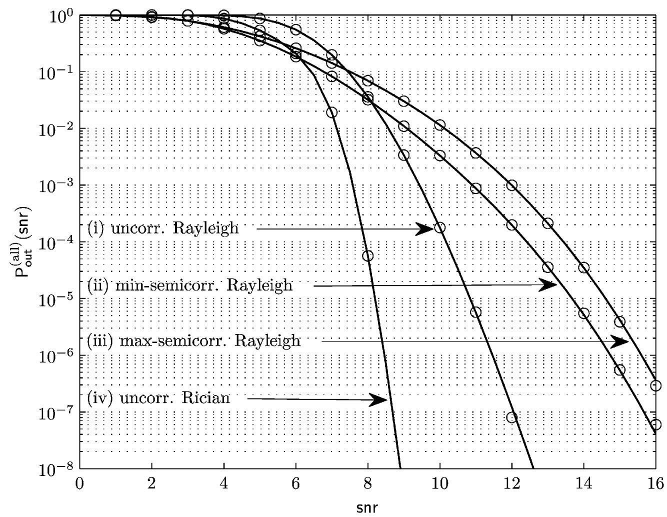


Fig. 2. Exact (—) and simulated (∘) global all-outage probability of a 6×4 MIMO system with $\kappa = 4$, $\phi_k = 1/4$, and using a QPSK modulation.

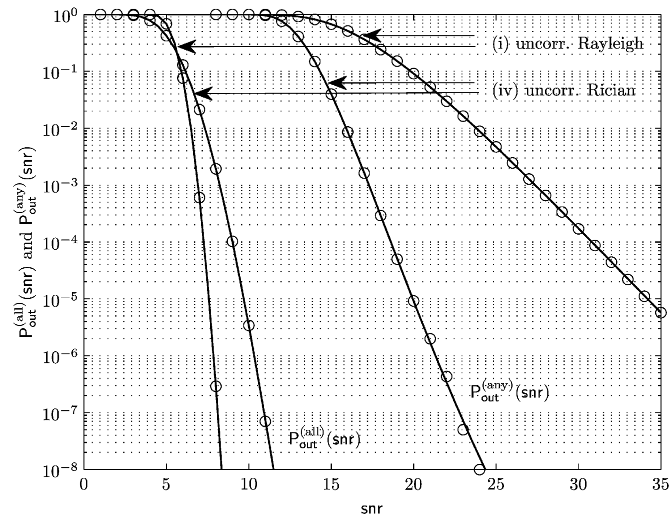


Fig. 3. Exact (—) and simulated (∘) global all-outage and any-outage probabilities of a 6×4 MIMO system with $\kappa = 4$, $\phi = 1/4$, and using a QPSK modulation.

is used, i.e., $\phi_k = 1/\kappa$, the outage probability in (83) is simply given by

$$\mathsf{P}_{\mathsf{out}}^{(\mathrm{all})}(\mathsf{snr}) = \mathsf{Pr}(\rho_1 \leq \bar{\rho}) = F_{\lambda_1}\left(\frac{\kappa\bar{\rho}}{\mathsf{snr}}\right) \qquad (84)$$

where $F_{\lambda_1}(\cdot)$ denotes the marginal cdf of the largest channel eigenvalue.

In Fig. 2, we provide the global all-outage probability defined in (83) of a spatial multiplexing MIMO system with $n_{\mathsf{T}} = 6$ and $n_{\mathsf{R}} = 4$ antennas when $\kappa = 4$ substreams are established, a QPSK modulation is used on each substream, and the power is uniformly allocated ($\phi_k = 1/4$). The target BERs have been established as $\overline{\mathsf{BER}}_1 = 10^{-5}$, $\overline{\mathsf{BER}}_2 = 10^{-4}$, $\overline{\mathsf{BER}}_3 = 10^{-3}$, and $\overline{\mathsf{BER}}_4 = 10^{-2}$ and the parameters of the MIMO channel models have been chosen as defined in the simulations of the previous section.

Any-Outage Probability: Assume that the quality of all substreams has to be simultaneously guaranteed, then a global outage event is declared whenever at least one of the used channel eigenmodes cannot offer the desired performance. In this case, the global outage probability in defined as

$$\begin{aligned}\mathsf{P}_{\mathsf{out}}^{(\mathrm{any})}(\mathsf{snr}) &\triangleq 1 - \mathsf{Pr}(\rho_1 > \bar{\rho}_1, \ldots, \rho_\kappa > \bar{\rho}_\kappa) \\ &= 1 - CF_{\boldsymbol{\lambda}}\left(\frac{\bar{\rho}_1}{\phi_1\mathsf{snr}}, \ldots, \frac{\bar{\rho}_\kappa}{\phi_\kappa\mathsf{snr}}, 0, \ldots, 0\right) \end{aligned} \qquad (85)$$

where $CF_{\boldsymbol{\lambda}}(\cdot)$ denotes the joint complementary cdf of the ordered channel eigenvalues and can be obtained with techniques similar to those used to derive the joint cdf given in Theorem 3.1.

When equal target performances are imposed on all the established substreams and a uniform power allocation is used, the outage probability in (85) is simply given by

$$\mathsf{P}_{\mathsf{out}}^{(\mathrm{any})}(\mathsf{snr}) = 1 - \mathsf{Pr}(\rho_\kappa > \bar{\rho}) = F_{\lambda_\kappa}\left(\frac{\kappa\bar{\rho}}{\mathsf{snr}}\right) \qquad (86)$$

where $F_{\lambda_\kappa}(\cdot)$ denotes the marginal cdf of the κth largest channel eigenvalue.

In Fig. 3, we compare the global all-outage and any-outage probabilities defined in (84) and (86) of a spatial multiplexing MIMO system with $n_{\mathsf{T}} = 6$ and $n_{\mathsf{R}} = 4$ antennas when $\kappa = 4$ substreams are established, a QPSK modulation is used on each substream, and the power is uniformly allocated ($\phi_k = 1/4$). The target BERs is equal for all established substreams, $\overline{\mathsf{BER}} = 10^{-3}$ and the parameters of the MIMO channel models have been chosen as defined in the simulations of the previous section.

VI. Conclusion

The probabilistic characterization of the eigenvalues of Wishart, Pseudo-Wishart, and Quadratic form distributions is critical in the performance evaluation of many communication and signal processing applications. In particular, the performance of MIMO systems without CSI at the transmitter demanded the characterization of the unordered eigenvalues whereas the techniques that employed CSI at the transmitter required the evaluation of probabilities associated with one or several of the eigenvalues in some specific order (often the highest or smallest but sometimes any one in particular within the ordered set). Many different contributions, as early as the 1960s in the mathematical literature and much more recently in the signal processing community, provided partial characterizations for specific problems. However, the unified perspective provided by this paper was missing and can, not only fill the gap of the currently unknown results, but even more importantly, provide a solid framework for the understanding and direct derivation of all the previously derived results.

Appendix A
Preliminary Definitions and Results

A.1 Determinants and Matrices

In this section we review some basic results of matrices and determinants required in the derivations presented in the paper.

Definition A.1 (Determinant [83, Sec. 0.3.2]): The determinant of a matrix $\mathbf{A}$ $(n \times n)$, denoted by $|\mathbf{A}|$, is defined as

$$|\mathbf{A}| = \sum_{\boldsymbol{\mu}} \text{sgn}(\boldsymbol{\mu}) \prod_{k=1}^{n} [\mathbf{A}]_{\mu_k, k} = \text{sgn}(\boldsymbol{\nu}) \sum_{\boldsymbol{\mu}} \text{sgn}(\boldsymbol{\mu}) \prod_{k=1}^{n} [\mathbf{A}]_{\mu_k, \nu_k} \quad (87)$$

where the summation over $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is for all permutations of the integers $(1, \ldots, n)$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ is any arbitrary fixed permutation of the integers $(1, \ldots, n)$, and $\text{sgn}(\cdot)$ denotes the sign of the permutation. Alternatively, we also use the common compact notation of the determinant $|\mathbf{A}|$ in terms of its (i, j)th element: $|\mathbf{A}| = |a_{ij}|$.

Lemma A.1 (Derivative of a Determinant [84, eq. (6.5.9)]): The derivative of the determinant of matrix $\mathbf{A}(\eta)$ $(n \times n)$ is given by

$$\frac{d}{d\eta} |\mathbf{A}(\eta)| = \sum_{t=1}^{n} |\mathbf{A}(t; \eta)| \quad (88)$$

where $\mathbf{A}(t; \eta)$ coincides with $\mathbf{A}(\eta)$ except that every entry in the tth column is differentiated with respect to η.

Definition A.2 (Vandermonde Matrix [84, eq. (6.1.32)]): The nth order Vandermonde matrix in $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, denoted by $\mathbf{V}(\boldsymbol{\lambda})$ $(n \times n)$, is defined as

$$[\mathbf{V}(\boldsymbol{\lambda})]_{i,j} = \lambda_j^{i-1} \qquad \text{for } i, j = 1, \ldots, n. \quad (89)$$

Lemma A.2 (Vandermonde Determinant [84, eq. (6.1.33)]): The determinant of the nth order Vandermonde matrix introduced in Definition A.2 is given by

$$|\mathbf{V}(\boldsymbol{\lambda})| = \prod_{i<j} (\lambda_j - \lambda_i). \quad (90)$$

The following operator will prove useful to express a sum of determinants compactly.

Definition A.3: The operator $\mathcal{T}\{\cdot\}$ over a tensor $\mathbf{T}$ $(n \times n \times n)$ is defined as[12]

$$\mathcal{T}\{\mathbf{T}\} = \sum_{\boldsymbol{\mu}, \boldsymbol{\nu}} \text{sgn}(\boldsymbol{\mu}) \text{sgn}(\boldsymbol{\nu}) \prod_{k=1}^{n} [\mathbf{T}]_{\mu_k, \nu_k, k} \quad (91)$$

where the summation over $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is for all permutations of the integers $(1, \ldots, n)$ and $\text{sgn}(\cdot)$ denotes the sign of the permutation.

Remark A.1: The operator $\mathcal{T}\{\cdot\}$ introduced in Definition A.3 can be alternatively expressed as

$$\mathcal{T}\{\mathbf{T}\} = \sum_{\boldsymbol{\mu}, \boldsymbol{\nu}} \text{sgn}(\boldsymbol{\mu}) \prod_{k=1}^{n} [\mathbf{T}]_{\mu_k, k, \nu_k} = \sum_{\boldsymbol{\nu}} |\mathbf{A}(\boldsymbol{\nu})| \quad (92)$$

where matrix $\mathbf{A}(\boldsymbol{\nu})$ $(n \times n)$ is defined as

$$[\mathbf{A}(\boldsymbol{\nu})]_{i,j} = [\mathbf{T}]_{i,j,\nu_j} \qquad \text{for } i, j = 1, \ldots, n. \quad (93)$$

[12]Note that this operator was also introduced in [18, Def. 1].

A.2 Integral Functions

In this section, we introduce some functions defined in integral form, which, due to their importance, have been tabulated and are available as build-in functions in most common mathematical software packages such as MATLAB or Mathematica.

Definition A.4 (Lower Incomplete Gamma Function [84, eq. (6.5.2)]): The lower incomplete gamma function is defined as

$$\gamma(a, \lambda) = \int_0^{\lambda} e^{-x} x^{a-1} dx. \quad (94)$$

Definition A.5 (Upper Incomplete Gamma Function [79, eq. (6.5.3)]): The upper incomplete gamma function is defined as

$$\Gamma(a, \lambda) = \int_{\lambda}^{\infty} e^{-x} x^{a-1} dx = \Gamma(a) - \gamma(a, \lambda). \quad (95)$$

Definition A.6 (Nuttall Q-Function [82, eq. (4.104)]): The Nuttall Q-function is defined as

$$Q_{m,n}(a, b) = \int_b^{\infty} x^m e^{-(x^2 + a^2/2)} I_n(ax) dx \quad (96)$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of integer order n [79, eq. (9.6.19)].

Remark A.2 ([85]): The Nuttall Q-function is not considered to be a tabulated function. However, if $m + n$ is odd, i.e., $m = n + 2k + 1$ for $k = 0, 1, \ldots$, the Nuttall Q-function can be expressed as a weighted sum of $k + 1$ generalized Marcum Q-functions [82, eq. (4.60)] and modified Bessel functions of the first kind [79, eq. (9.6.19)].

APPENDIX B
ORDERED EIGENVALUES. PROOFS

B.1 Motivation of the Joint pdf Structure

The joint pdf of the ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ of a complex Hermitian random matrix $\mathbf{W}$ $(n \times n) > 0$ with pdf $f_{\mathbf{W}}(\mathbf{W})$ is given by [9, eq. (93)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)} \prod_{i<j} (\lambda_i - \lambda_j)^2 \int_{\mathcal{U}(n)} f_{\mathbf{W}} \left(\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^\dagger \right) d\mathbf{U} \quad (97)$$

where $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\mathbf{W} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^\dagger$ is the eigendecomposition of $\mathbf{W}$, and $d\mathbf{U}$ is the invariant measure on the unitary group $\mathcal{U}(n)$ normalized to make the total measure unity.

Typical univariate distributions such as the Chi-squared, Cauchy and Beta distributions involve Bessel and hypergeometric functions which can all be written as special cases, for particular integers p and q, of the generalized hypergeometric function of scalar arguments [76, eq. (9.14.1)]

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!} \quad (98)$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the Pochhammer's symbol [79, eq. (6.1.22)]. The corresponding complex multivariate distributions involve a generalization of this function to the case in which the variable x is replaced by an Hermitian matrix $\mathbf{X}$. The generalized hypergeometric function of Hermitian matrix argument is defined as [9, eq. (87)]

$$ {}_p\tilde{F}_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;\mathbf{X}\right) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(\mathbf{X})}{k!} \quad (99) $$

where the summation over κ is for all partitions of k into n parts and $\tilde{C}_\kappa(\mathbf{X})$ is an homogenous symmetric polynomial in the eigenvalues of $\mathbf{X}$, $x_1,\ldots,x_n$, known as zonal polynomial [9, eq. (85)]:

$$ \tilde{C}_\kappa(\mathbf{X}) = k!\frac{\prod_{i<j}^{n}(k_i-k_j-i+j)}{\prod_{i=1}^{n}\Gamma(n+k_i-i+1)}\frac{\left|\left(x_i^{k_j+n-j}\right)\right|}{\left|\left(x_i^{n-j}\right)\right|}. \quad (100) $$

Let us consider first that the distribution of the Hermitian random matrix $\mathbf{W}=\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$ can be written as

$$ f_{\mathbf{W}}(\mathbf{W}) = K_{\mathbf{W}}\,{}_p\tilde{F}_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;\mathbf{\Sigma}\mathbf{W}\right)\prod_{t=1}^{n}\varphi(\lambda_t) \quad (101) $$

where $K_{\mathbf{W}}$ is a normalization constant, $\mathbf{\Sigma}$ $(n\times n)$ is a deterministic Hermitian matrix with eigenvalues denoted by $\boldsymbol{\sigma}$, and $\varphi(\lambda)$ is an arbitrary function. This pdf expression holds for some cases of the complex Wishart and inverted Wishart distribution, complex matrix variate Cauchy, and Bessel distributions (see [9, Sec. 8] and [86]). The joint pdf of the ordered eigenvalues of $\mathbf{W}$ is then given by

$$ \begin{aligned} f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) &= K_{\mathbf{W}}\frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)}|\mathbf{V}(\boldsymbol{\lambda})|^2\prod_{t=1}^{n}\varphi(\lambda_t)\int_{\mathcal{U}(n)} \\ &\quad\times {}_p\tilde{F}_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;\mathbf{\Sigma}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger\right)d\mathbf{U} \quad (102) \\ &= K_{\mathbf{W}}\frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)}|\mathbf{V}(\boldsymbol{\lambda})|^2\prod_{t=1}^{n}\varphi(\lambda_t) \\ &\quad\times {}_p\tilde{F}_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;\boldsymbol{\sigma},\boldsymbol{\lambda}\right) \quad (103) \end{aligned} $$

where ${}_p\tilde{F}_q\left(\cdot;\cdot;\cdot,\cdot\right)$ denotes the hypergeometric function of two Hermitian matrix arguments [9, eq. (87)] and (103) follows from the splitting property in [9, eq. (92)]. Hypergeometric functions of two Hermitian matrix arguments are defined similarly to (99) as an infinite series of zonal polynomials but can be alternatively expressed in terms of a quotient of determinants including generalized hypergeometric functions of scalar arguments [87], [88]. Using [87, Lem. 3], [88, Th. 4.2] it follows that [see (104) and (105) at the bottom of the page], where $|\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})|$ $(n\times n)$ is defined as

$$ [\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})]_{i,j} = {}_pF_q\left(a_1-n+1,\ldots,a_p-n+1;\; b_1-n+1,\ldots,b_q-n+1;\lambda_i\sigma_j\right). \quad (106) $$

Observe that the joint pdf of the ordered eigenvalues in (105) coincide with Assumption 3.1 if we let the set $\mathcal{I}$ be a singleton.

Let us now consider an $m\times m$ Hermitian random matrices of rank n $(n<m)$ such as Pseudo-Wishart matrices. In this case, the same procedure can be followed but interchanging n by m and taking the limit $\{\lambda_i\}_{i=n+1,\ldots,m}\to 0$ [30], [53], [89]. Hence, matrix $\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})$ is $m\times m$ but only the first n rows depend on $\{\lambda_i\}_{i=1,\ldots,n}$. Performing the Laplace expansion of the determinant $|\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})|$ over the last $(m-n)$ rows, the joint pdf of the ordered eigenvalues can be expressed as in Assumption 3.1 using the sum over the set $\mathcal{I}$ to include this sum of determinants.

Finally, in a more general case, such as the Quadratic form distributions, the hypergeometric function of one Hermitian matrix argument in (101) is substituted by an hypergeometric function of more Hermitian matrix arguments [90, eq. (2.30)]. Since the splitting property of hypergeometric functions also holds, the structure of the joint pdf of the ordered eigenvalues is maintained (see, e.g., [53, eq. (45)]). Although a determinantal expression for hypergeometric function of more than two Hermitian matrix arguments is not known, the corresponding series expansion in terms of zonal polynomials can still be included in the general expression of Assumption 3.1 using the summation over the set $\mathcal{I}$ (see an example in Appendix C).

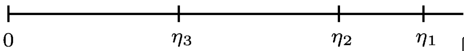
In this Appendix, we have justified the adoption of Assumption 3.1 based on the results in [87] and [88] to express hypergeometric functions of Hermitian matrix arguments as a (finite) sum of determinants. Hypergeometric functions result from integrating over the unitary group when deriving the joint pdf of the ordered eigenvalues from the corresponding random matrix distribution as in (97) and are the common representation used in multivariate analysis. An alternative (and even more direct) approach it to attack such integrals from the group theoretic point-of-view as done in [91]. This method was recently applied in [30] to derive the joint pdf of the ordered eigenvalues of the same kind of MIMO channels discussed in this paper.

B.2 Proof of Theorem 3.1

Proof: The joint cdf of the ordered eigenvalues $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta})$ can be obtained from the joint pdf of the ordered eigenvalues $f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ as

$$ F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \int_{\mathcal{D}_{\mathrm{ord}}(\boldsymbol{\eta})} f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})d\boldsymbol{\lambda} \quad (107) $$

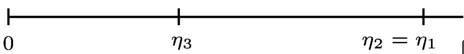
$$ \begin{aligned} f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) &= K_{\mathbf{W}}\frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)}\frac{\tilde{\Gamma}_n(n)}{\pi^{n(n-1)/2}}\frac{\prod_{i=1}^{n}\prod_{j=1}^{q}(b_j-i+1)^{i-1}}{\prod_{i=1}^{n}\prod_{j=1}^{p}(a_j-i+1)^{i-1}}\frac{|\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})|}{|\mathbf{V}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\sigma})|}|\mathbf{V}(\boldsymbol{\lambda})|^2\prod_{t=1}^{n}\varphi(\lambda_t) \quad (104) \\ &= K_{\boldsymbol{\lambda}}|\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})||\mathbf{V}(\boldsymbol{\lambda})|\prod_{t=1}^{n}\varphi(\lambda_t) \quad (105) \end{aligned} $$



λ_3	λ_2	λ_1
$\lambda_3 \leq \lambda_2$		λ_1
λ_3	$\lambda_2 \leq \lambda_1$	
$\lambda_3 \leq \lambda_2$	λ_1	
$\lambda_3 \leq \lambda_2 \leq \lambda_1$		

$\boldsymbol{i}$	$\tau(\boldsymbol{i})$
$(1,2,3)$	1!1!1!
$(1,3,3)$	1!0!2!
$(2,2,3)$	0!2!1!
$(2,3,3)$	0!1!2!
$(3,3,3)$	0!0!3!

(a) $n=3$ and $\eta_1 > \eta_2 > \eta_3$.



λ_3	$\lambda_2 \leq \lambda_1$
$\lambda_3 \leq \lambda_2$	λ_1
$\lambda_3 \leq \lambda_2 \leq \lambda_1$	

$\boldsymbol{i}$	$\tau(\boldsymbol{i})$
$(2,2,3)$	0!2!1!
$(2,3,3)$	0!1!2!
$(3,3,3)$	0!0!3!

(b) $n=3$ and $\eta_1 = \eta_2 > \eta_3$.

Fig. 4. Integration region $\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})$ and normalizing constant $\tau(\boldsymbol{i})$.

where

$$\mathcal{D}_{\rm ord}(\boldsymbol{\eta}) = \{0 \leq \lambda_1 \leq \eta_1, \ldots, 0 \leq \lambda_n \leq \eta_n\} \cap \{\lambda_1 \geq \cdots \geq \lambda_n\} \quad (108)$$

and recall that by assumption $(\eta_1 \geq \cdots \geq \eta_n > 0)$.

Let $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ and $\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta}) = \mathcal{D}(\boldsymbol{i};\boldsymbol{\eta}) \cap \{\lambda_1 \geq \cdots \geq \lambda_n\}$ with

$$\mathcal{D}(\boldsymbol{i};\boldsymbol{\eta}) = \{\eta_{i_1+1} < \lambda_1 \leq \eta_{i_1}, \eta_{i_2+1} < \lambda_2 \leq \eta_{i_2}, \ldots, 0 < \lambda_n \leq \eta_{i_n}\}. \quad (109)$$

Then, $\mathcal{D}_{\rm ord}(\boldsymbol{\eta})$ can be expressed as a union of non-overlapping domains $\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})$, i.e.,

$$\mathcal{D}_{\rm ord}(\boldsymbol{\eta}) = \bigcup_{\mathbf{i} \in \mathcal{S}} \mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta}) \quad (110)$$

where the set $\mathcal{S}$ is defined as $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ with[13]

$$\mathcal{S}_1 = \{\boldsymbol{i} \in \mathbb{N}^n | \max(i_{s-1}, s) \leq i_s \leq n\} \quad (111)$$
$$\mathcal{S}_2 = \{\boldsymbol{i} \in \mathbb{N}^n | i_s \neq r \text{ if } \eta_r = \eta_{r+1}\}. \quad (112)$$

Observe that the set $\mathcal{S}_1$ is such that $\bigcup_{\boldsymbol{i} \in \mathcal{S}_1} \mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})$ only includes domains $\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})$ in which each λ_k belongs to one of the $(n-k+1)$ possible non-overlapping partitions of the interval $[0,\eta_k]$, i.e., $[0,\eta_n], (\eta_n, \eta_{n-1}], \ldots, (\eta_{k-1}, \eta_k]$ (see a representation for $n=3$ in Fig. 4). Then, by intersecting $\mathcal{S}_1$ with $\mathcal{S}_2$ we eliminate all domains $\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})$ with empty intervals, i.e., intervals such that $\eta_i = \eta_{i+1}$ (compare representations for $n=3$ in Fig. 4).

From (107) and (110), the joint cdf $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta})$ can be rewritten as

$$F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \sum_{\mathbf{i} \in \mathcal{S}} \mathcal{I}(\boldsymbol{i};\boldsymbol{\eta}) \quad (113)$$

where

$$\mathcal{I}(\boldsymbol{i};\boldsymbol{\eta}) = \int_{\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})} f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}. \quad (114)$$

[13]Recall that by definition $i_0 = 0$, $i_{n+1} = n+1$ and $\eta_{n+1} = 0$.

Now, expanding the determinants of $\mathbf{E}(\boldsymbol{\lambda})$ and $\mathbf{V}(\boldsymbol{\lambda})$ (see Definitions A.1 and A.2) in the joint pdf expression in (24), we can rewrite $f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ as

$$\begin{aligned} f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) &= \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \operatorname{sgn}(\boldsymbol{\mu}) \operatorname{sgn}(\boldsymbol{\nu}) \prod_{t=1}^{n} [\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})]_{\mu_t,t} \\ &\quad \times [\mathbf{V}(\boldsymbol{\lambda})]_{\nu_t,t} \varphi(\lambda_t) \quad (115) \\ &= \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \operatorname{sgn}(\boldsymbol{\mu}) \operatorname{sgn}(\boldsymbol{\nu}) \\ &\quad \times \prod_{t=1}^{n} \zeta_{\mu_t}^{(\boldsymbol{\iota})}(\lambda_t) \varphi(\lambda_t) \lambda_t^{\nu_t - 1} \quad (116) \end{aligned}$$

where the summation over $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ and over $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is for all permutations of the integers $(1, \ldots, n)$ and $\operatorname{sgn}(\cdot)$ denotes the sign of the permutation. Substituting (116) back in (114) and defining $\xi_{u,v}^{(\boldsymbol{\iota})} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_t)\varphi(\lambda_t)\lambda_t^{v-1}$, it follows that

$$\mathcal{I}(\boldsymbol{i};\boldsymbol{\eta}) = \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} \int_{\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \operatorname{sgn}(\boldsymbol{\mu}) \operatorname{sgn}(\boldsymbol{\nu}) \times \prod_{t=1}^{n} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda_t) d\boldsymbol{\lambda}. \quad (117)$$

Using the symmetry of the integrand in (117) with respect to $\boldsymbol{\lambda}$, i.e.

$$\sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \operatorname{sgn}(\boldsymbol{\mu}) \operatorname{sgn}(\boldsymbol{\nu}) \prod_{t=1}^{n} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda_t) = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \operatorname{sgn}(\boldsymbol{\mu}) \operatorname{sgn}(\boldsymbol{\nu}) \prod_{t=1}^{n} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda_{\sigma_t}) \quad (118)$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ is any arbitrary fixed permutation of the integers $(1, \ldots, n)$, the domain $\mathcal{D}_{\rm ord}(\boldsymbol{i};\boldsymbol{\eta})$ in (117) can be replaced with the unordered domain $\mathcal{D}(\boldsymbol{i};\boldsymbol{\eta})$ (see a similar result in [80, Lem. 2]) by properly normalizing the result of the integral with

$$\tau(\boldsymbol{i}) = \prod_{s=1}^{o(\boldsymbol{i})} \tau_s(\boldsymbol{i})! = \prod_{u=1}^{n} (1 - \delta_{i_u, i_{u+1}}) \left(\sum_{v=1}^{u} \delta_{i_u, i_v} \right)! \quad (119)$$

where $o(\boldsymbol{i})$ denotes the number of different integration intervals in $\mathcal{D}(\boldsymbol{i};\boldsymbol{\eta})$, $\tau_s(\boldsymbol{i})$ is the number of ordered variables integrated in the sth one of these intervals (see Fig. 4) and $\delta_{u,v}$ denotes the Kronecker delta. Then, it holds that

$$\begin{aligned} \mathcal{I}(\boldsymbol{i};\boldsymbol{\eta}) &= \sum_{\boldsymbol{\iota} \in \mathcal{I}} \frac{K^{(\boldsymbol{\iota})}}{\tau(\boldsymbol{i})} \int_{\mathcal{D}(\boldsymbol{i};\boldsymbol{\eta})} \prod_{t=1}^{n} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda_t) d\boldsymbol{\lambda} \\ &= \sum_{\boldsymbol{\iota} \in \mathcal{I}} \frac{K^{(\boldsymbol{\iota})}}{\tau(\boldsymbol{i})} \prod_{t=1}^{n} \int_{\eta_{i_t+1}}^{\eta_{i_t}} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda) d\lambda. \quad (120) \end{aligned}$$

Finally, $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta})$ is given by

$$F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \sum_{\boldsymbol{\iota} \in \mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\mathbf{i} \in \mathcal{S}} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \operatorname{sgn}(\boldsymbol{\mu}) \operatorname{sgn}(\boldsymbol{\nu}) \times \frac{1}{\tau(\boldsymbol{i})} \prod_{t=1}^{n} \int_{\eta_{i_t+1}}^{\eta_{i_t}} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda) d\lambda \quad (121)$$

and the proof is completed by using again the operator $\mathcal{T}\{\cdot\}$.

B.3 Proof of Theorem 3.2

Proof: The marginal cdf of the kth largest eigenvalue is

$$F_{\lambda_k}(\eta) = \Pr(\lambda_k \leq \eta) = \Pr(\lambda_1 \leq \infty, \dots, \lambda_{k-1} \leq \infty, \lambda_k \leq \eta, \dots, \lambda_n \leq \eta). \quad (122)$$

Hence, it can be obtained from the joint cdf of the ordered eigenvalues $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta})$ for $\{\eta_i\}_{i=1,\dots,k-1} = \infty$ and $\{\eta_i\}_{i=k,\dots,n} = \eta > 0$. Particularizing the expression of $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta})$ given in Theorem 3.1, we have that the set $\mathcal{S}$ reduces to

$$\mathcal{S}_k = \{\boldsymbol{i} \in \{k-1, n\}^n | i_s = n \text{ if } s \geq k\} \quad (123)$$

which only contains k elements and $F_{\lambda_k}(\eta)$ is given by

$$F_{\lambda_k}(\eta) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\mathbf{i}\in\mathcal{S}_k} \frac{1}{\tau(\boldsymbol{i})} \mathcal{T}\{\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i};\boldsymbol{\eta}(\eta))\} \quad (124)$$

where $\boldsymbol{\eta}(\eta) = (\infty, \dots, \infty, \eta, \dots, \eta)$. Let us denote by a unique index i $(i = 1, \dots, k)$ each element of $\mathcal{S}_k$ such that $\boldsymbol{i}(i)$ has the first $i-1$ components equal to $k-1$ and the rest equal to n. Noting that

$$\tau(\boldsymbol{i}(i)) = (i-1)!(n-i+1)! \quad (125)$$

and using the alternative expression of the operator $\mathcal{T}(\cdot)$ in Remark A.1, it follows that

$$F_{\lambda_k}(\eta) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{i=1}^{k} \frac{1}{(i-1)!(n-i+1)!} \sum_{\boldsymbol{\mu}} |\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)| \quad (126)$$

where the summation over $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ is for all permutations of the integers $(1, \dots, n)$ and the $n \times n$ matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)$ is defined as [see (29)]

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)]_{u,\mu_v} = [\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, \boldsymbol{i}(i); \boldsymbol{\eta}(\eta))]_{u,\mu_v,v} = \begin{cases} \int_\eta^\infty \xi_{u,\mu_v}^{(\boldsymbol{\iota})}(\lambda)d\lambda & 1 \leq v < i \\ \int_0^\eta \xi_{u,\mu_v}^{(\boldsymbol{\iota})}(\lambda)d\lambda & i \leq v \leq n \end{cases} \quad (127)$$

for $u, v = 1, \dots, n$. Observing that $\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta) = \mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\nu}, i; \eta)$ if $\boldsymbol{\nu} = (\pi(\mu_1, \dots, \mu_{i-1}), \pi(\mu_i, \dots, \mu_n))$ where $\pi(\cdot)$ denotes permutation, it suffices to calculate one determinant for all these $(i-1)!(n-i+1)!$ permutations, for instance, $|\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)|$ where $\boldsymbol{\mu}$ is such that $(\mu_1 < \cdots < \mu_{i-1})$ and $(\mu_i < \cdots < \mu_n)$. Finally, we have that

$$\frac{1}{(i-1)!(n-i+1)!} \sum_{\boldsymbol{\mu}} |\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)| = \sum_{\boldsymbol{\mu}\in\mathcal{P}(i)} |\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)| \quad (128)$$

and this completes the proof.

B.4 Proof of Corollary 3.1

Proof: Particularizing Theorem 3.2 for $k = 1$, it follows that

$$F_{\lambda_1}(\eta) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\boldsymbol{\mu}\in\mathcal{P}(1)} |\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, 1; \eta)|. \quad (129)$$

Observe now that $\mathcal{P}(1)$ only contains the element $(1, \dots, n)$. Thus, using (31), we define the $n \times n$ matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\eta)$ as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\eta)]_{u,v} = [\mathbf{F}^{(\boldsymbol{\iota})}((1, \dots, n), 1; \eta)]_{u,v} = \int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda \qquad \text{for } u, v = 1, \dots, n \quad (130)$$

and this completes the proof.

B.5 Proof of Corollary 3.2

Proof: This proof could be done by particularizing Theorem 3.2 for $k = n$ and simplifying the resulting expression. However, it is easier to obtain $F_{\lambda_n}(\eta)$ directly as

$$F_{\lambda_n}(\eta) = 1 - \Pr(\lambda_n > \eta) = 1 - \Pr(\lambda_1 > \eta, \dots, \lambda_n > \eta) = 1 - \int_{\mathcal{D}_{\text{ord}}(\eta)} f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})d\boldsymbol{\lambda} \quad (131)$$

where

$$\mathcal{D}_{\text{ord}}(\eta) = \{\lambda_1 > \eta, \dots, \lambda_n > \eta\} \cap \{\lambda_1 > \cdots > \lambda_n\}. \quad (132)$$

Then, using the expression for the joint pdf $f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ given in Assumption 3.1 and substituting operator $\mathcal{T}\{\cdot\}$ by its definition (see Definition A.3), it follows that

$$F_{\lambda_n}(\eta) = 1 - \sum_{\iota\in\mathrm{I}} K^{(\boldsymbol{\iota})} \int_{\mathcal{D}_{\text{ord}}(\eta)} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \text{sgn}(\boldsymbol{\mu})\text{sgn}(\boldsymbol{\nu}) \times \prod_{t=1}^{n} \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda_t)d\boldsymbol{\lambda} \quad (133)$$

where the summation over $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ is for all permutations of the integers $(1, \dots, n)$ and $\text{sgn}(\cdot)$ denotes the sign of the permutation. Noting the symmetry of the integrand in (133)

$$F_{\lambda_n}(\eta) = 1 - \sum_{\boldsymbol{\iota}\in\mathcal{I}} \frac{K^{(\boldsymbol{\iota})}}{n!} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \text{sgn}(\boldsymbol{\mu})\text{sgn}(\boldsymbol{\nu}) \times \prod_{t=1}^{n} \int_\eta^\infty \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda_t)d\lambda \quad (134)$$

$$= 1 - \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})}\text{sgn}(\boldsymbol{\mu}) \sum_{\boldsymbol{\nu}} \text{sgn}(\boldsymbol{\nu}) \times \prod_{t=1}^{n} \int_\eta^\infty \xi_{\mu_t,\nu_t}^{(\boldsymbol{\iota})}(\lambda)d\lambda \quad (135)$$

and using the definition of determinant in Definition A.1 the proof is completed.

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \frac{(-1)^{n(n-1)/2}}{\prod_{i=1}^{n}(\psi_i\sigma_i)^n}\sum_{k=0}^{\infty}\sum_{\boldsymbol{\kappa}\in\mathcal{K}(k)}\frac{\prod_{i=1}^{n}(-1)^{\kappa_i}}{\prod_{i=1}^{n}\kappa_i!|\mathbf{V}(\boldsymbol{\kappa})|}\frac{|\mathbf{K}^{(\boldsymbol{\kappa})}(\boldsymbol{\psi}^{-1})||\mathbf{K}^{(\boldsymbol{\kappa})}(\boldsymbol{\sigma}^{-1})|}{|\mathbf{V}(\boldsymbol{\psi}^{-1})||\mathbf{V}(\boldsymbol{\sigma}^{-1})|}|\mathbf{K}^{(\boldsymbol{\kappa})}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \quad (138)$$

B.6 Proof of Corollary 3.3

Proof: The marginal pdf of the kth largest eigenvalue can be obtained from its marginal cdf as

$$f_{\lambda_k}(\eta) = \frac{d}{d\eta}F_{\lambda_k}(\eta). \quad (136)$$

Then, the proof follows from using the expression for the marginal cdf of the kth largest eigenvalue in Theorem 3.2 and the derivative of a determinant in Lemma A.1.

Appendix C
Note on Double-Correlated Rayleigh Fading MIMO Channels

Consider a double-correlated Rayleigh fading MIMO channel, i.e.,

$$\mathbf{H} = \begin{cases}\boldsymbol{\Psi}^{1/2}\mathbf{H}_{\mathsf{w}}\boldsymbol{\Sigma}^{1/2} & n_{\mathsf{R}} \le n_{\mathsf{T}}\\ \boldsymbol{\Sigma}^{1/2}\mathbf{H}_{\mathsf{w}}\boldsymbol{\Psi}^{1/2} & n_{\mathsf{R}} > n_{\mathsf{T}}\end{cases} \quad (137)$$

where $\boldsymbol{\Psi} = (\boldsymbol{\Psi}^{1/2})(\boldsymbol{\Psi}^{1/2})^\dagger$ is the $n \times n$ positive definite correlation matrix with $n = \min(n_{\mathsf{T}}, n_{\mathsf{R}})$, $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}^{1/2})(\boldsymbol{\Sigma}^{1/2})^\dagger$ is the $m \times m$ positive definite correlation matrix with $m = \max(n_{\mathsf{T}}, n_{\mathsf{R}})$, and $\mathbf{H}_{\mathsf{w}}$ is a $n_{\mathsf{R}} \times n_{\mathsf{T}}$ random channel matrix with i.i.d. zero-mean unit-variance complex Gaussian entries. Then, the joint eigenvalue distribution of the matrix $\mathbf{H}^\dagger\mathbf{H}$ for the case[14] $n = m$ is given by [30, eq. (56)], [53, eq. (57)] [see (138) at the top of the page], where the summation over $\boldsymbol{\kappa} \in \mathcal{K}(k)$ is for all strictly ordered partitions $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_n)$ with $\kappa_1 > \cdots > \kappa_n$ and $\kappa_1 + \cdots + \kappa_n = k$, $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Definition A.2), matrix $\mathbf{K}^{(\boldsymbol{\kappa})}(\cdot)$ $(n \times n)$ is defined as

$$[\mathbf{K}^{(\boldsymbol{\kappa})}(\mathbf{x})]_{u,v} = x_u^{\kappa_v} \quad \text{for } u, v = 1, \ldots, n \quad (139)$$

and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$ denote the eigenvalues of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$ ordered such that $(\sigma_1 > \cdots > \sigma_n > 0)$ and $(\psi_1 > \cdots > \psi_n > 0)$. Identifying terms, the expression in (138) coincides with the general pdf given in Assumption 3.1 by defining the set $\mathcal{I}$ as

$$\mathcal{I} = \{(\iota_1, \ldots, \iota_n) \in \mathbb{N}^n | (\iota_1 > \cdots > \iota_n) \text{ and } (\iota_1 + \cdots + \iota_n = k) \text{ for } k = 0, 1, \ldots\} \quad (140)$$

with cardinality $|\mathcal{I}| = \sum_{k=0}^{\infty}|\mathcal{K}(k)|$, the constant $K^{(\boldsymbol{\iota})}$ as

$$K^{(\boldsymbol{\iota})} = \frac{(-1)^{n(n-1)/2}\prod_{i=1}^{n}(-1)^{\iota_i}}{\prod_{i=1}^{n}\iota_i!\prod_{i<j}^{n}(\iota_j - \iota_i)}\frac{|\mathbf{K}^{(\boldsymbol{\iota})}(\boldsymbol{\psi}^{-1})||\mathbf{K}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma}^{-1})|}{|\mathbf{V}(\boldsymbol{\psi}^{-1})||\mathbf{V}(\boldsymbol{\sigma}^{-1})|} \times \prod_{i=1}^{n}(\psi_i\sigma_i)^{-n} \quad (141)$$

the function $\varphi(\lambda) = 1$, and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda}) = \mathbf{K}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$. Hence, it follows that

$$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1} = \lambda^{\iota_u + v - 1}. \quad (142)$$

[14]The case $n < m$ follows from applying the standard limiting approach (see [53, App. A]).

In order to derive the marginal cdf and pdf of the kth largest eigenvalue using the results presented in Section III-B, we only have to particularize $\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda$ and $\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda$. Unfortunately, integrating over the λ_i's before summing over $\iota \in \mathcal{I}$ seems problematic since some of the integrals are unbounded above. However, as noted in [30, Lem. 5], the joint pdf of the ordered eigenvalues in (138) is bounded by an exponential function of any λ_i as λ_i becomes arbitrarily large and hence integrable. To circumvent this discrepancy we can introduce a cutoff function $g(\lambda)$ which is unity as $\lambda \to 0$ and tends to zero faster than a power law as $\lambda \to \infty$. This makes all terms finite and allows us to freely interchange the order of summation and integration. For instance, we can choose $g(\lambda) = e^{-\delta\lambda}$ and calculate the integrals for any positive δ. Then we only have to sum the infinite series using the Cauchy-Binet Theorem [30, Lem. 3] and set $\delta = 0$ at the end of the calculation (see an example of this procedure in [30, Sec. 5]).

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