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# Contents

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# Part I

# 1 Competitive Optimization of Cognitive Radio MIMO Systems via Game Theory

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Game theory is a field of applied mathematics that describes and analyzes scenarios with interactive decisions. In recent years, there has been a growing interest in adopting cooperative and non-cooperative game theoretic approaches to model many communications and networking problems, such as power control and resource sharing in wireless/wired and peer-to-peer networks. In this chapter we show how many challenging unsolved resource allocation problems in the emerging field of Cognitive Radio (CR) networks fit naturally in the game theoretical paradigm. This provides us with all the mathematical tools necessary to analyze the proposed equilibrium problems for CR systems (e.g., existence and uniqueness of the solution) and to devise distributed algorithms along with their convergence properties. The proposed algorithms differ in performance, level of protection of the primary users, computational effort and signaling among primary and secondary users, convergence analysis, and convergence speed; which makes them suitable for many different CR systems. We also propose a more general framework suitable for investigating and solving more sophisticated equilibrium problems in CR systems when classical game theory may fail, based on variation inequality (VI for short) that constitutes a very general class of problems in nonlinear analysis.

## 1.1 Introduction and Motivation

In recent years, increasing demand of wireless services has made the radio spectrum a very scarce and precious resource. Moreover, most current wireless networks characterized by fixed spectrum assignment policies are known to be very inefficient considering that licensed bandwidth demands are highly varying along the time or space dimensions (according to Federal Communications Commission (FCC), only 15% to 85% of the licensed spectrum is utilized on the average [1]). Many recent works [2, 3, 4] have recognized that the most appropriate approach to tackle the great spectrum variability in time and space calls for *dynamic* access strategies that adapt transmission parameters (e.g., operating spectrum, modulation, transmission power and communication technology) based on knowledge of the electromagnetic environment.

Cognitive Radio (CR) originated as a possible solution to this problem [5] obtained by endowing the radio nodes with "cognitive capabilities", e.g., the ability to sense the electromagnetic environment, make short term predictions, and react intelligently in order to optimize the usage of the available resources. Multiple debated positions have been proposed for implementing the CR idea [2, 3, 4], depending on the policy to be followed with respect to the *licensed* users, i.e., the users who have acquired the right to transmit over specific portions of the spectrum buying the corresponding license. The most common strategies adopt a hierarchical access structure, distinguishing between *primary* users, or legacy spectrum holders, and *secondary* users, who access the licensed spectrum dynamically, under the constraint of not inducing any significant Quality of Service (QoS) degradations to the primary users.

Within this context, adopting a general multiple input-multiple output (MIMO) channel, is natural to model the system of cognitive secondary users as vector interference channel, where the transmission over the generic $q$-th MIMO channel with $n_{T_q}$ transmit and $n_{R_q}$ receive dimensions is given by the following baseband complex-valued signal model:

$$\mathbf{y}_q = \mathbf{H}_{qq}\mathbf{x}_q + \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{x}_r + \mathbf{n}_q, \tag{1.1}$$

where $\mathbf{x}_q \in \mathbb{C}^{n_{T_q}}$ is the signal transmitted by source $q$, $\mathbf{y}_q \in \mathbb{C}^{n_{R_q}}$ is the received signal by destination $q$, $\mathbf{H}_{qq} \in \mathbb{C}^{n_{R_q}\times n_{T_q}}$ is the channel matrix between the $q$-th transmitter and the intended receiver, $\mathbf{H}_{rq} \in \mathbb{C}^{n_{R_q}\times n_{T_r}}$ is the cross-channel matrix between source $r$ and destination $q$, and $\mathbf{n}_q \in \mathbb{C}^{n_{R_q}}$ is a zero-mean circularly symmetric complex Gaussian noise vector with arbitrary (nonsingular) covariance matrix $\mathbf{R}_{n_q}$, collecting the effect of both thermal noise and interference generated by the primary users. The first term on the right-hand side of (1.1) is the useful signal for link $q$, the second and third terms represent the Multi-User Interference (MUI) received by secondary user $q$ and generated by the other secondary users and the primary users, respectively. The power constraint for each transmitter is

$$\mathcal{E}\left\{\|\mathbf{x}_q\|_2^2\right\} = \mathsf{Tr}\left(\mathbf{Q}_q\right) \leq P_q, \tag{1.2}$$

where $\mathcal{E}\left\{\cdot\right\}$ denotes the expectation value, $\mathsf{Tr}\left(\cdot\right)$ is the trace operator, $\mathbf{Q}_q$ is the covariance matrix of the transmitted signal by user $q$, and $P_q$ is the transmit power in units of energy per transmission.

The model in (1.1) represents a fairly general MIMO setup, describing multiuser transmissions (e.g., peer-to-peer links, multiple access, or broadcast channels) over multiple channels, which may represent frequency channels (as in OFDM systems) [6, 7, 8, 9], time slots (as in TDMA systems) [6, 7, 9], or spatial channels (as in transmit/receive beamforming systems) [10].

Due to the distributed nature of the CR system, with neither a centralized control nor coordination among the secondary users, we focus on transmission techniques where no interference cancellation is performed and the MUI is treated as additive colored noise at each receiver. Each channel is assumed to change sufficiently slowly to be considered fixed during the whole transmission. Moreover, perfect channel state information at both transmitter and receiver sides of each link is assumed. This includes the direct channel $\mathbf{H}_{qq}$ (but not the cross-channels $\{\mathbf{H}_{rq}\}_{r\neq q}$ from the other users) as well as the covariance matrix of noise plus MUI

$$\mathbf{R}_{-q}(\mathbf{Q}_{-q}) \triangleq \mathbf{R}_{n_q} + \sum_{r\neq q} \mathbf{H}_{rq}\mathbf{Q}_r\mathbf{H}_{rq}^H. \tag{1.3}$$

Within the assumptions made above, the maximum information rate on link $q$ for a given set of user covariance matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_Q$, is [11]

$$R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) = \log\det\left(\mathbf{I} + \mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{Q}_q\right) \tag{1.4}$$

where $\mathbf{Q}_{-q} \triangleq (\mathbf{Q}_r)_{r\neq q}$ is the set of all the users covariance matrices, except the $q$-th one.

In this chapter, we focus on opportunistic resource allocation techniques in hierarchical CR systems as given in (1.1). In particular, our interest is in devising the most appropriate form of concurrent communications of cognitive users competing over the physical resources that primary users make available, under the constraint that the degradation induced on the primary users' performance is null or tolerable [2, 3]. While the definition of degradation may be formulated mathematically in a number of ways, one common definition involves the imposition of some form of interference constraints on the secondary users, whose choice and implementation are a complex and open regulatory issue. Both deterministic and probabilistic interference constraints have been suggested in the literature [2, 3]. In this chapter, we will consider in detail deterministic interference constraints, as described next .

### 1.1.1 Interference constraints: individual and conservative versus global and flexible

We envisage two classes of interference constraints termed *individual conservative* constraints and *global flexible* constraints.

**Individual conservative constraints:** These constraints are defined individually for each secondary user (with the disadvantage that sometimes may result too conservative) to control the overall interference caused on the primary receivers. Specifically, we have

- *Null shaping constraints*:

$$\mathbf{U}_q^H\mathbf{Q}_q = \mathbf{0}, \tag{1.5}$$

where $\mathbf{U}_q \in \mathbb{C}^{n_{T_q} \times r_{U_q}}$ is a tall matrix whose columns represent the spatial and/or the frequency "directions" along which user $q$ is not allowed to transmit.

- *Soft and peak power shaping constraints*:

$$\mathsf{Tr}\left(\mathbf{G}_q^H \mathbf{Q}_q \mathbf{G}_q\right) \leq P_{\mathrm{SU},q}^{\mathrm{ave}} \quad \text{and} \quad \lambda_{\max}\left(\mathbf{G}_q^H \mathbf{Q}_q \mathbf{G}_q\right) \leq P_{\mathrm{SU},q}^{\mathrm{peak}} \tag{1.6}$$

  which represent a relaxed version of the null constraints with a constraint on the total average and peak average power radiated along the range space of matrix $\mathbf{G}_q \in \mathbb{C}^{n_{T_q} \times n_{G_q}}$, where $P_{\mathrm{SU},q}^{\mathrm{ave}}$ and $P_{\mathrm{SU},q}^{\mathrm{peak}}$ are the maximum average and average peak power respectively that can be transmitted along the spatial and/or the frequency directions spanned by $\mathbf{G}_q$.

The null constraints are motivated in practice by the interference-avoiding paradigm in CR communications (also called white-space filling approach) [4, 12]: CR nodes sense the spatial, temporal or spectral voids and adjust their transmission strategy to fill in the sensed white spaces. This white-space filling strategy is often considered to be the key motivation for the introduction and development of CR idea and has already been adopted as a core platform in emerging wireless access standards such as the IEEE 802.22-Wireless Regional Area Networks (WRANs) [13]. Observe that the structure of the null constraints in (1.5) has a very general form and includes, as particular cases, the imposition of nulls over: 1) frequency bands occupied by the primary users (the range space of $\mathbf{U}_q$ coincides with the subspace spanned by a set of IDFT vectors); 2) the time slots used by the primary users (the set of canonical vectors); 3) angular directions identifying the primary receivers as observed from the secondary transmitters (the set of steering vectors representing the directions of the primary receivers as observed from the secondary transmitters).

Opportunistic communications allow simultaneous transmissions between primary and secondary users, provided that the required QoS of the primary users is preserved (also called interference-temperature controlled transmissions [2, 12, 14]). This can be done using the individual soft shaping constraints expressed in (1.6) that represent a constraint on the total average and peak average power allowed to be radiated (projected) along the directions spanned by the column space of matrix $\mathbf{G}_q$. For example, in a MIMO setup, the matrix $\mathbf{G}_q$ in (1.6) may contain, in its columns, the steering vectors identifying the directions of the primary receivers. By using these constraints, we assume that the power thresholds $P_{\mathrm{SU},q}^{\mathrm{ave}}$ and $P_{\mathrm{SU},q}^{\mathrm{peak}}$ at each secondary transmitter have been fixed in advance (imposed, e.g., by the network service provider, or legacy systems, or the spectrum body agency) so that the interference temperature limit constraints at the primary receivers are met. For example, a possible (but conservative) choice for $P_{\mathrm{SU},q}^{\mathrm{ave}}$'s and $P_{\mathrm{SU},q}^{\mathrm{peak}}$'s is $P_{\mathrm{SU},q}^{\mathrm{ave}} = P_{\mathrm{PU}}^{\mathrm{ave}}/Q$ and $P_{\mathrm{SU},q}^{\mathrm{peak}} = P_{\mathrm{PU}}^{\mathrm{peak}}/Q$ for all $q$, where $Q$ is the number of active secondary users, and $P_{\mathrm{PU}}^{\mathrm{ave}}$ and $P_{\mathrm{PU}}^{\mathrm{peak}}$ are the overall maximum average and peak average interference tolerable by the primary user. The assumption made above is motivated by all the practical CR scenarios where primary terminals are oblivious to the presence of secondary users, thus behaving as if no secondary activity was present (also called *commons model*).

The imposition of the individual interference constraints requires an opportunity identification phase, through a proper sensing mechanism: Secondary users need to reliably detect weak primary signals of possibly different type over a targeted region and wide frequency band in order to identify white-space halls. Examples of solutions to this problem have recently been proposed in [3, 15, 14, 16]. The study of sensing in CR networks goes beyond the scope of this chapter. Thus, hereafter, we assume perfect sensing from the secondary users.

Individual interference constraints (possibly in addition with the null constraints) lead to totally distributed algorithms with no coordination between the primary and the secondary users, as we will show in the forthcoming sections. However, sometimes, they may be too restrictive and thus marginalize the potential gains offered by the dynamic resource assignment mechanism. Since the interference temperature limit [2] is given by the *aggregate* interference induced by *all* the active secondary users to the primary users' receivers, it seems natural to limit instead such an aggregate interference, rather than the individual soft power and peak power constraints. This motivates the following global interference constraints.

**Global flexible constraints**: These constraints, as opposed to the individual ones, are defined globally over all the secondary users:

$$\sum_{q=1}^{Q} \mathsf{Tr}\left(\mathbf{G}_{q,p}^{H}\mathbf{Q}_{q}\mathbf{G}_{q,p}\right) \leq P_{\mathrm{PU},p}^{\mathrm{ave}} \qquad \text{and} \qquad \sum_{q=1}^{Q} \lambda_{\max}\left(\mathbf{G}_{q,p}^{H}\mathbf{Q}_{q}\mathbf{G}_{q,p}\right) \leq P_{\mathrm{PU},p}^{\mathrm{peak}}, \tag{1.7}$$

where $P_{\mathrm{PU},p}^{\mathrm{ave}}$ and $P_{\mathrm{PU},p}^{\mathrm{peak}}$ are the maximum average and peak average interference tolerable by the $p$-th primary user. As we will show in the forthcoming sections, these constraints in general lead to better performance of secondary users than imposing the conservative individual constraints. However, this gain comes at a price: The resulting algorithms require some signaling (albeit very reduced) from the primary to the secondary users. They can be employed in all CR networks where an interaction between the primary and the secondary users is allowed, as, e.g., in the so-called *property-right* CR model (or *spectrum leasing*), where primary users own the spectral resource and possibly decide to lease part of it to secondary users in exchange for appropriate remuneration.

### 1.1.2 System design: A game theoretical approach

Given the CR model in (1.1), the system design consists in finding out the set of covariance matrices of the secondary users satisfying a prescribed optimality criterion, under power and interference constraints in (1.2) and (1.5)-(1.7). One approach would be to design the transmission strategies of the secondary users using global optimization techniques. However, this has some practical issues that are insurmountable in the CR context. First of all, it requires the presence of a central node having full knowledge of all the channels and interference structure at every receiver. But this poses a serious implementation problem in terms of scalability and amount of signaling to be exchanged among the nodes. The required extra signaling could, in the end, jeopardize the promise for higher efficiency. On top of that, recent results in [17] have shown that the network utility maximization based on the rate functions is an NP-hard problem, under different choices of the system utility function; which means that there is no hope to obtain an algorithm, even centralized, that can efficiently compute a globally optimal solution. Consequently, suboptimal algorithms have been proposed (see, e.g., [18, 19]), but they are centralized and may converge to poor spectrum sharing strategies, due to the nonconvexity of the optimization problem. Thus, it seems natural to concentrate on decentralized strategies, where the cognitive users are able to self-enforce the negotiated agreements on the usage of the available resources (time, frequency, and space) without the intervention of a centralized authority. The philosophy underlying this approach is a *competitive optimality* criterion, as every user aims for the transmission strategy that unilaterally maximizes his own payoff function. This form of equilibrium is, in fact, the well-known concept of Nash Equilibrium (NE) in game theory.

Because of the inherently competitive nature of multi-user systems, it is not surprising indeed that game theory has been already adopted to solve distributively many resource allocation problems in communications. An early application of game theory in a communication system is [20], where the information rates of the users were maximized with respect to the power allocation in a DSL system modeled as a frequency-selective (in practice, multicarrier) Gaussian interference channel. Extension of the basic problem to ad-hoc frequency-selective and MIMO networks were given in [6, 7, 8, 9, 21] and [10, 22, 23, 24], respectively. However, results in the cited papers have been recognized not to be applicable to CR systems because they do not provide any mechanism to control the amount of interference generated by the secondary users on the primary users [2].

### 1.1.3 Outline

Within the CR context introduced so far, we formulate in the next sections the optimization problem for the transmission strategies of the secondary users under different combinations of power and individual/global interference constraints. Using the game-theoretic concept of NE as competitive optimality criterion, we propose various equilibrium problems that differ in the achievable trade-off between performance and amount of signaling among primary and secondary users. Using results from game theory and VI theory, we study, for each equilibrium problem, properties of the solution (e.g., existence and uniqueness) and propose many iterative, possibly asynchronous, distributed algorithms along with their convergence properties.

The rest of the chapter is organized as follows. Section 1.2 introduces some basic concepts and results on non-cooperative strategic form games that will be used extensively through the whole chapter. Section 1.3 deals with transmissions over unlicensed bands, where there are no constraints on the interference generated by the secondary users on the primary users. Section 1.4 considers CR systems under different individual interference constraints and proposes various NE problems. Section 1.5 focuses on the more challenging design of CR systems under global interference constraints and studies the NE problem using VI theory. Finally, Section 1.6 draws some conclusions.

### 1.1.4 Notation

The following notation is used in the chapter. Uppercase and lowercase boldface denote matrices and vectors respectively. The operators $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^\sharp$, $\mathcal{E}\{\cdot\}$, and $\mathrm{Tr}(\cdot)$ are conjugate, Hermitian, Moore-Penrose pseudoinverse [25], expectation, and trace operators, respectively. The range space and null space are denoted by $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$, respectively. The set of eigenvalues of a $n\times n$ Hermitian matrix $\mathbf{A}$ is denoted by $\{\lambda_i(\mathbf{A})\}_{i=1}^n$, whereas the maximum and the minimum eigenvalue are denoted by $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$, respectively. The operators $\leq$ and $\geq$ for vectors and matrices are defined component-wise, while $\mathbf{A}\succeq\mathbf{B}$ (or $\mathbf{A}\preceq\mathbf{B}$) means that $\mathbf{A}-\mathbf{B}$ is positive (or negative) semidefinite. The operator $\mathrm{Diag}(\cdot)$ is the diagonal matrix with the same diagonal elements as the matrix (or vector) argument; $\mathrm{bdiag}(\mathbf{A},\mathbf{B},\ldots)$ is the diagonal matrix, whose diagonal blocks are the matrices $\mathbf{A}$, $\mathbf{B}$, ...; the operator $\perp$ for vector and matrices means that two vectors $\mathbf{x}$ and $\mathbf{y}$ or two matrices $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, i.e., $\mathbf{x}\perp\mathbf{y}\Leftrightarrow\mathbf{x}^H\mathbf{y}=0$ and $\mathbf{A}\perp\mathbf{B}\Leftrightarrow\mathrm{Tr}(\mathbf{A}^H\mathbf{B})=0$ (note that $\mathrm{Tr}(\mathbf{A}^H\mathbf{B})=0\Leftrightarrow\mathbf{A}^H\mathbf{B}=\mathbf{0}$ if $\mathbf{A},\mathbf{B}\succeq\mathbf{0}$). The operators $(\cdot)^+$ and $[\,\cdot\,]_a^b$, with $0\leq a\leq b$, are defined as $(x)^+\triangleq\max(0,x)$ and $[\,\cdot\,]_a^b\triangleq\min\left(b,\max(x,a)\right)$, respectively; when the argument of the operators is a vector or a matrix, then they are assumed to be applied component-wise. The spectral radius of a matrix $\mathbf{A}$ is denoted by $\rho(\mathbf{A})$, and is defined as $\rho\left(\mathbf{A}\right)\triangleq\max\{|\lambda|:\lambda\in\sigma(\mathbf{A})\}$, with $\sigma(\mathbf{A})$ denoting the spectrum (set of eigenvalues) of $\mathbf{A}$ [26]. The operator $\mathbf{P}_{\mathcal{N}(\mathbf{A})}$ (or $\mathbf{P}_{\mathcal{R}(\mathbf{A})}$) denotes the orthogonal projection onto the null space (or the range space) of matrix $\mathbf{A}$ and it is given by $\mathbf{P}_{\mathcal{N}(\mathbf{A})}=\mathbf{N}_A(\mathbf{N}_A^H\mathbf{N}_A)^{-1}\mathbf{N}_A^H$ (or $\mathbf{P}_{\mathcal{R}(\mathbf{A})}=\mathbf{R}_A(\mathbf{R}_A^H\mathbf{R}_A)^{-1}\mathbf{R}_A^H$), where $\mathbf{N}_A$ (or $\mathbf{R}_A$) is any matrix whose columns are linear independent vectors spanning $\mathcal{N}(\mathbf{A})$ (or $\mathcal{R}(\mathbf{A})$) [26]. The operator $[\mathbf{X}]_\mathcal{Q}=\mathrm{argmin}_{\mathbf{Z}\in\mathcal{Q}}\left\|\mathbf{Z}-\mathbf{X}\right\|_F$ denotes the matrix projection with respect to the Frobenius norm of matrix $\mathbf{X}$ onto the (convex) set $\mathcal{Q}$, where $\|\mathbf{X}\|_F$ is defined as $\|\mathbf{X}\|_F\triangleq\left(\mathrm{Tr}(\mathbf{X}^H\mathbf{X})\right)^{1/2}$ [26]. We denote by $\mathbf{I}_n$ the $n\times n$ identity matrix and by $r_X\triangleq\mathrm{rank}(\mathbf{X})$ the rank of matrix $\mathbf{X}$. The sets $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$, $\mathbb{R}_{++}$, $\mathbb{N}_+$, $\mathbb{S}^n$, and $\mathbb{S}_+^n$ (or $\mathbb{S}_{++}^n$) stand for the set of complex, real, nonnegative real, nonpositive real, positive real, nonnegative integer numbers, and $n \times n$ complex Hermitian, and positive semidefinite (or definite) matrices, respectively.

## 1.2 Strategic Non-cooperative Games: Basic Solution Concepts and Algorithms

In this section we introduce non-cooperative strategic form games and provide some basic results dealing with the solution concept of Nash equilibrium (NE). We do not attempt to cover such topics in encyclopedic depth. We have restricted our exposition only to those results (not necessary the most general ones in the literature of game theory) that will be used in the forthcoming sections to solve the proposed CR problems and make this chapter self-contained. The literature on pure Nash equilibrium problem is enormous; we refer the interested reader to [27, 28, 29, 30, 31, 32] as entry points. A more recent survey on current state-of-the-art results on non-cooperative games is [33].

A *non-cooperative strategic form* game models a scenario where all players act independently and simultaneously according to their own self-interests and with no a priori knowledge of other players strategies. Stated in mathematical terms, we have the following.

**Definition 1.1.** *A strategic form game is a triplet* $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$, *where:*

- $\Omega = \{1, 2, \ldots, Q\}$ *is the (finite) set of players;*
- $\mathcal{Q}_i$ *is a non-empty set of the available (pure) strategies (actions) for player* $i$, *also called admissible strategy set of player* $i$ *(assumed here to be independent of the other players' strategies*[1]*);*
- $u_i : \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_Q \to \mathbb{R}$ *is the payoff (utility) function of player* $i$ *that depends in general on the strategies of all players.*

*We denote by* $\mathbf{x}_i \in \mathcal{Q}_i$ *a feasible strategy profile of player* $i$, *by* $\mathbf{x}_{-i} = (\mathbf{x}_j)_{j\neq i}$ *a tuple of strategies of all players except the* $i$*-th, and by* $\mathcal{Q} = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_Q$ *the set of feasible strategy profiles of all players. We use the notation* $\mathcal{Q}_{-i} = \mathcal{Q}_1 \times \mathcal{Q}_{i-1}, \mathcal{Q}_{i+1}, \cdots \times \mathcal{Q}_Q$ *to define the set of feasible strategy profiles of all players except the* $i$*-th. If all the strategy sets* $\mathcal{Q}_i$ *are finite, the game is called* finite*; otherwise* infinite.

The non-cooperative paradigm postulates the rationality of players' behaviors: Each player $i$ competes against the others by choosing a strategy profile $\mathbf{x}_i \in \mathcal{Q}_i$

[1] The focus on more general games where the strategy set of the players may depend on the other players' actions (usually termed as generalized Nash equilibrium problem) goes beyond the scope of this section. We refer the interested reader to [33] and references therein.

that maximizes his own payoff function $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$, given the actions $\mathbf{x}_{-i} \in \mathcal{Q}_{-i}$ of the other players. A non-cooperative strategic form game can be then represented as a set of *coupled* optimization problems

$$(\mathcal{G}) : \quad \begin{array}{ll} \underset{\mathbf{x}_i}{\text{maximize}} & u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{Q}_i, \end{array} \qquad \forall i \in \Omega. \tag{1.8}$$

The problem of the $i$-th player in (1.8) is to determine, for each fixed but arbitrary tuple $\mathbf{x}_{-i}$ of the other players' strategies, an optimal strategy $\mathbf{x}_i^\star$ that solves the maximization problem in the variable $\mathbf{x}_i \in \mathcal{Q}_i$.

A desirable solution to (1.8) is one in which every (rational) player acts in accordance with his incentives, maximizing his own payoff function. This idea is best captured by the notion of Nash equilibrium, formally defined next.

**Definition 1.2.** *Given a strategic form game $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$, an action profile $\mathbf{x}^\star \in \mathcal{Q}$ is a pure strategy Nash equilibrium of $\mathcal{G}$ if the following condition holds for all $i \in \Omega$:*

$$u_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \geq u_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{x}_i \in \mathcal{Q}_i. \tag{1.9}$$

In words, a Nash equilibrium is a (self-enforcing) strategy profile with the property that no *single* player can unilaterally benefit from a deviation from it, given that all the other players act according to it. It is useful to restate the definition of NE in terms of a fixed-point solution to the best-response multifunction (i.e., point-to-set map).

**Definition 1.3.** *Let $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$ be a strategic form game. For any given $\mathbf{x}_{-i} \in \mathcal{Q}_{-i}$, define the best-response multifunction $\mathcal{B}_i(\mathbf{x}_{-i})$ of player $i$ as*

$$\mathcal{B}_i(\mathbf{x}_{-i}) \triangleq \left\{ \mathbf{x}_i \in \mathcal{Q}_i \,|\, u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \geq u_i(\mathbf{y}_i, \mathbf{x}_{-i}), \ \forall \mathbf{y}_i \in \mathcal{Q}_i \right\}, \tag{1.10}$$

*i.e., the set of the optimal solutions to the $i$-th optimization problem in (1.8), given $\mathbf{x}_{-i} \in \mathcal{Q}_{-i}$ (assuming that the maximum in (1.10) exists). We also introduce the multifunction mapping $\mathcal{B} : \mathcal{Q} \rightrightarrows \mathcal{Q}$ defined as $\mathcal{B}(\mathbf{x}) : \mathcal{Q} \ni \mathbf{x} \rightrightarrows \mathcal{B}_1(\mathbf{x}_{-1}) \times \mathcal{B}_2(\mathbf{x}_{-2}) \times \cdots \times \mathcal{B}_Q(\mathbf{x}_{-Q})$. A strategy profile $\mathbf{x}^\star \in \mathcal{Q}$ is a pure strategy NE of $\mathcal{G}$ if and only if*

$$\mathbf{x}^\star \in \mathcal{B}(\mathbf{x}^\star). \tag{1.11}$$

*If $\mathcal{B}(\mathbf{x})$ is a single-valued function (denoted, in such a case, as $\mathsf{B}(\mathbf{x})$), then $\mathbf{x}^\star \in \mathcal{Q}$ is a pure strategy NE if and only if $\mathbf{x}^\star = \mathsf{B}(\mathbf{x}^\star)$.*

This alternative formulation of the equilibrium solution may be useful to address some essential issues of the equilibrium problems, such as the existence and uniqueness of solutions, stability of equilibria, design of effective algorithms for finding equilibrium solutions, thus paving the way to the application of the fixed-point machinery. In fact, in general, the uniqueness or even the existence of a pure strategy Nash equilibrium is not guaranteed; neither is convergence to an equilibrium when one exists (some basic existence and uniqueness results in the form useful for our purposes will be discussed in Section 1.2.1). Sometimes, however, the structure of a game is such that one is able to establish one or more of these desirable properties, as for example happens in potential games [34] or supermodular games [35], which have recently received some attention in the signal processing and communication communities as a useful tool to solve various power control problems in wireless communications [36, 37, 38].

Finally, it is important to remark that, even when the NE is unique, it need not be Pareto efficient.

**Definition 1.4.** *Given a strategic form game* $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$*, and two action profiles* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathcal{Q}$*,* $\mathbf{x}^{(1)}$ *is said to be Pareto-dominant on* $\mathbf{x}^{(2)}$ *if* $u_i(\mathbf{x}^{(1)}) \geq u_i(\mathbf{x}^{(2)})$ *for all* $i \in \Omega$*, and* $u_j(\mathbf{x}^{(1)}) > u_j(\mathbf{x}^{(2)})$ *for at least one* $j \in \Omega$*. A strategy profile* $\mathbf{x} \in \mathcal{Q}$ *is Pareto efficient (optimal) if there exists no other feasible strategy that dominates* $\mathbf{x}$*.*

This means that there might exist proper coalitions among the players yielding an outcome of the game with the property that there is always (at least) one player who cannot profit by deviating by that action profile. In other words, a NE may be vulnerable to deviations by coalitions of players, even if it is not vulnerable to unilateral deviation by a single player. However, Pareto optimality in general comes at the price of a centralized optimization, which requires the full knowledge of the strategy sets and the payoff functions of all players. Such a centralized approach is not applicable in many practical applications in signal processing and communications, e.g., in emerging wireless networks, such as sensor networks, ad-hoc networks, cognitive radio systems, and pervasive computing systems. The NE solutions, instead, are more suitable to be computed using a decentralized approach that requires no exchange of information among the players. Different refinements of the NE concept have also been proposed in the literature to overcome some shortcomings of the NE solution (see, e.g., [29, 39]).

The definition of NE as given in Definition 1.2 covers only pure strategies. One can restate the NE concept to contain mixed strategies, i.e. the possibility of choosing a randomization over a set of pure strategies. A mixed strategy NE of a strategic game is then defined as a NE of its mixed extension (see, e.g., [27, 40] for details). An interesting result dealing with Nash equilibria in mixed strategy is that every *finite* strategic game has a mixed strategy NE [41], which in general does not hold for pure strategies. In this chapter, we focus only on pure strategy Nash equilibria of non-cooperative strategic form games with infinite strategy sets.

### 1.2.1 Existence and uniqueness of the NE

Several different approaches have been proposed in the literature to study properties of the Nash solutions, such as existence, (local/global) uniqueness, and devise numerical algorithms to solve the NE problem. The three most frequent methods are: i) interpreting the Nash equilibria as fixed-point solutions, ii) reducing the NE problem to a variational inequality problem, and iii) transforming the equilibrium problem into an optimization problem. Each of these methods leads to alternative conditions and algorithms. We focus next only on the former approach and refer the interest reader to [32, 33, 42] and [27, 43] as examples of the application of the other techniques.

**Existence of a Nash solution.** The study of the existence of equilibria under weaker and weaker assumptions has been investigated extensively in the literature (see, e.g., [41, 44, 45, 46, 47]). A good overview of the relevant literature is [33]. For the purpose of this chapter, it is enough to recall an existence result that is one of the simplest of the genre, based on the interpretation of the NE as a fixed-point of the best-response multifunction (cf. Definition 1.3) and the existence result from the Kakutani fixed-point theorem.

**Theorem 1.1** (Kakutani's Fixed Point Theorem)**.** *Given* $\mathcal{X} \subseteq \mathbb{R}^n$*, let* $\mathcal{S}(\mathbf{x}) : \mathcal{X} \ni \mathbf{x} \rightrightarrows \mathcal{S}(\mathbf{x}) \subseteq \mathcal{X}$ *be a multifunction. Suppose that the following hold:*

**(a)** $\mathcal{X}$ *is a nonempty, compact, and convex set;*
**(b)** $\mathcal{S}(\mathbf{x})$ *is a convex-valued correspondence (i.e.,* $\mathcal{S}(\mathbf{x})$ is a convex set for all $\mathbf{x} \in \mathcal{X}$*) and has a closed graph (i.e., if* $\{\mathbf{x}^{(n)}, \mathbf{y}^{(n)}\} \to \{\mathbf{x}, \mathbf{y}\}$ with $\mathbf{y}^{(n)} \in \mathcal{S}(\mathbf{x}^{(n)})$, then $\mathbf{y} \in \mathcal{S}(\mathbf{x})$*).*

*Then, there exists a fixed-point of* $\mathcal{S}(\mathbf{x})$*.*

Given $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$ with best-response $\mathcal{B}(\mathbf{x})$, it follows from Definition 1.3 and Theorem 1.1 that conditions (a) and (b) applied to $\mathcal{B}(\mathbf{x})$ are sufficient to guarantee the existence of a NE. To make condition (b) less abstract, we use Theorem 1.1 in a simplified form, which provides a set of sufficient conditions for assumption (b) that represent classical existence results in the game theory literature [44, 45, 46, 47].

**Theorem 1.2** (Existence of a NE)**.** *Consider a strategic form game* $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$*, where* $\Omega$ *is a finite set. Suppose that*

**(a)** *each* $\mathcal{Q}_i$ *is a non-empty, compact, and convex subset of a finite-dimensional Euclidean space;*
**(b)** *one of the two following conditions holds:*
  **1.** *each payoff function* $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ *is continuous on* $\mathcal{Q}$*, and, for any given* $\mathbf{x}_{-i} \in \mathcal{Q}_{-i}$*, it is quasi-concave on* $\mathcal{Q}_i$*;*

**2.** *each payoff function $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is continuous on $\mathcal{Q}$, and, for any given $\mathbf{x}_{-i} \in \mathcal{Q}_{-i}$, the following optimization problem*

$$\max_{\mathbf{x}_i \in \mathcal{Q}_i} u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \tag{1.12}$$

*admits a unique (globally) optimal solution.*

*Then, game $\mathcal{G}$ admits a pure strategy NE.*

The assumptions in Theorem 1.2 are only sufficient for the existence of a fixed point. However, this does not mean that some of them can be relaxed. For example, the convexity assumption in the existence condition (Theorem 1.1(a) and Theorem 1.2(a)) cannot, in general, be removed, as the simple one-dimensional example $f(x) = -x$ and $\mathcal{X} = \{-c, c\}$, with $c \in \mathbb{R}$, shows. Furthermore, a pure strategy NE may fail to exist if the quasi-concavity assumption (Theorem 1.2(b.1)) is relaxed, as shown in the following example. Consider a two-player game, where the payers pick points $\mathbf{x}_1$ and $\mathbf{x}_2$ on the unit circle, and the payoff functions of the two players are $u_1(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_2\|$ and $u_2(\mathbf{x}_1, \mathbf{x}_2) = -\|\mathbf{x}_1 - \mathbf{x}_2\|$, where $\|\cdot\|$ denotes the Euclidean norm. In this game there is no pure strategy NE. In fact, if both players pick the same location, player 1 has incentive to deviate; whereas if they pick different locations, player 2 has incentive to deviate.

The relaxation of the assumptions in Theorem 1.2 has been the subject of a fairly intense study. Relaxations of the (i) continuity assumptions; (ii) compactness assumptions and (iii) quasi-concavity assumption have all been considered in the literature. The relevant literature is discussed in detail in [33]. More recent advanced results based on degree theory can be found in [42].

**Uniqueness of a Nash solution.** The study of uniqueness of a solution for the Nash problem is more involved and available results are scarce. Some classical papers on the subject are [46, 48, 49] and more recently [43, 50, 51], where different uniqueness conditions have been derived, most of them valid for games having special structure. Since the games considered in this chapter satisfy Theorem 1.2.(b.2), in the following we focus on this special class of games and provide some basic results, based on the uniqueness of fixed-points of single-valued functions. A simple uniqueness result is given in the following (see, e.g., [52, 53]).

**Theorem 1.3** (Uniqueness of the NE)**.** Let $\mathsf{B}(\mathbf{x}) : \mathcal{X} \ni \mathbf{x} \to \mathsf{B}(\mathbf{x}) \in \mathcal{X}$ *be a function, mapping $\mathcal{X} \subseteq \mathbb{R}^n$ into itself. Suppose that* $\mathsf{B}$ is a contraction in some vector norm $\|\cdot\|$, with modulus $\alpha \in [0, 1)$, i.e.,

$$\left\|\mathsf{B}(\mathbf{x}^{(1)}) - \mathsf{B}(\mathbf{x}^{(2)})\right\| \leq \alpha \left\|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right\|, \quad \forall \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathcal{X}. \tag{1.13}$$

*Then, there exists at most one fixed-point of* $\mathsf{B}$. *If, in addition, $\mathcal{X}$ is closed, then there exists a unique fixed-point.* □

Alternative sufficient conditions still requiring some properties on the best-response function $\mathsf{B}$ can be obtained, by observing that the fixed-point of $\mathsf{B}$ is unique if the function $\mathsf{T}(\mathbf{x}) \triangleq \mathbf{x} - \mathsf{B}(\mathbf{x})$ is one-to-one. Invoking results from mathematical analysis, many conditions can be obtained guaranteeing that $\mathsf{T}$ is one-to-one. For example, assuming that $\mathsf{T}$ is continuously differentiable and denoting by $\mathbf{J}(\mathbf{x})$ the Jacobian matrix of $\mathsf{T}$ at $\mathbf{x}$, some frequently applied conditions are the following: i) all leading principal minors of $\mathbf{J}(\mathbf{x})$ are positive (i.e., $\mathbf{J}(\mathbf{x})$ is a P-matrix [54]); ii) all leading principal minors of $\mathbf{J}(\mathbf{x})$ are negative (i.e., $\mathbf{J}(\mathbf{x})$ is a N-matrix [54]); iii) matrix $\mathbf{J}(\mathbf{x}) + \mathbf{J}(\mathbf{x})^T$ is positive (or negative) semidefinite, and between any pair of points $\mathbf{x}^{(1)} \neq \mathbf{x}^{(2)}$ there is a point $\mathbf{x}^{(0)}$ such that $\mathbf{J}(\mathbf{x}^{(0)}) + \mathbf{J}(\mathbf{x}^{(0)})^T$ is positive (or negative) definite [53].

### 1.2.2 Convergence to a fixed-point

We focus on asynchronous iterative algorithms, since they are particularly suitable for CR applications. More specifically, we consider a general fixed-point problem−the NE problem in (1.11)−and describe a fairly general class of totally asynchronous algorithms following [52], along with a convergence theorem of broad applicability. According to the totally asynchronous scheme, all the players of game $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega} \right\rangle$ maximize their own payoff function in a *totally asynchronous* way, meaning that some players are allowed to update their strategy more frequently than the others, and they might perform their updates using *outdated* information about the strategy profile used by the others. To provide a formal description of the algorithm, we need to introduce some preliminary definitions, as given next.

We assume w.l.o.g. that the set of times at which one or more players update their strategies is the discrete set $\mathcal{T} = \mathbb{N}_+ = \{0, 1, 2, \ldots\}$. Let $\mathbf{x}_i^{(n)}$ denote the strategy profile of user $i$ at the $n$-th iteration, and let $\mathcal{T}_i \subseteq \mathcal{T}$ be the set of times at which player $i$ updates his own strategy $\mathbf{x}_i^{(n)}$ (thus, implying that, at time $n \notin \mathcal{T}_i$, $\mathbf{x}_i^{(n)}$ is left unchanged). Let $\tau_j^i(n)$ denote the most recent time at which the strategy profile from player $j$ is perceived by player $i$ at the $n$-th iteration (observe that $\tau_j^i(n)$ satisfies $0 \leq \tau_j^i(n) \leq n$). Hence, if player $i$ updates its strategy at the $n$-th iteration, then he maximizes his payoff function using the following outdated strategy profile of the other players:

$$\mathbf{x}_{-i}^{(\boldsymbol{\tau}^i(n))} \triangleq \left( \mathbf{x}_1^{(\tau_1^i(n))}, \ldots, \mathbf{x}_{i-1}^{(\tau_{i-1}^i(n))}, \mathbf{x}_{i+1}^{(\tau_{i+1}^i(n))}, \ldots, \mathbf{x}_Q^{(\tau_Q^i(n))} \right). \tag{1.14}$$

The overall system is said to be totally asynchronous if the following assumptions are satisfied for each $i$: A1) $0 \leq \tau_j^i(n) \leq n$; A2) $\lim_{k\to\infty} \tau_j^i(n_k) = +\infty$; and A3) $|\mathcal{T}_i| = \infty$; where $\{n_k\}$ is a sequence of elements in $\mathcal{T}_i$ that tends to infinity. Assumptions (A1)−(A3) are standard in asynchronous convergence theory [52], and they are fulfilled in any practical implementation. In fact, (A1) simply indicates that, at any given iteration $n$, each player $i$ can use only the strategy profile $\mathbf{x}_{-i}^{(\boldsymbol{\tau}^i(n))}$ adopted by the other players in the previous iterations (to preserve causality). Assumption (A2) states that, for any given iteration index $n_k$, the values of the components of $\mathbf{x}_{-i}^{(\boldsymbol{\tau}^i(n))}$ in (1.14) generated prior to $n_k$, are not used in the updates of $\mathbf{x}_i^{(n)}$, when $n$ becomes sufficiently larger than $n_k$; which guarantees that old information is eventually purged from the system. Finally, assumption (A3) indicates that no player fails to update his own strategy as time $n$ goes on.

Using the above definitions, the totally asynchronous algorithm based on the multifunction $\mathcal{B}(\mathbf{x})$ is described in Algorithm 1. Observe that Algorithm 1 contains as special cases a plethora of algorithms, each one obtained by a possible choice of the scheduling of the users in the updating procedure (i.e., the parameters $\{\tau_r^q(n)\}$ and $\{\mathcal{T}_q\}$). Examples are the the *sequential* (Gauss-Seidel scheme) and the *simultaneous* (Jacobi scheme) updates, where the players update their own strategies *sequentially* and *simultaneously,* respectively. Moreover, variations of such a totally asynchronous scheme, e.g., including constraints on the maximum tolerable delay in the updating and on the use of the outdated information (which leads to the so-called *partially* asynchronous algorithms), can also be considered [52]. A fairly general convergence theorem for Algorithm 1 is given in Theorem 1.4, whose proof is based on [52].

**Algorithm 1: Totally asynchronous algorithm**

1 : Set $n = 0$ and choose any feasible $\mathbf{x}_i^{(0)}, \ \forall i \in \Omega$;

2 : `repeat`

3 : $$\mathbf{x}_i^{(n+1)} = \begin{cases} \mathbf{x}_i^\star \in \mathcal{B}_i\left(\mathbf{x}_{-i}^{(\boldsymbol{\tau}^i(n))}\right), & \text{if } n \in \mathcal{T}_i, \\ \mathbf{x}_i^{(n)}, & \text{otherwise}, \end{cases} \qquad \forall i \in \Omega; \tag{1.15}$$

4 : `until the prescribed convergence criterion is satisfied`

**Theorem 1.4** (Asynchronous Convergence Theorem)**.** *Given Algorithm 1 based on a multifunction* $\mathcal{B}(\mathbf{x}) : \mathcal{X} \ni \mathbf{x} \rightrightarrows \mathcal{B}_1(\mathbf{x}_{-1}) \times \mathcal{B}_2(\mathbf{x}_{-2}) \times \cdots \times \mathcal{B}_Q(\mathbf{x}_{-Q}) \subseteq \mathcal{X}$, *suppose that assumptions* (A1)-(A3) *hold true and that there exists a sequence of nonempty sets* $\{\mathcal{X}(n)\}$ *with*

$$\ldots \subset \mathcal{X}(n+1) \subset \mathcal{X}(n) \subset \ldots \subset \mathcal{X}, \tag{1.16}$$

*satisfying the next two conditions:*

**(a)** (Synchronous Convergence Condition) For all $\mathbf{x} \in \mathcal{X}(n)$ *and* $n$,

$$\mathcal{B}(\mathbf{x}) \subseteq \mathcal{X}(n+1). \tag{1.17}$$

*Furthermore, if* $\{\mathbf{x}^{(n)}\}$ *is a sequence such that* $\mathbf{x}^{(n)} \in \mathcal{X}(n)$, *for every* $n$, *then every limit point of* $\{\mathbf{x}^{(n)}\}$ *is a fixed point of* $\mathcal{B}(\cdot)$.

**(b)** (Box Condition) *For every $n$, there exist sets $\mathcal{X}_i(n) \subset \mathcal{X}_i$ such that $\mathcal{X}(n)$ can be written as a Cartesian product*

$$\mathcal{X}(n) = \mathcal{X}_1(n) \times \ldots \times \mathcal{X}_Q(n). \tag{1.18}$$

*Then, every limit point of $\{\mathbf{x}^{(n)}\}$ generated by Algorithm 1 and starting from $\mathbf{x}^{(0)} \in \mathcal{X}(0)$ is a fixed point of $\mathcal{B}(\,\cdot\,)$.*

The challenge in applying the Asynchronous Convergence Theorem is to identify a suitable sequence of sets $\{\mathcal{X}(n)\}$. This is a reminiscent of the process of identifying a Lyapunov function in the stability analysis of nonlinear dynamic systems (the sets $\mathcal{X}(k)$ play conceptually the role of the level set of a Lyapunov function). For the purpose of this chapter, it is enough to restrict our focus to single-value best-response functions and consider sufficient conditions for (1.16)-(1.18) in Theorem 1.4, as detailed next.

Given the game $\mathcal{G} = \left\langle \Omega, (\mathcal{Q}_i)_{i\in\Omega}, (u_i)_{i\in\Omega}\right\rangle$ with the best-response function $\mathsf{B}(\mathbf{x}) = (\mathsf{B}_i(\mathbf{x}_{-i}))_{i\in\Omega}$, where each $\mathsf{B}_i(\mathbf{x}_{-i}) : \mathcal{Q}_{-i} \ni \mathbf{x}_{-i} \to \mathsf{B}_i(\mathbf{x}_{-i}) \in \mathcal{Q}_i$, let us introduce the following block-maximum vector norm $\|\cdot\|_{\text{block}}$ on $\mathbb{R}^n$, defined as

$$\|\mathsf{B}\|_{\text{block}} = \max_{i\in\Omega} \|\mathsf{B}_i\|_i\,, \tag{1.19}$$

where $\|\cdot\|_i$ is any vector norm on $\mathbb{R}^{n_i}$. Suppose that each $\mathcal{Q}_i$ is a closed subset of $\mathbb{R}^{n_i}$ and that $\mathsf{B}(\mathbf{x})$ is a contraction with respect to the block-maximum norm, i.e.,

$$\left\|\mathsf{B}(\mathbf{x}^{(1)}) - \mathsf{B}(\mathbf{x}^{(2)})\right\|_{\text{block}} \le \alpha \left\|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right\|_{\text{block}}, \quad \forall \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathcal{Q}, \tag{1.20}$$

with $\alpha \in [0,1)$. Then, there exists a unique fixed-point $\mathbf{x}^\star$ of $\mathsf{B}(\mathbf{x})$ (cf. Theorem 1.3)−the NE of $\mathcal{G}$−and the Asynchronous Convergence theorem holds. In fact, it is not difficult to show that, under (1.20), conditions (1.16)-(1.18) in Theorem 1.4 are satisfied with the following choice for the sets $\mathcal{X}_i(k)$:

$$\mathcal{X}_i(k) = \left\{\mathbf{x} \in \mathcal{Q} \mid \|\mathbf{x}_i - \mathbf{x}_i^\star\|_2 \le \alpha^k\, w_i \left\|\mathbf{x} - \mathbf{x}^{(0)}\right\|_{\text{block}}\right\} \subset \mathcal{Q}_i, \qquad k \ge 1, \tag{1.21}$$

where $\mathbf{x}^{(0)} \in \mathcal{Q}$ is the initial point of the algorithm. Note that, because of the uniqueness of the fixed-point of $\mathsf{B}(\mathbf{x})$ under (1.20), the statement on convergence in Theorem 1.4 can be made stronger: For any initial vector $\mathbf{x}^{(0)} \in \mathcal{Q}$, the sequence $\{\mathbf{x}^{(n)}\}$ generated by Algorithm 1 converges to the fixed-point of $\mathsf{B}(\mathbf{x})$.

## 1.3 Opportunistic Communications over Unlicensed Bands

We start considering the CR system in (1.1), under the transmit power constraints (1.2) only. This models transmissions over unlicensed bands, where multiple systems coexist, thus interfering with each other, and there are no constraints on the maximum amount of interference that each transmitter can generated.

The results obtained in this case provide the building blocks instrumental to study the equilibrium problems including interference constraints, as described in the next sections.

The rate maximization game among the secondary users in the presence of the power constraints (1.2) is formally defined as

$$(\mathcal{G}_{\text{pow}}): \quad \begin{array}{ll} \underset{\mathbf{Q}_q}{\text{maximize}} & R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) \\ \text{subject to} & \mathbf{Q}_q \in \mathcal{Q}_q, \end{array} \quad \forall q \in \Omega, \tag{1.22}$$

where $\Omega \triangleq \{1, 2, \cdots, Q\}$ is the set of players (the secondary users), $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$ is the payoff function of player $q$, defined in (1.4), and $\mathcal{Q}_q$ is the set of admissible strategies (the covariance matrices) of player $q$, defined as

$$\mathcal{Q}_q \triangleq \left\{ \mathbf{Q} \in \ \mathbb{S}_+^{n_{T_q}} \,|\, \text{Tr}\{\mathbf{Q}\} = P_q \right\}. \tag{1.23}$$

Observe that there is no loss of generality in considering in (1.23) the power constraint with equality rather than inequality as stated in (1.2), since at the optimum to each problem in (1.22), the power constraint must be satisfied with equality. To write the Nash equilibria of game $\mathcal{G}_{\text{pow}}$ in a convenient form, we introduce the MIMO waterfilling operator. Given $q \in \Omega$, $n_q \in \{1, 2, \ldots, n_{T_q}\}$, and some $\mathbf{X} \in \mathbb{S}_+^{n_q}$, the MIMO waterfilling function $\mathsf{WF}_q : \mathbb{S}_+^{n_q} \ni \mathbf{X} \rightarrow \mathbb{S}_+^{n_q}$ is defined as

$$\mathsf{WF}_q\,(\mathbf{X}) \triangleq \mathbf{U}_X \left(\mu_{q,X} \mathbf{I}_{r_X} - \mathbf{D}_X^{-1}\right)^+ \mathbf{U}_X^H, \tag{1.24}$$

where $\mathbf{U}_X \in \mathbb{C}^{n_q \times r_X}$ and $\mathbf{D}_X > \mathbf{0}$ are the (semi-)unitary matrix of the eigenvectors and the diagonal matrix of the $r_X \triangleq \text{rank}(\mathbf{X}) \leq n_q$ (positive) eigenvalues of $\mathbf{X}$, respectively, and $\mu_{q,X} > 0$ is the water-level chosen to satisfy $\text{Tr}\left\{(\mu_{q,X}\mathbf{I}_{r_X} - \mathbf{D}_X^{-1})^+\right\} = P_q$. Using the above definitions, the solution to the single-user optimization problem in (1.22)—the best-response of player $q$ for any given $\mathbf{Q}_{-q} \succeq \mathbf{0}$—is the well-known waterfilling solution (e.g., [11])

$$\mathbf{Q}_q^\star = \mathsf{WF}_q(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}), \tag{1.25}$$

implying that the Nash equilibria of game $\mathcal{G}_{\text{pow}}$ are the solutions of the following fixed-point matrix equation (cf. Definition 1.3):

$$\mathbf{Q}_q^\star = \mathsf{WF}_q(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}^\star)\mathbf{H}_{qq}), \quad \forall q \in \Omega. \tag{1.26}$$

*Remark 1 - On the Nash equilibria:* The main difficulty in the analysis of the solutions to (1.26) comes from the fact that the optimal eigenvector matrix $\mathbf{U}_q^\star = \mathbf{U}_q(\mathbf{Q}_{-q}^\star)$ of each user $q$ (see (1.24)) depends, in general, on the strategies $\mathbf{Q}_{-q}^\star$ of all the other users, through a very complicated implicit relationship—the eigendecomposition of the equivalent channel matrix $\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_q^\star)\mathbf{H}_{qq}$. To overcome this issue, we provide next an equivalent expression of the waterfilling solution enabling us to express the Nash equilibria in (1.26) as a fixed-point of a more tractable mapping. This alternative expression is based on the recent interpretation of the MIMO waterfilling mapping as a proper projector operator [7, 10, 55]. Based on this result, we can then derive sufficient conditions for the uniqueness of the NE and convergence of asynchronous distributed algorithms, as detailed in Sections 1.3.4 and 1.3.5, respectively.

### 1.3.1 Properties of the multiuser waterfilling mapping

In this section we derive some interesting properties of the multiuser MIMO waterfilling mapping. These results will be instrumental to study the games we propose in this chapter. The main result of the section is a contraction theorem for the multiuser MIMO waterfilling mapping, valid for arbitrary channel matrices. Results in this section are based on recent works [10, 24].

For the sake of notation, through the whole section we refer to the best-response $\mathsf{WF}_q(\mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq})$ of each user $q$ in (1.25) as $\mathsf{WF}_q(\mathbf{Q}_{-q})$, making explicitly only the dependence on the strategy profile $\mathbf{Q}_{-q}$ of the other players.

### 1.3.2 MIMO waterfilling as a projector

The interpretation of the MIMO waterfilling solution as a matrix projection is based on the following result.

**Lemma 1.1.** *Let* $\mathbb{S}^n \ni \mathbf{X}_0 = \mathbf{U}_0\mathbf{D}_0\mathbf{U}_0^H$, *where* $\mathbf{U}_0 \in \mathbb{C}^{n\times n}$ *is unitary and* $\mathbf{D}_0 = \mathrm{diag}(\{d_{0,k}\}_{k=1}^n)$, *and let* $\mathcal{Q}$ *be the convex set defined as*

$$\mathcal{Q} \triangleq \left\{\mathbf{Q} \in \mathbb{S}_+^n \mid \mathrm{Tr}\{\mathbf{Q}\} = P_T\right\}. \tag{1.27}$$

*The matrix projection* $[\mathbf{X}_0]_{\mathcal{Q}}$ *of* $\mathbf{X}_0$ *onto* $\mathcal{Q}$ *with respect to the Frobenius norm, defined as*

$$[\mathbf{X}_0]_{\mathcal{Q}} = \underset{\mathbf{X}\in\mathcal{Q}}{\mathrm{argmin}} \left\|\mathbf{X} - \mathbf{X_0}\right\|_F^2 \tag{1.28}$$

*takes the following form:*

$$[\mathbf{X}_0]_{\mathcal{Q}} = \mathbf{U}_0\left(\mathbf{D}_0 - \mu_0\mathbf{I}\right)^+\mathbf{U}_0^H, \tag{1.29}$$

*where* $\mu_0$ *satisfies the constraint* $\mathrm{Tr}\{(\mathbf{D}_0 - \mu_0\mathbf{I})^+\} = P_T$.

*Proof.* Using $\mathbf{X}_0 = \mathbf{U}_0\mathbf{D}_0\mathbf{U}_0^H$, the objective function in (1.28) becomes

$$\left\|\mathbf{X} - \mathbf{X_0}\right\|_F^2 = \left\|\widetilde{\mathbf{X}} - \mathbf{D}_0\right\|_F^2, \tag{1.30}$$

where $\widetilde{\mathbf{X}}$ is defined as $\widetilde{\mathbf{X}} \triangleq \mathbf{U}_0^H\mathbf{X}\mathbf{U}_0$ and we used the unitary invariance of the Frobenius norm [26]. Since

$$\left\|\widetilde{\mathbf{X}} - \mathbf{D}_0\right\|_F^2 \geq \left\|\mathrm{Diag}(\widetilde{\mathbf{X}}) - \mathbf{D}_0\right\|_F^2, \tag{1.31}$$

with equality if and only if $\widetilde{\mathbf{X}}$ is diagonal, and the power constraint $\text{Tr}\{\mathbf{X}\} = \text{Tr}\{\widetilde{\mathbf{X}}\} = P_T$ depends only on the diagonal elements of $\widetilde{\mathbf{X}}$, it follows that the optimal $\widetilde{\mathbf{X}}$ must be diagonal, i.e., $\widetilde{\mathbf{X}} = \text{diag}(\{d_k\}_{k=1}^n)$. The matrix-valued problem in (1.28) reduces then to the following vector (strictly) convex optimization problem

$$\begin{aligned} \underset{\mathbf{d}\geq\mathbf{0}}{\text{minimize}} \quad & \sum_{k=1}^{n} (d_k - d_{0,k})^2 \\ \text{subject to} \quad & \sum_{k=1}^{n} d_k = P_T, \end{aligned} \tag{1.32}$$

whose unique solution $\{d_k^\star\}$ is given by $d_k^\star = (d_{0,k} - \mu_0)^+$, with $k = 1, \cdots, n$, where $\mu_0$ is chosen to satisfy $\sum_{k=1}^{n} (d_{0,k} - \mu_0)^+ = P_T$. □

Using the above result we can obtain the alternative expression of the waterfilling solution $\text{WF}_q(\mathbf{Q})$ in (1.25) as given next.

**Lemma 1.2** (MIMO waterfiling as a projector)**.** *The MIMO waterfilling operator* $\mathsf{WF}_q(\mathbf{Q}_{-q})$ *in (1.25) can be equivalently written as*

$$\mathsf{WF}_q(\mathbf{Q}_{-q}) = \left[ -\left( \left( \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}) \mathbf{H}_{qq} \right)^\sharp + c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} \right) \right]_{\mathcal{Q}_q}, \tag{1.33}$$

*where* $c_q$ *is a positive constant that can be chosen independent of* $\mathbf{Q}_{-q}$ *(c.f. [24]), and* $\mathcal{Q}_q$ *is defined in* (1.23).

*Proof.* Given $q \in \Omega$ and $\mathbf{Q}_{-q} \in \mathcal{Q}_{-q}$, using the eigendecomposition $\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}) \mathbf{H}_{qq} = \mathbf{U}_{q,1} \mathbf{D}_{q,1} \mathbf{U}_{q,1}^H$, where $\mathbf{U}_{q,1} = \mathbf{U}_{q,1}(\mathbf{Q}_{-q}) \in \mathbb{C}^{n_{T_q} \times r_{H_{qq}}}$ is semi-unitary and $\mathbf{D}_{q,1} = \mathbf{D}_{q,1}(\mathbf{Q}_{-q}) = \text{diag}(\{\lambda_i\}_{i=1}^{r_{H_{qq}}}) > \mathbf{0}$ (we omit in the following the dependence of $\mathbf{Q}_{-q}$ for the sake of notation), and introducing the unitary matrix $\mathbf{U}_q \triangleq (\mathbf{U}_{q,1}, \mathbf{U}_{q,2}) \in \mathbb{C}^{n_{T_q} \times n_{T_q}}$ (note that $\mathcal{R}(\mathbf{U}_{q,2}) = \mathcal{N}(\mathbf{H}_{qq})$), we have, for any given $c_q \in \mathbb{R}$,

$$\left( \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1} \mathbf{H}_{qq} \right)^\sharp + c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} = \mathbf{U}_q \begin{pmatrix} \mathbf{D}_{q,1}^{-1} & \mathbf{0} \\ \mathbf{0} & c_q \mathbf{I}_{n_{T_q} - r_{H_{qq}}} \end{pmatrix} \mathbf{U}_q^H \triangleq \mathbf{U}_q \widetilde{\mathbf{D}}_q^{-1} \mathbf{U}_q^H, \tag{1.34}$$

where $\widetilde{\mathbf{D}}_q^{-1} \triangleq \text{bdiag}(\mathbf{D}_q^{-1}, c_q \mathbf{I}_{n_{T_q} - r_{H_{qq}}})$. It follows from Lemma 1.1 that, for any given $c_q \in \mathbb{R}_{++}$,

$$\left[ -\left( \left( \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1} \mathbf{H}_{qq} \right)^\sharp + c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} \right) \right]_{\mathcal{Q}_q} = \mathbf{U}_q \left( \mu_q \mathbf{I}_{n_{T_q}} - \widetilde{\mathbf{D}}_q^{-1} \right)^+ \mathbf{U}_q^H \tag{1.35}$$

where $\mu_q$ is chosen to satisfy the constraint $\text{Tr}((\mu_q \mathbf{I}_{n_{T_q}} - \widetilde{\mathbf{D}}_q^{-1})^+) = P_q$. Since each $P_q < \infty$, there exists a (sufficiently large) constant $0 < c_q < \infty$, such that $(\mu_q - c_q)^+ = 0$, and thus the RHS of (1.35) becomes

$$\left[ -\left( \left( \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1} \mathbf{H}_{qq} \right)^\sharp + c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} \right) \right]_{\mathcal{Q}_q} = \mathbf{U}_{q,1} \left( \mu_q \mathbf{I}_{r_{H_{qq}}} - \mathbf{D}_{q,1}^{-1} \right)^+ \mathbf{U}_{q,1}^H \tag{1.36}$$

which coincides with the desired solution in (1.25). □

Observe that, for each $q \in \Omega$, $\mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})}$ in (1.33) depends only on the channel matrix $\mathbf{H}_{qq}$ (through the right singular vectors of $\mathbf{H}_{qq}$ corresponding to the zero singular values) and not on the strategies of the other users, since $\mathbf{R}_{-q}(\mathbf{Q}_{-q})$ is positive definite for all $\mathbf{Q}_{-q} \in \mathcal{Q}_{-q}$.

Lemma 1.2 can be further simplified if the (direct) channels $\mathbf{H}_{qq}$'s are full column-rank matrices: Given the nonsingular matrix $\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}$, the MIMO waterfilling operator $\mathsf{WF}_q(\mathbf{Q}_{-q})$ in (1.25) can be equivalently written as

$$\mathsf{WF}_q(\mathbf{Q}_{-q}) = \left[-\left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\right)^{-1}\right]_{\mathcal{Q}_q}. \tag{1.37}$$

*Non-expansive property of the waterfilling operator:* Thanks to the interpretation of the MIMO waterfilling in (1.25) as a projector, building on [52, Prop. 3.2], one can easily obtain the following non-expansive property of the waterfilling function.

**Lemma 1.3.** *The matrix projection* $[\cdot]_{\mathcal{Q}_q}$ *onto the convex set* $\mathcal{Q}_q$ *defined in* (1.23) *satisfies the following non-expansive property*:

$$\left\| [\mathbf{X}]_{\mathcal{Q}_q} - [\mathbf{Y}]_{\mathcal{Q}_q} \right\|_F \leq \|\mathbf{X} - \mathbf{Y}\|_F, \quad \forall\, \mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n_{T_q} \times n_{T_q}}. \tag{1.38}$$

### 1.3.3 Contraction properties of the multiuser MIMO waterfilling mapping

Building on the interpretation of the waterfilling operator as a projector, we can now focus on the contraction properties of the multiuser MIMO waterfilling operator. We will consider w.l.o.g. only the case where all the direct channel matrices are either full row-rank or full column-rank. The rank deficient case in fact can be cast into the full column-rank case by a proper transformation of the original rank deficient channel matrices into a lower-dimensional full column-rank matrices, as shown in Section 1.3.4.

#### 1.3.3.1 Intermediate definitions

To derive the contraction properties of the MIMO waterfilling mapping we need the following intermediate definitions. Given the multiuser waterfilling mapping

$$\mathsf{WF}(\mathbf{Q}) = (\mathsf{WF}_q(\mathbf{Q}_{-q}))_{q \in \Omega} : \mathcal{Q} \mapsto \mathcal{Q}, \tag{1.39}$$

where $\mathsf{WF}_q(\mathbf{Q}_{-q})$ is defined in (1.25), we introduce the following block-maximum norm on $\mathbb{C}^{n \times n}$, with $n = n_{T_1} + \ldots + n_{T_Q}$, defined as

$$\|\mathsf{WF}(\mathbf{Q})\|_{F,\text{block}}^{\mathbf{w}} \triangleq \max_{q \in \Omega} \frac{\|\mathsf{WF}_q(\mathbf{Q}_{-q})\|_F}{w_q}, \tag{1.40}$$

where $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T > \mathbf{0}$ is any given positive weight vector. Let $\|\cdot\|_{\infty,\text{vec}}^{\mathbf{w}}$ be the *vector* weighted maximum norm, defined as

$$\|\mathbf{x}\|_{\infty,\text{vec}}^{\mathbf{w}} \triangleq \max_{q\in\Omega} \frac{|x_q|}{w_q}, \quad \text{for} \quad \mathbf{w} > \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^Q, \tag{1.41}$$

and let $\|\cdot\|_{\infty,\text{mat}}^{\mathbf{w}}$ denote the *matrix* norm induced by $\|\cdot\|_{\infty,\text{vec}}^{\mathbf{w}}$, given by [26]

$$\|\mathbf{A}\|_{\infty,\text{mat}}^{\mathbf{w}} \triangleq \max_{q} \frac{1}{w_q} \sum_{r=1}^{Q} |[\mathbf{A}]_{qr}|\, w_r, \quad \text{for} \quad \mathbf{A} \in \mathbb{R}^{Q\times Q}. \tag{1.42}$$

Finally, we introduce the nonnegative matrices $\mathbf{S}_{\text{pow}}, \mathbf{S}_{\text{pow}}^{\text{up}}, \widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}} \in \mathbb{R}_+^{Q\times Q}$ defined as

$$[\mathbf{S}_{\text{pow}}]_{qr} \triangleq \begin{cases} \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp} \mathbf{H}_{rq}\right), & \text{if } r \neq q, \\ 0, & \text{otherwise}, \end{cases} \tag{1.43}$$

$$\left[\mathbf{S}_{\text{pow}}^{\text{up}}\right]_{qr} \triangleq \begin{cases} \text{innr}_q \cdot \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{rq}\right) \rho\left(\mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp}\right), & \text{if } r \neq q, \\ 0, & \text{otherwise} \end{cases} \tag{1.44}$$

$$\left[\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}\right]_{qr} \triangleq \begin{cases} [\mathbf{S}_{\text{pow}}]_{qr}, & \text{if } \operatorname{rank}(\mathbf{H}_{qq}) = n_{R_q}, \\ \left[\mathbf{S}_{\text{pow}}^{\text{up}}\right]_{qr}, & \text{otherwise}, \end{cases} \tag{1.45}$$

where the interference-plus-noise to noise ratio $\text{innr}_q$ is given by

$$\text{innr}_q \triangleq \frac{\rho\left(\mathbf{R}_{n_q} + \sum_{r\neq q} P_r \mathbf{H}_{rq} \mathbf{H}_{rq}^H\right)}{\lambda_{\min}\left(\mathbf{R}_{n_q}\right)} \geq 1, \quad q \in \Omega. \tag{1.46}$$

Note that $\mathbf{S}_{\text{pow}} < \mathbf{S}_{\text{pow}}^{\text{up}} < \widetilde{\mathbf{S}}_{\text{pow}}$ implying $\|\mathbf{S}_{\text{pow}}\|_{\infty,\text{mat}}^{\mathbf{w}} < \|\mathbf{S}_{\text{pow}}^{\text{up}}\|_{\infty,\text{mat}}^{\mathbf{w}} < \|\widetilde{\mathbf{S}}_{\text{pow}}\|_{\infty,\text{mat}}^{\mathbf{w}}$, for all $\mathbf{w} > \mathbf{0}$.

#### 1.3.3.2 Case of full row-rank (fat/square) channel matrices

We start assuming that the channel matrices $\{\mathbf{H}_{qq}\}_{q\in\Omega}$ are full row-rank. The contraction property of the waterfilling mapping is given in the following.

**Theorem 1.5** (Contraction property of WF mapping)**.** *Suppose that* $\operatorname{rank}(\mathbf{H}_{qq}) = n_{R_q}$, $\forall q \in \Omega$. *Then, for any given* $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T > \mathbf{0}$, *the* WF *mapping defined in* (1.39) *is Lipschitz continuous on* $\mathcal{Q}$*:*

$$\left\|\mathsf{WF}(\mathbf{Q}^{(1)}) - \mathsf{WF}(\mathbf{Q}^{(2)})\right\|_{F,\text{block}}^{\mathbf{w}} \leq \|\mathbf{S}_{\text{pow}}\|_{\infty,\text{mat}}^{\mathbf{w}} \left\|\mathbf{Q}^{(1)} - \mathbf{Q}^{(2)}\right\|_{F,\text{block}}^{\mathbf{w}}, \tag{1.47}$$

*for all* $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$, *where* $\|\cdot\|_{F,\text{block}}^{\mathbf{w}}$, $\|\cdot\|_{\infty,\text{mat}}^{\mathbf{w}}$ *and* $\mathbf{S}_{\text{pow}}$ *are defined in* (1.40), (1.42) *and* (1.43), *respectively. Furthermore, if the following condition is satisfied*

$$\|\mathbf{S}_{\text{pow}}\|_{\infty,\text{mat}}^{\mathbf{w}} < 1, \qquad \text{for some } \mathbf{w} > \mathbf{0}, \tag{1.48}$$

*then, the* WF *mapping is a* block-contraction *with modulus* $\beta = \|\mathbf{S}_{\text{pow}}\|_{\infty,\text{mat}}^{\mathbf{w}}$.

*Proof.* Given $\mathbf{Q}^{(1)} = \left(\mathbf{Q}_q^{(1)}, \ldots, \mathbf{Q}_Q^{(1)}\right) \in \mathcal{Q}$ and $\mathbf{Q}^{(2)} = \left(\mathbf{Q}_1^{(2)}, \ldots, \mathbf{Q}_Q^{(2)}\right) \in \mathcal{Q}$, let define, for each $q \in \Omega$,

$$e_{\mathsf{WF}_q} \triangleq \left\| \mathsf{WF}_q\left(\mathbf{Q}_{-q}^{(1)}\right) - \mathsf{WF}_q\left(\mathbf{Q}_{-q}^{(2)}\right) \right\|_F \text{ and } e_q \triangleq \left\| \mathbf{Q}_q^{(1)} - \mathbf{Q}_q^{(2)} \right\|_F \tag{1.49}$$

where, according to Lemma 1.2, each component $\mathsf{WF}_q(\mathbf{Q}_{-q})$ can be rewritten as in (1.33). Then, we have:

$$\begin{aligned}
e_{\mathsf{WF}_q} = & \left\| \left[ -\left(\mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(1)})\mathbf{H}_{qq}\right)^{\sharp} - c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} \right]_{\mathcal{Q}_q} \right. \\
& \left. - \left[ -\left(\mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(2)})\mathbf{H}_{qq}\right)^{\sharp} - c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} \right]_{\mathcal{Q}_q} \right\|_F & (1.50) \\
\leq & \left\| \left(\mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(1)})\mathbf{H}_{qq}\right)^{\sharp} - \left(\mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(2)})\mathbf{H}_{qq}\right)^{\sharp} \right\|_F & (1.51) \\
= & \left\| \mathbf{H}_{qq}^{\sharp} \left( \sum_{r \neq q} \mathbf{H}_{rq} \left(\mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)}\right) \mathbf{H}_{rq}^H \right) \mathbf{H}_{qq}^{\sharp H} \right\|_F & (1.52) \\
\leq & \sum_{r \neq q} \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp} \mathbf{H}_{rq}\right) \left\| \mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)} \right\|_F & (1.53) \\
= & \sum_{r \neq q} [\mathbf{S}_{\text{pow}}]_{qr} \left\| \mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)} \right\|_F = \sum_{r \neq q} [\mathbf{S}_{\text{pow}}]_{qr}\, e_r, \quad \forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}, & (1.54)
\end{aligned}$$

where (1.50) follows from (1.33) (Lemma 1.2); (1.51) follows from the non-expansive property of the projector with respect to the Frobenius norm as given in (1.38) (Lemma 1.3); (1.52) follows from the reverse order law for Moore-Penrose pseudoinverses (see, e.g., [56]), valid under the assumption $\text{rank}(\mathbf{H}_{qq}) = n_{R_q}$, $\forall q \in \Omega$;[2] (1.53) follows from the triangle inequality [26] and

$$\left\| \mathbf{A}\mathbf{X}\mathbf{A}^H \right\|_F \leq \rho\left(\mathbf{A}^H\mathbf{A}\right) \|\mathbf{X}\|_F, \tag{1.55}$$

and in (1.54) we have used the definition of $\mathbf{S}_{\text{pow}}$ given in (1.43).

Introducing the vectors

$$\mathbf{e}_{\mathsf{WF}} \triangleq [e_{\mathsf{WF}_1}, \ldots, e_{\mathsf{WF}_Q}]^T, \quad \text{and} \quad \mathbf{e} \triangleq [e_1, \ldots, e_Q]^T, \tag{1.56}$$

with $e_{\mathsf{WF}_q}$ and $e_q$ defined in (1.49), the set of inequalities in (1.54) can be rewritten in vector form as

$$\mathbf{0} \leq \mathbf{e}_{\mathsf{WF}} \leq \mathbf{S}_{\text{pow}}\, \mathbf{e}, \quad \forall \mathbf{Q}^{(1)},\ \mathbf{Q}^{(2)} \in \mathcal{Q}. \tag{1.57}$$

[2] Note that in the case of (strictly) *full column-rank* matrix $\mathbf{H}_{qq}$, the reverse order law for $(\mathbf{H}_{qq}^H \mathbf{R}_q^{-1} \mathbf{H}_{qq})^{\sharp}$ does not hold true (the necessary and sufficient conditions given in [56, Th. 2.2] are not satisfied). In fact, in such a case, it follows from (the matrix version of) the Kantorovich inequality [57, Ch. 11] that $\left(\mathbf{H}^H\mathbf{R}\mathbf{H}\right)^{\sharp} \preceq \mathbf{H}^{\sharp}\mathbf{R}\mathbf{H}^{\sharp H}$.

Using the weighted maximum norm $\|\cdot\|_{\infty,\mathrm{vec}}^{\mathbf{w}}$ defined in (1.41) in combination with (1.57), we have, for any given $\mathbf{w} > \mathbf{0}$ (recall that $\|\cdot\|_{\infty,\mathrm{vec}}^{\mathbf{w}}$ is a monotonic norm[3]),

$$\|\mathbf{e}_{\mathsf{WF}}\|_{\infty,\mathrm{vec}}^{\mathbf{w}} \leq \|\mathbf{S}_{\mathrm{pow}}\mathbf{e}\|_{\infty,\mathrm{vec}}^{\mathbf{w}} \leq \|\mathbf{S}_{\mathrm{pow}}\|_{\infty,\mathrm{mat}}^{\mathbf{w}} \|\mathbf{e}\|_{\infty,\mathrm{vec}}^{\mathbf{w}}\,, \tag{1.58}$$

$\forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$, where $\|\cdot\|_{\infty,\mathrm{mat}}^{\mathbf{w}}$ is the matrix norm induced by the vector norm $\|\cdot\|_{\infty,\mathrm{vec}}^{\mathbf{w}}$ in (1.41) and defined in (1.42) [26]. Finally, introducing (1.40) in (1.58), we obtain the desired result as stated in (1.47). □

*Negative result:* As stated in Theorem 1.5, the waterfilling mapping $\mathsf{WF}$ satisfies the Lipschitz property in (1.47) if the channel matrices $\{\mathbf{H}_{qq}\}_{q\in\Omega}$ are full row-rank. Surprisingly, if the channels are not full row-rank matrices, the property in (1.47) *does not* hold for *every* given set of matrices $\{\mathbf{H}_{qq}\}_{q\in\Omega}$, implying that the $\mathsf{WF}$ mapping is not a contraction under (1.48) for all $\{\mathbf{H}_{qq}\}_{q\in\Omega}$ and stronger conditions are needed, as given in the next section. A simple counter-example is given in [24].

#### 1.3.3.3 Case of full column-rank (tall) channel matrices

The main difficulty in deriving contraction properties of the MIMO multiuser waterfilling mapping in the case of (strictly) tall channel matrices $\{\mathbf{H}_{qq}\}_{q\in\Omega}$ is that one cannot use the reverse order law of generalized inverses, as done in the proof of Theorem 1.5 (see (1.51)-(1.52)). To overcome this issue, we develop a different approach based on the mean-value theorem for complex matrix-valued functions, as detailed next.

*Mean-value theorem for complex matrix-valued functions*: The mean value theorem for scalar real functions is one of the most important and basic theorems in functional analysis (see, e.g., [57, Ch.5-Th.10], [58, Th.5.10]). The generalization of the (differential version of the) theorem to vector-valued real functions that one would expect does not hold, meaning that for real vector-valued functions $\mathbf{f} : \mathcal{D} \subseteq \mathbb{R}^m \mapsto \mathbb{R}^n$ in general

$$\nexists\, t \in (0,1) \mid \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \mathbf{D}_{\mathbf{x}}\mathbf{f}(t\,\mathbf{y} + (1-t)\,\mathbf{x})(\mathbf{y} - \mathbf{x}), \tag{1.59}$$

for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ and $\mathbf{x} \neq \mathbf{y}$, where $\mathbf{D}_{\mathbf{x}}\mathbf{f}$ denotes the Jacobian matrix of $\mathbf{f}$. One of the simplest examples to illustrate (1.59) is the following. Consider the real vector-valued function $\mathbf{f}(x) = [x^{\alpha}, x^{\beta}]^T$, with $x \in \mathbb{R}$ and, e.g., $\alpha = 2$, $\beta = 3$. There exists no value of $t \in (0,1)$ such that $\mathbf{f}(1) = \mathbf{f}(0) + \mathbf{D}_t\mathbf{f}(t)$.

Many extensions and variations of the main value theorem exist in the literature, either for (real/ complex) scalar or real vector-valued functions (see, e.g., [59, 60], [53, Ch. 3.2]). Here, we focus on the following.

[3] A vector norm $\|\cdot\|$ is monotonic if $\mathbf{x} \geq \mathbf{y}$ implies $\|\mathbf{x}\| \geq \|\mathbf{y}\|$.

**Lemma 1.4** ([24])**.** *Let* $\mathbf{F}(\mathbf{X}) : \mathcal{D} \subseteq \mathbb{C}^{m\times n} \mapsto \mathbb{C}^{p\times q}$ *be a complex matrix-valued function defined on a convex set* $\mathcal{D}$*, assumed to be continuous on* $\mathcal{D}$ *and differentiable on the interior of* $\mathcal{D}$*, with Jacobian matrix* $\mathbf{D_X F}(\mathbf{X})$*.*[4] *Then, for any given* $\mathbf{X}, \mathbf{Y} \in \mathcal{D}$*, there exists some* $t \in (0, 1)$ *such that*

$$\|\mathbf{F}(\mathbf{Y}) - \mathbf{F}(\mathbf{X})\|_F \leq \|\mathbf{D_X F}((t\,\mathbf{Y} + (1-t)\,\mathbf{X}))\,\mathrm{vec}(\mathbf{Y}-\mathbf{X})\|_2 \tag{1.60}$$

$$\leq \|\mathbf{D_X F}\left((t\,\mathbf{Y} + (1-t)\,\mathbf{X})\right)\|_{2,\mathrm{mat}}\,\|\mathbf{Y}-\mathbf{X}\|_F\,, \tag{1.61}$$

*where* $\|\mathbf{A}\|_{2,\mathrm{mat}} \triangleq \sqrt{\rho(\mathbf{A}^H\mathbf{A})}$ *denotes the spectral norm of* $\mathbf{A}$.

We can now provide the contraction theorem for the $\mathsf{WF}$ mapping valid also for the case in which the channels $\{\mathbf{H}_{qq}\}_{q\in\Omega}$ are full column-rank matrices.

**Theorem 1.6** (Contraction property of $\mathsf{WF}$ mapping)**.** *Suppose that* $\mathrm{rank}(\mathbf{H}_{qq}) = n_{T_q}$, $\forall q \in \Omega$*. Then, for any given* $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T > \mathbf{0}$*, the mapping* $\mathsf{WF}$ *defined in* (1.39) *is Lipschitz continuous on* $\mathcal{Q}$:

$$\left\|\mathsf{WF}(\mathbf{Q}^{(1)}) - \mathsf{WF}(\mathbf{Q}^{(2)})\right\|_{F,\mathrm{block}}^{\mathbf{w}} \leq \|\mathbf{S}_{\mathrm{pow}}^{\mathrm{up}}\|_{\infty,\mathrm{mat}}^{\mathbf{w}} \left\|\mathbf{Q}^{(1)} - \mathbf{Q}^{(2)}\right\|_{F,\mathrm{block}}^{\mathbf{w}}, \tag{1.62}$$

*for all* $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$*, where* $\|\cdot\|_{F,\mathrm{block}}^{\mathbf{w}}$, $\|\cdot\|_{\infty,\mathrm{mat}}^{\mathbf{w}}$ *and* $\mathbf{S}_{\mathrm{pow}}^{\mathrm{up}}$ *are defined in* (1.40), (1.42) *and* (1.44)*, respectively. Furthermore, if the following condition is satisfied*[5]

$$\|\mathbf{S}_{\mathrm{pow}}^{\mathrm{up}}\|_{\infty,\mathrm{mat}}^{\mathbf{w}} < 1, \qquad \text{for some } \mathbf{w} > \mathbf{0}, \tag{1.63}$$

*then, the mapping* $\mathsf{WF}$ is a *block-contraction with modulus* $\beta = \|\mathbf{S}_{\mathrm{pow}}^{\mathrm{up}}\|_{\infty,\mathrm{mat}}^{\mathbf{w}}$.

*Proof.* The proof follows the same guidelines of that of Theorem 1.5, with the key difference that, in the case of (strictly) full column-rank direct channel matrices, we cannot use the reverse order law of pseudoinverses as done to obtain (1.51)-(1.52) in the proof of Theorem 1.5. We apply instead the mean-value theorem in Lemma 1.4, as detailed next. For technical reasons, we introduce first a proper complex matrix-valued function $\mathbf{F}_q(\mathbf{Q}_{-q})$ related to the MIMO multiuser waterfilling mapping $\mathsf{WF}_q(\mathbf{Q}_{-q})$ in (1.24) and, using Lemma 1.4, we study the Lipschitz properties of the function on $\mathcal{Q}_{-q}$. Then, building on this result, we show that the $\mathsf{WF}$ mapping satisfies (1.62).

[4] We define the Jacobian matrix of a differentiable complex matrix-valued function following the approach in [61], meaning that we treat the complex differential of the complex variable and its complex conjugate as independent. This approach simplifies the derivation of many complex derivative expressions. We refer the interested reader to [24, 61] for details.

[5] Milder conditions than (1.63) are given in [24], whose proof is much more involved and thus omitted here because of the space limitation.

Given $q \in \Omega$, let us introduce the following complex matrix-valued function $\mathbf{F}_q : \mathcal{Q}_{-q} \ni \mathbf{Q}_{-q} \mapsto \mathbb{S}_{++}^{n_{T_q}}$, defined as:

$$\mathbf{F}_q(\mathbf{Q}_{-q}) \triangleq \left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\right)^{-1}. \tag{1.64}$$

Observe that the function $\mathbf{F}_q(\mathbf{Q}_{-q})$ is continuous on $\mathcal{Q}_{-q}$ (implied from the continuity of $\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})$ at any $\mathbf{Q}_{-q} \succeq \mathbf{0}$ [25, Th. 10.7.1]) and differentiable on the interior of $\mathcal{Q}_{-q}$. The Jacobian matrix of $\mathbf{F}_q(\mathbf{Q}_{-q})$ is [24]:

$$\begin{aligned}\mathbf{D}_{\mathbf{Q}_{-q}}\mathbf{F}(\mathbf{Q}_{-q}) = \big[&\mathbf{G}_{1\,q}^*(\mathbf{Q}_{-q}) \otimes \mathbf{G}_{1\,q}(\mathbf{Q}_{-q}), \ldots, \mathbf{G}_{q-1\,q}^*(\mathbf{Q}_{-q}) \otimes \mathbf{G}_{q-1\,q}(\mathbf{Q}_{-q}), \ldots, \\ &\mathbf{G}_{q+1\,q}^*(\mathbf{Q}_{-q}) \otimes \mathbf{G}_{q+1\,q}(\mathbf{Q}_{-q}), \ldots, \mathbf{G}_{Q\,q}^*(\mathbf{Q}_{-q}) \otimes \mathbf{G}_{Q\,q}(\mathbf{Q}_{-q})\big],\end{aligned} \tag{1.65}$$

where

$$\mathbf{G}_{rq}(\mathbf{Q}_{-q}) \triangleq \left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\right)^{-1} \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{rq}. \tag{1.66}$$

It follows from Lemma 1.4 that, for any two different points $\mathbf{Q}_{-q}^{(1)}, \mathbf{Q}_{-q}^{(2)} \in \mathcal{Q}_{-q}$, with $\mathbf{Q}_{-q}^{(i)} = [\mathbf{Q}_1^{(i)}, \ldots, \mathbf{Q}_{q-1}^{(i)}, \mathbf{Q}_{q+1}^{(i)}, \ \ldots, \mathbf{Q}_Q^{(i)}]$ for $i = 1, 2$, there exists some $t \in (0, 1)$ such that, introducing

$$\mathbf{\Delta} \triangleq t\mathbf{Q}_{-q}^{(1)} + (1-t)\mathbf{Q}_{-q}^{(2)}, \tag{1.67}$$

we have:

$$\begin{aligned}\left\|\mathbf{F}_q\left(\mathbf{Q}_{-q}^{(1)}\right) - \mathbf{F}_q\left(\mathbf{Q}_{-q}^{(2)}\right)\right\|_F &\leq \left\|\mathbf{D}_{\mathbf{Q}_{-q}}\mathbf{F}_q(\mathbf{\Delta})\,\mathrm{vec}\left(\mathbf{Q}_{-q}^{(1)} - \mathbf{Q}_{-q}^{(2)}\right)\right\|_2 &(1.68)\\ &\leq \sum_{r \neq q} \left\|\mathbf{G}_{rq}^*(\mathbf{\Delta}) \otimes \mathbf{G}_{rq}(\mathbf{\Delta})\right\|_{2,\mathrm{mat}} \left\|\mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)}\right\|_F &(1.69)\\ &= \sum_{r \neq q} \rho\left(\mathbf{G}_{rq}^H(\mathbf{\Delta})\mathbf{G}_{rq}(\mathbf{\Delta})\right) \left\|\mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)}\right\|_F, &(1.70)\end{aligned}$$

where (1.68) follows from (1.60) (Lemma 1.4); (1.69) follows from the structure of $\mathbf{D}_{\mathbf{Q}_{-q}}\mathbf{F}_q$ (see (1.65)) and the triangle inequality [26]; and in (1.70) we used

$$\rho\left[\left(\mathbf{G}_{rq}^T \otimes \mathbf{G}_{rq}^H\right)\left(\mathbf{G}_{rq}^* \otimes \mathbf{G}_{rq}\right)\right] = \left(\rho\left[\mathbf{G}_{rq}^H \mathbf{G}_{rq}\right]\right)^2. \tag{1.71}$$

Observe that, differently from (1.53)-(1.54), the factor $\alpha_{rq}(\mathbf{\Delta}) \triangleq \rho\left[\mathbf{G}_{rq}^H(\mathbf{\Delta})\mathbf{G}_{rq}(\mathbf{\Delta})\right]$ in (1.70) depends, in general, on both $t \in (0,1)$ and the covariance matrices $\mathbf{Q}_{-q}^{(1)}$ and $\mathbf{Q}_{-q}^{(2)}$ through $\mathbf{\Delta}$ (see (1.67)):

$$\begin{aligned}\alpha_{rq}(\mathbf{\Delta}) = \rho\Big[&\mathbf{H}_{rq}^H \mathbf{R}_{-q}^{-1}(\mathbf{\Delta})\mathbf{H}_{qq}\left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{\Delta})\mathbf{H}_{qq}\right)^{-1} \\ &\times \left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{\Delta})\mathbf{H}_{qq}\right)^{-1}\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{\Delta})\mathbf{H}_{rq}\Big]\end{aligned} \tag{1.72}$$

where in (1.72) we used (1.66). Interestingly, in the case of square (nonsingular) channel matrices $\mathbf{H}_{qq}$, (1.72) reduces to $\alpha_{rq}(\mathbf{\Delta}) = \rho\left[\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{\sharp^H} \mathbf{H}_{qq}^{\sharp} \mathbf{H}_{rq}\right] = [\mathbf{S}_{\mathrm{pow}}]_{qr}$, where $\mathbf{S}_{\mathrm{pow}}$ is defined in (1.43), thus recovering the same contraction factor for the WF mapping as in Theorem 1.5. In the case of (strictly) full column-rank matrices $\mathbf{H}_{qq}$, an upper bound of $\alpha_{rq}(\mathbf{\Delta})$, independent of $\mathbf{\Delta}$ is [24]

$$\alpha_{rq}(\boldsymbol{\Delta}) < \text{innr}_q \cdot \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{rq}\right) \rho\left(\mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp}\right) \tag{1.73}$$

where $\text{innr}_q$ is defined in (1.46). The Lipschitz property of the WF mapping as given in (1.62) comes from (1.70) and (1.73), using the same steps as in the proof of Theorem 1.5. □

Comparing Theorems 1.5 and 1.6, one infers that conditions for the multiuser MIMO waterfilling mapping to be a block-contraction in the case of (strictly) full column-rank channel matrices are stronger than those required when the channels are full row-rank matrices.

#### 1.3.3.4 Case of full rank channel matrices

In the case in which the (direct) channel matrices are either full row-rank or full column-rank, we have the following contraction theorem for the WF mapping.

**Theorem 1.7** (Contraction property of WF mapping)**.** *Suppose that, for each* $q \in \Omega$, *either* $\text{rank}(\mathbf{H}_{qq}) = n_{R_q}$ *or* $\text{rank}(\mathbf{H}_{qq}) = n_{T_q}$. *Then, for any given* $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T > \mathbf{0}$, *the* WF *mapping defined in* (1.39) *is Lipschitz continuous on* $\mathcal{Q}$*:*

$$\left\|\mathsf{WF}(\mathbf{Q}^{(1)}) - \mathsf{WF}(\mathbf{Q}^{(2)})\right\|_{F,\text{block}}^{\mathbf{w}} \leq \|\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}\|_{\infty,\text{mat}}^{\mathbf{w}} \left\|\mathbf{Q}^{(1)} - \mathbf{Q}^{(2)}\right\|_{F,\text{block}}^{\mathbf{w}}, \tag{1.74}$$

*for all* $\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$, *where* $\|\cdot\|_{F,\text{block}}^{\mathbf{w}}$, $\|\cdot\|_{\infty,\text{mat}}^{\mathbf{w}}$ *and* $\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}$ *are defined in* (1.40), (1.42) *and* (1.45), *respectively. Furthermore, if the following condition is satisfied*

$$\|\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}\|_{\infty,\text{mat}}^{\mathbf{w}} < 1, \qquad \text{for some } \mathbf{w} > \mathbf{0}, \tag{1.75}$$

*then, the* WF *mapping is a block-contraction with modulus* $\beta = \|\widetilde{\mathbf{S}}_{\text{pow}}\|_{\infty,\text{mat}}^{\mathbf{w}}$.

### 1.3.4 Existence and uniqueness of the Nash equilibrium

We can now study game $\mathcal{G}_{\text{pow}}$ and derive conditions for the uniqueness of the NE, as given next.

**Theorem 1.8.** *Game* $\mathcal{G}_{\text{pow}}$ *always admits a NE, for any set of channel matrices and transmit power of the users. Furthermore, the NE is unique if*[6]

$$\rho(\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}) < 1, \tag{C1}$$

*where* $\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}$ *is defined in (1.45).*

*Proof.* The existence of a NE of game $\mathcal{G}_{\text{pow}}$ for any set of channel matrices and power budget follows directly from Theorem 1.2 (i.e., compact convex strategy

[6] Milder conditions are given in [24].

sets and continuous quasiconcave payoff functions). As far as the uniqueness of the NE is concerned, a sufficient condition is that the waterfilling mapping in (1.24) be a contraction with respect to some norm (Theorem 1.3). Hence, the sufficiency of (C1) in the case of full (column/row) rank channel matrices $\{\mathbf{H}_{qq}\}_{q\in\Omega}$ comes from Theorem 1.7 and the equivalence of the following two statements [52, Cor. 6.1]: i) there exists some $\mathbf{w} > \mathbf{0}$ such that $\|\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}\|_{\infty,\text{mat}}^{\mathbf{w}} < 1$; and ii) $\rho(\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}}) < 1$.

We focus now on the more general case in which the channel matrices $\mathbf{H}_{qq}$ may be rank deficient and prove that condition (C1) is still sufficient to guarantee the uniqueness of the NE. For any $q \in \overline{\Omega} \triangleq \{q \in \Omega \,|\, r_{H_{qq}} \triangleq \text{rank}(\mathbf{H}_{qq}) < \min(n_{T_q}, n_{R_q})\}$ and $\mathbf{Q}_{-q} \succeq \mathbf{0}$, the best-response $\mathbf{Q}_q^\star = \mathsf{WF}_q(\mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq})$—the solution to the rate-maximization problem in (1.22) for a given $\mathbf{Q}_{-q} \succeq \mathbf{0}$—will be orthogonal to the null space of $\mathbf{H}_{qq}$, implying $\mathbf{Q}_q^\star = \mathbf{V}_{q,1}\overline{\mathbf{Q}}_q^\star\mathbf{V}_{q,1}^H$ for some $\overline{\mathbf{Q}}_q^\star \in \mathbb{S}_+^{r_{H_{qq}}}$ such that $\text{Tr}(\overline{\mathbf{Q}}_q^\star) = P_q$, where $\mathbf{V}_{q,1} \in \mathbb{C}^{n_{T_q}\times r_{H_{qq}}}$ is a semiunitary matrix such that $\mathcal{R}(\mathbf{V}_{q,1}) = \mathcal{N}(\mathbf{H}_{qq})^\perp$. Thus, the best-response of each user $q \in \overline{\Omega}$ belongs to the following class of matrices:

$$\mathbf{Q}_q = \mathbf{V}_{q,1}\overline{\mathbf{Q}}_q\mathbf{V}_{q,1}^H, \qquad \text{with} \qquad \overline{\mathbf{Q}}_q \in \overline{\mathcal{Q}}_q \triangleq \left\{\mathbf{X} \in \mathbb{S}_+^{r_{H_{qq}}} \mid \text{Tr}(\mathbf{X}) = P_q\right\}. \tag{1.76}$$

Using (1.76) and introducing the (possibly) lower-dimensional covariance matrices $\overline{\mathbf{Q}}_q$'s and the modified channel matrices $\widetilde{\mathbf{H}}_{rq}$'s, defined respectively as

$$\widetilde{\mathbf{Q}}_q \triangleq \begin{cases} \overline{\mathbf{Q}}_q \in \mathbb{S}_+^{r_{H_{qq}}}, & \text{if } q \in \overline{\Omega}, \\ \mathbf{Q}_q \in \mathbb{S}_+^{n_{T_q}}, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \widetilde{\mathbf{H}}_{rq} \triangleq \begin{cases} \mathbf{H}_{rq}\mathbf{V}_{r,1}, & \text{if } r \in \overline{\Omega}, \\ \mathbf{H}_{rq}, & \text{otherwise,} \end{cases} \tag{1.77}$$

game $\mathcal{G}_{\text{pow}}$ can be recast in the following lower-dimensional game $\widetilde{\mathcal{G}}_{\text{pow}}$, defined as

$$(\widetilde{\mathcal{G}}_{\text{pow}}): \quad \begin{array}{ll} \underset{\widetilde{\mathbf{Q}}_q \succeq \mathbf{0}}{\text{maximize}} & \log\det\left(\mathbf{I} + \widetilde{\mathbf{H}}_{qq}^H\widetilde{\mathbf{R}}_{-q}^{-1}(\widetilde{\mathbf{Q}}_{-q})\widetilde{\mathbf{H}}_{qq}\widetilde{\mathbf{Q}}_q\right) \\ \text{subject to} & \text{Tr}(\widetilde{\mathbf{Q}}_q) = P_q, \end{array} \qquad \forall q \in \Omega, \tag{1.78}$$

where $\widetilde{\mathbf{R}}_{-q}(\widetilde{\mathbf{Q}}_{-q}) \triangleq \mathbf{R}_{n_{R_q}} + \sum_{r\neq q} \widetilde{\mathbf{H}}_{rq}\widetilde{\mathbf{Q}}_r\widetilde{\mathbf{H}}_{rq}^H$. It turns out that conditions guaranteeing the uniqueness of the NE of game $\widetilde{\mathcal{G}}_{\text{pow}}$ are sufficient also for the uniqueness of the NE of $\mathcal{G}_{\text{pow}}$.

Observe that, in the game $\widetilde{\mathcal{G}}_{\text{pow}}$, all channel matrices $\widetilde{\mathbf{H}}_{qq}$ are full rank matrices. We can thus use Theorem 1.7 and obtain the following sufficient condition for the uniqueness of the NE of both games $\widetilde{\mathcal{G}}_{\text{pow}}$ and $\mathcal{G}_{\text{pow}}$:

$$\rho(\widetilde{\mathbf{S}}_{\text{pow}}) < 1, \tag{1.79}$$

with

$$\left[\widetilde{\mathbf{S}}_{\text{pow}}\right]_{qr} \triangleq \begin{cases} \rho\left(\widetilde{\mathbf{H}}_{rq}^H\mathbf{H}_{qq}^{\sharp H}\mathbf{H}_{qq}^{\sharp}\widetilde{\mathbf{H}}_{rq}\right), & \text{if } r \neq q,\ r_{H_{qq}} = n_{R_q}, \\ \widetilde{\text{innr}}_q \cdot \rho\left(\widetilde{\mathbf{H}}_{rq}^H\widetilde{\mathbf{H}}_{rq}\right)\rho\left(\widetilde{\mathbf{H}}_{qq}^{\sharp H}\widetilde{\mathbf{H}}_{qq}^{\sharp}\right), & \text{if } r \neq q,\ r_{H_{qq}} < n_{R_q}, \\ 0 & \text{if } r = q, \end{cases} \tag{1.80}$$

and $\widetilde{\text{innr}}_q$ is defined as in (1.46), where each $\mathbf{H}_{rq}$ is replaced by $\widetilde{\mathbf{H}}_{rq}$. The sufficiency of (1.79) for (C1) follows from $\mathbf{0} \leq \widetilde{\mathbf{S}}_{\text{pow}} \leq \widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}} \Longrightarrow \rho(\widetilde{\mathbf{S}}_{\text{pow}}) \leq \rho(\widetilde{\mathbf{S}}_{\text{pow}}^{\text{up}})$ [54, Cor. 2.2.22]; which completes the proof. □

To give additional insight into the physical interpretation of sufficient conditions for the uniqueness of the NE, we provide the following.

**Corollary 1.1.** *If* $\text{rank}(\mathbf{H}_{qq}) = n_{R_q}$ *for all* $q \in \Omega$, *then a sufficient condition for (C1) in Theorem 1.8 is given by one of the two following set of conditions:*

$$\text{Low received MUI:} \quad \frac{1}{w_q} \sum_{r \neq q} \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp} \mathbf{H}_{rq}\right) w_r < 1, \quad \forall q \in \Omega, \quad \text{(C2)}$$

$$\text{Low generated MUI:} \quad \frac{1}{w_r} \sum_{q \neq r} \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp} \mathbf{H}_{rq}\right) w_q < 1, \quad \forall r \in \Omega, \quad \text{(C3)}$$

*where* $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T$ *is any positive vector.*

*If* $\text{rank}(\mathbf{H}_{qq}) \leq n_{T_q}$, *for all* $q \in \Omega$, *then a sufficient condition for (C1) is given by one of the two following set of conditions:*[7]

$$\text{Low received MUI:} \quad \frac{1}{w_q} \sum_{r \neq q} \text{innr}_q \cdot \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{rq}\right) \rho\left(\mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp}\right) w_r < 1, \ \forall q \in \Omega, \quad \text{(C4)}$$

$$\text{Low generated MUI:} \quad \frac{1}{w_r} \sum_{q \neq r} \text{innr}_q \cdot \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{rq}\right) \rho\left(\mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp}\right) w_q < 1, \ \forall r \in \Omega, \quad \text{(C5)}$$

*where the* $\text{innr}_q$*'s are defined in (1.46).* □

**Remark 2 - On the uniqueness conditions.** Conditions (C2)-(C3) and (C4)-(C5) provide a physical interpretation of the uniqueness of the NE: as expected, the uniqueness of the NE is ensured if the interference among the links is sufficiently small. The importance of (C2)-(C3) and (C4)-(C5) is that they quantify how small the interference must be to guarantee that the equilibrium is indeed unique. Specifically, conditions (C2) and (C4) can be interpreted as a constraint on the maximum amount of interference that each receiver can tolerate, whereas (C3) and (C5) introduce an upper bound on the maximum level of interference that each transmitter is allowed to generate. Surprisingly, the above conditions differ if the channel matrices $\{\mathbf{H}_{qq}\}_{q \in \Omega}$ are (strictly) tall or fat.

### 1.3.5 Distributed algorithms

In this section we focus on distributed algorithms that converge to the NE of game $\mathcal{G}_{\text{pow}}$. We consider totally asynchronous distributed algorithms, as

[7] The case in which some channel matrices $\mathbf{H}_{qq}$ are (strictly) tall and some others are fat or there are rank deficient channel matrices can be similarly addressed (c.f. [24]).

described in Section 1.2.2. Using the same notation as introduced in Section 1.2.2, the asynchronous MIMO IWFA is formally described in Algorithm 2, where $\mathbf{Q}_q^{(n)}$ denote the covariance matrix of the vector signal transmitted by user $q$ at the $n$-th iteration, and $\mathsf{T}_q(\mathbf{Q}_{-q})$ in (1.82) is the best-response function of user $q$:

$$\mathsf{T}_q(\mathbf{Q}_{-q}) \triangleq \mathsf{WF}_q\left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\right), \tag{1.81}$$

with $\mathsf{WF}_q(\cdot)$ defined in (1.24). The algorithm is totally asynchronous, meaning that one can use any arbitrary schedule $\{\tau_r^q(n)\}$ and $\{\mathcal{T}_q\}$ satisfying the standard assumptions (A1)-(A3), given in Section 1.2.2.

**Algorithm 2: MIMO Asynchronous IWFA**

1 : Set $n = 0$ and choose any feasible $\mathbf{Q}_q^{(0)}$;
2 : `repeat`
3 :
$$\mathbf{Q}_q^{(n+1)} = \begin{cases} \mathsf{T}_q\left(\mathbf{Q}_{-q}^{(\boldsymbol{\tau}^q(n))}\right), & \text{if } n \in \mathcal{T}_q, \\ \mathbf{Q}_q^{(n)}, & \text{otherwise;} \end{cases} \quad \forall q \in \Omega \tag{1.82}$$
4 : `until the prescribed convergence criterion is satisfied`

Sufficient conditions guaranteeing the global convergence of the algorithm are given in Theorem 1.9, whose proof follows from results given in Section 1.2.2.

**Theorem 1.9.** *Suppose that condition* (*C1*) *of Theorem 1.8 is satisfied. Then, as* n $\rightarrow +\infty$, *the asynchronous MIMO IWFA, described in Algorithm 2, converges to the unique NE of game* $\mathcal{G}_{\text{pow}}$, *for any set of feasible initial conditions and updating schedule satisfying* (*A1*)*-*(*A3*).

**Remark 3 - Features of Algorithm** 2. Algorithm 2 contains as special cases a plethora of algorithms, each one obtained by a possible choice of the scheduling of the users in the updating procedure (i.e., the parameters $\{\tau_r^q(n)\}$ and $\{\mathcal{T}_q\}$). Two well-known special cases are the *sequential* and the *simultaneous* MIMO IWFA, where the users update their own strategies *sequentially* [7, 8, 10, 21] and *simultaneously* [7, 8, 10, 62, 55], respectively. Interestingly, since condition (C1) does not depend on the particular choice of $\{\mathcal{T}_q\}$ and $\{\tau_r^q(n)\}$, the important result coming from the convergence analysis is that all the algorithms resulting as special cases of the asynchronous MIMO IWFA are guaranteed to globally converge to the unique NE of the game, under the same set of convergence conditions. Moreover they have the following desired properties:

- *Low complexity and distributed nature*: The algorithm can be implemented in a distributed way, since each user, to compute his best response $\mathsf{T}_q(\cdot)$ in (1.81), only needs to measure the overall interference-plus-noise covariance matrix $\mathbf{R}_{-q}(\mathbf{Q}_{-q})$ and waterfill over the equivalent channel $\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}$.

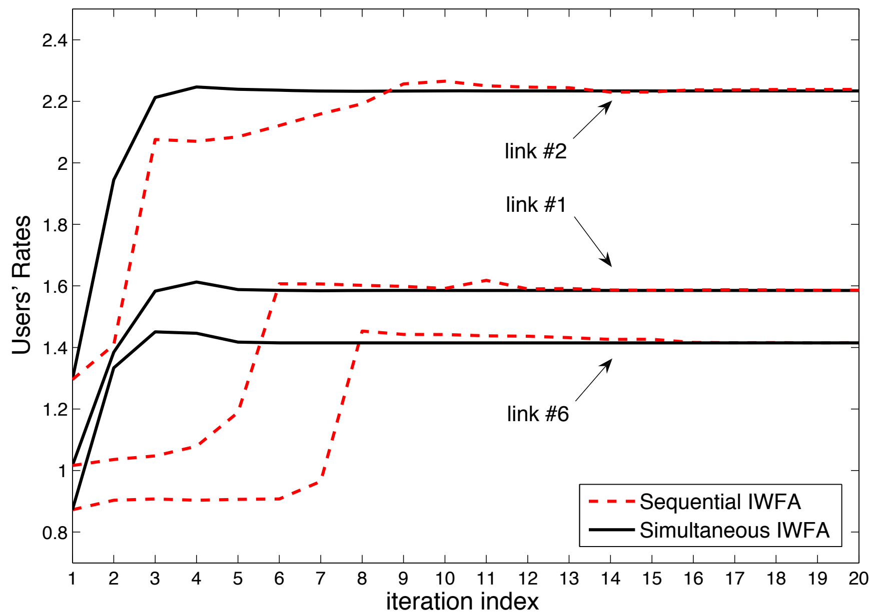


**Figure 1.1** Rates of the MIMO links versus iterations: sequential IWFA (dashed line curves) and simultaneous IWFA (solid line curves).

- *Robustness*: Algorithm 2 is robust against missing or outdated updates of secondary users. This feature strongly relaxes the constraints on the synchronization of the users' updates with respect to those imposed, for example, by the simultaneous or sequential updating schemes.

- *Fast convergence behavior:* The simultaneous version of the proposed algorithm converges in a very few iterations, even in networks with many active secondary users. As expected, the sequential IWFA is slower than the simultaneous IWFA, especially if the number of active secondary users is large, since each user is forced to wait for all the users scheduled ahead, before updating his own covariance matrix. As an example, in Figure 1.1 we compare the performance of the sequential and simultaneous IWFA, in terms of convergence speed, for a given set of MIMO channel realizations. We consider a cellular network composed by 7 (regular) hexagonal cells, sharing the same spectrum. For the sake of simplicity, we assume that in each cell there is only one active link, corresponding to the transmission from the BS (placed at the center of the cell) to a MT placed in a corner of the cell. The overall network is thus modeled as eight $4 \times 4$ MIMO interference wideband channels, according to (1.1). In Figure 1.1, we show the rate evolution of the links of three cells corresponding to the sequential IWFA and simultaneous IWFA as a function of the iteration index $n$ . To make the figure not excessively overcrowded, we plot only the curves of 3 out of 8 links.

## 1.4 Opportunistic Communications under Individual Interference Constraints

In this section, we focus now on the more general resource allocation problem under interference constraints as given in (1.6). We start considering power constraints (1.2) and individual null constraints (1.5), since they are suitable to model the white-space filling paradigm. More specifically, the problem formulated leads directly to what we call game $\mathcal{G}_{\text{null}}$. We also consider an alternative game formulation, $\mathcal{G}_{\infty}$, with improved convergence properties; however, it does not correspond to any physical scenario so it is a rather artificial formulation. The missing ingredient is provided by another game formulation, $\mathcal{G}_{\alpha}$, that does have a nice physical interpretation and asymptotically is equivalent to $\mathcal{G}_{\infty}$ (in the sense specified next); thus inheriting the improved convergence properties as well as the physical interpretation. After that, we consider more general opportunistic communications by allowing also soft shaping interference constraints (1.6) through the game $\mathcal{G}_{\text{soft}}$.

### 1.4.1 Game with null constraints

Given the rate functions in (1.4), the rate maximization game among the secondary users in the presence of the power constraints (1.2) and the null constraints (1.5) is formally defined as:

$$(\mathcal{G}_{\text{null}}): \quad \begin{array}{ll} \underset{\mathbf{Q}_q \succeq \mathbf{0}}{\text{maximize}} & R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) \\ \text{subject to} & \mathsf{Tr}\,(\mathbf{Q}_q) \leq P_q, \quad \mathbf{U}_q^H \mathbf{Q}_q = \mathbf{0} \end{array} \quad \forall q \in \Omega, \tag{1.83}$$

where $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$ is defined in (1.4). Without the null constraints, the solution of each optimization problem in (1.83) would lead to the MIMO waterfilling solution, as studied in Section 1.3. The presence of the null constraints modifies the problem and the solution for each user is not necessarily a waterfilling anymore. Nevertheless, we show now that introducing a proper projection matrix the solutions of (1.83) can still be efficiently computed via a waterfilling-like expression. To this end, we rewrite game $\mathcal{G}_{\text{null}}$ in the form of game $\mathcal{G}_{\text{pow}}$ in (1.22), as detailed next.

We need the following intermediate definitions. For any $q \in \Omega$, given $r_{H_{qq}} \triangleq \text{rank}(\mathbf{H}_{qq})$ and $r_{U_q} \triangleq \text{rank}(\mathbf{U}_q)$, with $r_{U_q} < n_{T_q}$ w.l.o.g., let $\mathbf{U}_q^{\perp} \in \mathbb{C}^{n_{T_q} \times r_{U_q^{\perp}}}$ be the semi-unitary matrix orthogonal to $\mathbf{U}_q$ (note that $\mathcal{R}(\mathbf{U}_q^{\perp}) = \mathcal{R}(\mathbf{U}_q)^{\perp}$), with $r_{U_q^{\perp}} \triangleq \text{rank}(\mathbf{U}_{qq}^{\perp}) = n_{T_q} - r_{U_q}$ and $\mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})} = \mathbf{U}_q^{\perp} \mathbf{U}_q^{\perp H}$ be the orthogonal projection onto $\mathcal{R}(\mathbf{U}_q^{\perp})$. We can then rewrite the null constraint $\mathbf{U}_q^H \mathbf{Q}_q = \mathbf{0}$ in (1.83) as

$$\mathbf{Q}_q = \mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})} \mathbf{Q}_q \mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}. \tag{1.84}$$

At this point, the problem can be further simplified by noting that the constraint $\mathbf{Q}_q = \mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}\,\mathbf{Q}_q\mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}$ in (1.83) is redundant, provided that the original channels $\mathbf{H}_{rq}$ are replaced with the modified channels $\mathbf{H}_{rq}\mathbf{P}_{\mathcal{R}(\mathbf{U}_r^{\perp})}$. The final formulation then becomes

$$\begin{array}{ll}\underset{\mathbf{Q}_q\succeq\mathbf{0}}{\text{maximize}} & \log\det\left(\mathbf{I}+\mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}\mathbf{H}_{qq}^{H}\tilde{\mathbf{R}}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}\mathbf{Q}_q\right)\\ \text{subject to} & \mathsf{Tr}(\mathbf{Q}_q)\leq P_q\end{array}\quad\forall q\in\Omega \tag{1.85}$$

where

$$\tilde{\mathbf{R}}_{-q}(\mathbf{Q}_{-q}) \triangleq \mathbf{R}_{n_q} + \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{P}_{\mathcal{R}(\mathbf{U}_r^{\perp})}\mathbf{Q}_r\mathbf{P}_{\mathcal{R}(\mathbf{U}_r^{\perp})}\mathbf{H}_{rq}^{H}\succ\mathbf{0}. \tag{1.86}$$

Indeed, for any user $q$, any optimal solution $\mathbf{Q}_q^{\star}$ in (1.85)—the MIMO waterfilling solution—will be orthogonal to the null space of $\mathbf{H}_{qq}\mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}$, whatever $\tilde{\mathbf{R}}_{-q}(\mathbf{Q}_{-q})$ is (recall that $\tilde{\mathbf{R}}_{-q}(\mathbf{Q}_{-q})\succ\mathbf{0}$ for all feasible $\mathbf{Q}_{-q}$), implying $\mathcal{R}(\mathbf{Q}_q^{\star})\subseteq\mathcal{R}(\mathbf{U}_q^{\perp})$.

Building on the equivalence of (1.83) and (1.85), we can focus on the game in (1.85) and apply the framework developed in Section 1.3.1 to fully characterize game $\mathcal{G}_{\text{null}}$, by deriving the structure of the Nash equilibria and the conditions guaranteeing the existence/uniqueness of the equilibrium and the global convergence of the proposed distributed algorithms. We address these issues in the next sections.

#### 1.4.1.1 Nash equilibria: existence and uniqueness

To write the Nash equilibria of game $\mathcal{G}_{\text{null}}$ in a convenient form, we need the following notations and definitions. Given the game in (1.85), we introduce the set $\tilde{\Omega}$ of user' indexes associated to the rank deficient matrices $\mathbf{H}_{qq}\mathbf{U}_q^{\perp}$, defined as

$$\tilde{\Omega}\triangleq\left\{q\in\Omega\,:\,r_{H_{qq}U_q^{\perp}}\triangleq\text{rank}\left(\mathbf{H}_{qq}\mathbf{U}_q^{\perp}\right)<\min\left(n_{R_q},r_{U_q^{\perp}}\right)\right\}, \tag{1.87}$$

and the semi-unitary matrices $\mathbf{V}_{q,1}\in\mathbb{C}^{r_{U_q^{\perp}}\times r_{H_{qq}U_q^{\perp}}}$ such that $\mathcal{R}(\mathbf{V}_{q,1}) = \mathcal{N}\left(\mathbf{H}_{qq}\mathbf{U}_q^{\perp}\right)^{\perp}$. To obtain weak conditions guaranteeing the uniqueness of the NE and convergence of the proposed algorithms, it is useful to introduce also: the modified channel matrices $\tilde{\mathbf{H}}_{rq}\in\mathbb{C}^{n_{R_q}\times r_{H_{rr}U_r^{\perp}}}$, defined as:

$$\tilde{\mathbf{H}}_{rq}=\begin{cases}\mathbf{H}_{rq}\mathbf{U}_r^{\perp}\mathbf{V}_{r,1}, & \text{if } r\in\tilde{\Omega},\\ \mathbf{H}_{rq}\mathbf{U}_r^{\perp}, & \text{otherwise,}\end{cases},\qquad\forall r,q\in\Omega, \tag{1.88}$$

the interference-plus-noise to noise ratios $\widetilde{\text{innr}}_q$, defined as

$$\widetilde{\text{innr}}_q\triangleq\frac{\rho\left(\mathbf{R}_{n_q}+\sum_{r\neq q}P_r\tilde{\mathbf{H}}_{rq}\tilde{\mathbf{H}}_{rq}^{H}\right)}{\lambda_{\min}\left(\mathbf{R}_{n_q}\right)}\geq 1 \tag{1.89}$$

and the nonnegative matrices $\mathbf{S}_{\text{null}} \in \mathbb{R}_{+}^{Q\times Q}$ defined as

$$[\mathbf{S}_{\text{null}}]_{qr} \triangleq \begin{cases} \widetilde{\text{innr}}_q \cdot \rho\left(\tilde{\mathbf{H}}_{rq}^H \tilde{\mathbf{H}}_{rq}\right) \rho\left(\tilde{\mathbf{H}}_{qq}^{\sharp H} \tilde{\mathbf{H}}_{qq}^{\sharp}\right), & \text{if } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \tag{1.90}$$

Using the above definitions, the full characterization of the Nash equilibria of $\mathcal{G}_{\text{null}}$ is stated in the following theorem, whose proof follows similar steps of that of Theorem 1.8 and thus is omitted.

**Theorem 1.10** (Existence and uniqueness of the NE of $\mathcal{G}_{\text{null}}$)**.** *Consider the game $\mathcal{G}_{\text{null}}$ in (1.83) and suppose w.l.o.g. that $r_{U_q} < n_{T_q}$, for all $q \in \Omega$. Then, the following hold:*

**(a)** *there always exists a NE, for any set of channel matrices, power constraints for the users, and null shaping constraints;*

**(b)** *all the Nash equilibria are the solutions to the following set of nonlinear matrix-value fixed-point equations:*

$$\mathbf{Q}_q^{\star} = \mathbf{U}_q^{\perp} \mathsf{WF}_q\left(\mathbf{U}_q^{\perp H} \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}^{\star}) \mathbf{H}_{qq} \mathbf{U}_q^{\perp}\right) \mathbf{U}_q^{\perp H}, \quad \forall q \in \Omega, \tag{1.91}$$

*with $\mathsf{WF}_q(\cdot)$ and $\mathbf{R}_{-q}(\mathbf{Q}_{-q})$ defined in (1.24) and (1.3), respectively;*

**(c)** *the NE is unique if* [8]

$$\rho(\mathbf{S}_{\text{null}}) < 1, \tag{C6}$$

*with $\mathbf{S}_{\text{null}}$ defined in (1.90).*

**Remark 4 - Structure of the Nash equilibria.** The structure of the Nash equilibria as given in (1.91) shows that the null constraints in the transmissions of secondary users can be handled without affecting the computational complexity: Given the strategies $\mathbf{Q}_{-q}$ of the others, the optimal covariance matrix $\mathbf{Q}_q^{\star}$ of each user $q$ can be efficiently computed via a MIMO waterfilling solution, provided that the original channel matrix $\mathbf{H}_{qq}$ is replaced by $\mathbf{H}_{qq}\mathbf{U}_q^{\perp}$. Observe that the structure of $\mathbf{Q}_q^{\star}$ in (1.91) has an intuitive interpretation: To guarantee that each user $q$ does not transmit over a given subspace (spanned by the columns of $\mathbf{U}_q$), *whatever* the strategies of the other users are, while maximizing his information rate, it is enough to induce in the original channel matrix $\mathbf{H}_{qq}$ a null space that (at least) coincides with the subspace where the transmission is not allowed. This is precisely what is done in the pay-off functions in (1.85) by replacing $\mathbf{H}_{qq}$ with $\mathbf{H}_{qq}\mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})}$.

**Remark 5 - Physical interpretation of uniqueness conditions.** Similarly to (C1), condition (C6) quantifies how small the interference among secondary users must be to guarantee the uniqueness of the NE of the game. What affects the uniqueness of the equilibrium is only the MUI generated by secondary users

[8] Milder (but less easy to check) uniqueness conditions than (C6) are given in [24].

in the subspaces orthogonal to $\mathcal{R}(\mathbf{U}_q)$'s, i.e., the subspaces where secondary users are allowed to transmit (note that all the Nash equilibria $\{\mathbf{Q}_q^\star\}_{q\in\Omega}$ satisfy $\mathcal{R}(\mathbf{Q}_q^\star) \subseteq \mathcal{R}(\mathbf{U}_q^\perp)$, for all $q \in \Omega$). Interestingly, one can also obtain uniqueness conditions that are independent of the null constraints $\{\mathbf{U}_q\}_{q\in\Omega}$: it is sufficient to replace in (C6) the modified channels $\widetilde{\mathbf{H}}_{rq}$'s with the original channel matrices $\mathbf{H}_{rq}$ [63]. This means that if the NE is unique in a game without null constraints, then it is also unique with null constraint, which is not a trivial statement.

Observe that all the conditions above depend, among all, on the interference generated by the primary users and the power budgets of the secondary users through the $\widetilde{\text{innr}}_q$'s; which is an undesired result. We overcome this issue in Section 1.4.2.

#### 1.4.1.2 Distributed algorithms

To reach the Nash equilibria of game $\mathcal{G}_{\text{null}}$ while satisfying the null constraints (1.5), one can use the asynchronous IWFA as given in Algorithm 2, where the best-response $\mathsf{T}_q(\mathbf{Q}_{-q})$ of each user $q$ in (1.82) is replaced by the following:

$$\mathsf{T}_q(\mathbf{Q}_{-q}) \triangleq \mathbf{U}_q^\perp \mathsf{WF}_q\left(\mathbf{U}_q^{\perp H}\mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{U}_q^\perp\right)\mathbf{U}_q^{\perp H}, \tag{1.92}$$

where the MIMO waterfilling operator $\mathsf{WF}_q$ is defined in (1.24). Observe that such an algorithm has the same nice properties of the algorithm proposed to reach the Nash equilibria of game $\mathcal{G}_{\text{pow}}$ in (1.22) (see Remark 4 in Section 1.3.5). In particular, even in the presence of null constraints, the best-response of each player $q$ can be efficiently and locally computed via a MIMO waterfilling-like solution, provided that each channel $\mathbf{H}_{qq}$ is replaced by the channel $\mathbf{H}_{qq}\mathbf{U}_q^\perp$. Furthermore, thanks to the inclusion of the null constraints in the game the game theoretical formulation, the proposed asynchronous IWFA based on the mapping $\mathsf{T}_q(\mathbf{Q}_{-q})$ in (1.92) does not suffer of the main drawback of the classical sequential IWFA [20, 62, 64], i.e., the violation of the interference temperature limits [2]. The convergence properties of the algorithm are given in the following theorem (the proof follows from results in Section 1.2.2).

**Theorem 1.11.** *Suppose that condition* (C6) *of Theorem 1.10 is satisfied. Then, as $n \to +\infty$, the asynchronous MIMO IWFA, described in Algorithm 1 and based on mapping in (1.92), converges to the unique NE of game $\mathcal{G}_{\text{null}}$, for any set of feasible initial conditions and updating schedule satisfying* (A1)-(A3).

### 1.4.2 Game with null constraints via virtual noise shaping

We have seen how to deal efficiently with null constraints in the rate maximization game. However, condition (C6) guaranteeing the uniqueness of the NE as well as the convergence of the asynchronous IWFA depends, among all, on the interference generated by the primary users (through the $\text{innr}_q$'s), which is an undesired result. In such a case, the NE might not be unique and there is no guarantee that the proposed algorithms converge to an equilibrium. To overcome this issue, we propose here an alternative approach to impose null constraints (1.5) on the transmissions of secondary users, based on the introduction of *virtual interferers*. This leads to a new game with more relaxed uniqueness and convergence conditions. The solutions of this new game are in general different to the Nash equilibria of $\mathcal{G}_{\text{null}}$, but the two games are numerically shown to have almost the same performance in terms of sum-rate.

The idea behind this alternative approach can be easily understood if one considers the transmission over SISO frequency-selective channels, where all the channel matrices have the same eigenvectors (the DFT vectors): to avoid the use of a given subchannel, it is sufficient to introduce a "virtual" noise with sufficiently high power over that subchannel. The same idea cannot be directly applied to the MIMO case, as arbitrary MIMO channel matrices have different right/left singular vectors from each other. Nevertheless, we show how to bypass this difficulty to design the covariance matrix of the virtual noise (to be added to the noise covariance matrix of each secondary receiver), so that all the Nash equilibria of the game satisfy the null constraints along the specified directions. For the sake of notation simplicity and because of the space limitation, we focus here only on the case of square nonsingular channel matrices $\mathbf{H}_{qq}$, i.e., $r_{H_{qq}} = n_{R_q} = n_{T_q}$ for all $q \in \Omega$. Let us consider the following strategic non-cooperative game:

$$(\mathcal{G}_\alpha): \quad \begin{array}{ll} \underset{\mathbf{Q}_q \succeq \mathbf{0}}{\text{maximize}} & \log\det\left(\mathbf{I} + \mathbf{H}_{qq}^H \mathbf{R}_{-q,\alpha}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{Q}_q\right) \\ \text{subject to} & \mathsf{Tr}(\mathbf{Q}_q) \le P_q \end{array} \quad \forall q \in \Omega, \tag{1.93}$$

where

$$\mathbf{R}_{-q,\alpha}(\mathbf{Q}_{-q}) \triangleq \mathbf{R}_{n_q} + \sum_{r \neq q} \mathbf{H}_{rq}\mathbf{Q}_r\mathbf{H}_{rq}^H + \alpha \hat{\mathbf{U}}_q \hat{\mathbf{U}}_q^H \succ \mathbf{0}, \tag{1.94}$$

denotes the MUI-plus-noise covariance matrix observed by secondary user $q$, plus the covariance matrix $\alpha \hat{\mathbf{U}}_q \hat{\mathbf{U}}_q^H$ of the virtual interference along $\mathcal{R}(\hat{\mathbf{U}}_q)$, where $\hat{\mathbf{U}}_q \in \mathbb{C}^{n_{R_q} \times r_{\hat{U}_q}}$ is a (strictly) tall matrix assumed to be full column-rank with $r_{\hat{U}_q} \triangleq \text{rank}(\hat{\mathbf{U}}_q) < r_{H_{qq}} (= n_{T_q} = n_{R_q})$ w.l.o.g., and $\alpha$ is a positive constant. Our interest is on deriving the asymptotic properties of the solutions of $\mathcal{G}_\alpha$, as $\alpha \to +\infty$, and the structure of $\hat{\mathbf{U}}_q$'s making the null constraints (1.5) satisfied.

To this end, we introduce the following intermediate definitions first. For each $q$, define the (strictly) tall full column-rank matrix $\hat{\mathbf{U}}_q^\perp \in \mathbb{C}^{n_{R_q} \times r_{\hat{U}_q^\perp}}$, with $r_{\hat{U}_q^\perp} = n_{R_q} - r_{\hat{U}_q} = \text{rank}(\hat{\mathbf{U}}_q^\perp)$ and such that $\mathcal{R}(\hat{\mathbf{U}}_q^\perp) = \mathcal{R}(\hat{\mathbf{U}}_q)^\perp$, and the modified (strictly fat) channel matrices $\hat{\mathbf{H}}_{rq} \in \mathbb{C}^{r_{\hat{U}_q^\perp} \times n_{T_r}}$:

$$\hat{\mathbf{H}}_{rq} = \hat{\mathbf{U}}_q^{\perp H} \mathbf{H}_{rq} \qquad \forall r,\, q \in \Omega. \tag{1.95}$$

We then introduce the auxiliary game $\mathcal{G}_\infty$, defined as:

$$(\mathcal{G}_\infty): \quad \begin{array}{ll} \underset{\mathbf{Q}_q \succeq \mathbf{0}}{\text{maximize}} & \log\det\left(\mathbf{I} + \hat{\mathbf{H}}_{qq}^H \hat{\mathbf{R}}_{-q}^{-1}(\mathbf{Q}_{-q})\hat{\mathbf{H}}_{qq}\mathbf{Q}_q\right) \\ \text{subject to} & \mathsf{Tr}(\mathbf{Q}_q) \le P_q \end{array} \qquad \forall q \in \Omega, \tag{1.96}$$

where

$$\hat{\mathbf{R}}_{-q}(\mathbf{Q}_{-q}) \triangleq \hat{\mathbf{U}}_q^{\perp H} \mathbf{R}_{n_q} \hat{\mathbf{U}}_q^{\perp} + \sum_{r \neq q} \hat{\mathbf{H}}_{rq} \mathbf{Q}_r \hat{\mathbf{H}}_{rq}^H. \tag{1.97}$$

Building on the results obtained in Section 1.3.1, we study both games $\mathcal{G}_\alpha$ and $\mathcal{G}_\infty$, and derive the relationship between the Nash equilibria of $\mathcal{G}_\alpha$ and $\mathcal{G}_\infty$, showing that, under milder conditions, the two games are asymptotically equivalent (in the sense specified next), which will provide an alternative way to impose the null constraints (1.5).

#### 1.4.2.1 Nash equilibria: existence and uniqueness

We introduce the nonnegative matrices $\mathbf{S}_{\infty,1}$, $\mathbf{S}_{\infty,2} \in \mathbb{R}_+^{Q \times Q}$, defined as

$$[\mathbf{S}_{\infty,1}]_{qr} \triangleq \begin{cases} \rho\left(\hat{\mathbf{H}}_{rq}^H \hat{\mathbf{H}}_{qq}^{\sharp H} \hat{\mathbf{H}}_{qq}^{\sharp} \hat{\mathbf{H}}_{rq}\right), & \text{if } r \neq q, \\ 0, & \text{otherwise}, \end{cases} \tag{1.98}$$

$$[\mathbf{S}_{\infty,2}]_{qr} \triangleq \begin{cases} \rho\left(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{P}_{\mathcal{R}(\mathbf{U}_q^{\perp})} \mathbf{H}_{qq}^{-1} \mathbf{H}_{rq}\right), & \text{if } r \neq q, \\ 0, & \text{otherwise}. \end{cases} \tag{1.99}$$

**Game $\mathcal{G}_\alpha$:** The full characterization of game $\mathcal{G}_\alpha$ is given in the following theorem, whose proof is based on existence and uniqueness results given in Section 1.2 and the contraction properties of the multiuser waterfilling mapping as derived in Section 1.3.1.

**Theorem 1.12** (Existence and uniqueness of the NE of $\mathcal{G}_\alpha$)**.** *Consider the game $\mathcal{G}_\alpha$ in (1.93), the following hold:*

**(a)** *there always exists a NE, for any set of channel matrices, transmit power of the users, virtual interference matrices $\hat{\mathbf{U}}_q \hat{\mathbf{U}}_q^H$'s, and $\alpha \ge 0$;*

**(b)** *all the Nash equilibria are the solutions to the following set of nonlinear matrix-value fixed-point equations:*

$$\mathbf{Q}_{q,\alpha}^\star = \mathsf{WF}_q\left(\mathbf{H}_{qq}^H \mathbf{R}_{-q,\alpha}^{-1}(\mathbf{Q}_{-q,\alpha}^\star)\mathbf{H}_{qq}\right), \quad \forall q \in \Omega, \tag{1.100}$$

*with $\mathsf{WF}_q(\cdot)$ defined in (1.24);*

**(c)** *the NE is unique if*

$$\rho(\mathbf{S}_{\text{pow}}) < 1, \tag{C7}$$

*with $\mathbf{S}_{\text{pow}}$ defined in (1.43).*

**Remark 6 - On the properties of game $\mathcal{G}_\alpha$.** Game $\mathcal{G}_\alpha$ has some interesting properties, namely: i) The Nash equilibria depend on $\alpha$ and the virtual interference covariance matrices $\hat{\mathbf{U}}_q\hat{\mathbf{U}}_q^H$'s, whereas uniqueness condition (C7) *does not*; and ii) as desired, the uniqueness of the NE is not affected by the presence of the primary users. Exploring this degree of freedom, one can thus choose, under condition (C7), the proper set of $\alpha$ and $\hat{\mathbf{U}}_q\hat{\mathbf{U}}_q^H$'s so that the (unique) NE of the game satisfies the null constraints (1.5), while keeping the uniqueness property of the equilibrium unaltered and independent of both $\hat{\mathbf{U}}_q\hat{\mathbf{U}}_q^H$'s and the interference level generated by the primary users. It is not difficult to realize that the optimal design of $\alpha$ and $\hat{\mathbf{U}}_q\hat{\mathbf{U}}_q^H$'s in $\mathcal{G}_\alpha$ passes through the properties of game $\mathcal{G}_\infty$, as detailed next.

**Game $\mathcal{G}_\infty$:** The properties of game $\mathcal{G}_\infty$ are given in the following.

**Theorem 1.13** (Existence and uniqueness of the NE of $\mathcal{G}_\infty$)**.** *Consider the game $\mathcal{G}_\infty$ in (1.96) and suppose w.l.o.g. that $r_{\hat{U}_q} < r_{H_{qq}} (= n_{R_q} = n_{T_q})$, for all $q \in \Omega$. Then, the following hold:*

**(a)** *there always exists a NE, for any set of channel matrices, transmit power of the users, and virtual interference matrices $\hat{\mathbf{U}}_q$'s.*

**(b)** *all the Nash equilibria are the solutions to the following set of nonlinear matrix-value fixed-point equations:*

$$\mathbf{Q}^\star_{q,\infty} = \mathsf{WF}_q\left(\hat{\mathbf{H}}_{qq}^H \hat{\mathbf{R}}_{-q}^{-1}(\mathbf{Q}^\star_{-q,\infty})\hat{\mathbf{H}}_{qq}\right), \quad \forall q \in \Omega, \tag{1.101}$$

*with $\mathsf{WF}_q(\cdot)$ defined in (1.24), and satisfy*

$$\mathcal{R}(\mathbf{Q}^\star_{q,\infty}) \perp \mathcal{R}(\mathbf{H}_{qq}^{-1}\hat{\mathbf{U}}_q), \quad \forall q \in \Omega; \tag{1.102}$$

**(c)** *the NE is unique if*

$$\rho(\mathbf{S}_{\infty,1}) < 1, \tag{C8}$$

*with $\mathbf{S}_{\infty,1}$ defined in (1.98).*

**Remark 7 - Null constraints and virtual noise directions**. Condition (1.102) provides the desired relationship between the directions of the virtual noise to be introduced in the noise covariance matrix of the user (see (1.97))—the matrix $\hat{\mathbf{U}}_q$—and the real directions along with user $q$ will not allocate any power, i.e., the matrix $\mathbf{U}_q$. It turns out that if user $q$ is not allowed to allocate power along $\mathbf{U}_q$, it is sufficient to choose in (1.97) $\hat{\mathbf{U}}_q \triangleq \mathbf{H}_{qq}\mathbf{U}_q$. Exploring this choice, the structure of the Nash equilibria of game $\mathcal{G}_\infty$ can be further simplified, as given next.

**Corollary 1.2.** *Consider the game $\mathcal{G}_\infty$ and the null constraints (1.5) with $r_{H_{qq}} = n_{R_q} = n_{T_q}$ and $\hat{\mathbf{U}}_q = \mathbf{H}_{qq}\mathbf{U}_q$, for all $q \in \Omega$. Then, the following hold:*

**(a)** *all the Nash equilibria are the solutions to the following set of nonlinear matrix-value fixed-point equations:*

$$\mathbf{Q}_{q,\infty}^{\star} = \mathbf{U}_q^{\perp} \mathsf{WF}_q \left( \left( \mathbf{U}_q^{\perp H} \mathbf{H}_{qq}^{-1} \mathbf{R}_{-q}(\mathbf{Q}_{-q,\infty}^{\star}) \mathbf{H}_{qq}^{-H} \mathbf{U}_q^{\perp} \right)^{-1} \right) \mathbf{U}_q^{\perp H}, \quad \forall q \in \Omega, \tag{1.103}$$

*with* $\mathsf{WF}_q(\cdot)$ *defined in (1.24);*

**(b)** *the NE is unique if*

$$\rho(\mathbf{S}_{\infty,2}) < 1, \tag{C9}$$

*with* $\mathbf{S}_{\infty,2}$ *defined in (1.99).*

Observe that, since $\mathcal{R}(\mathbf{Q}_{q,\infty}^{\star}) \subseteq \mathcal{R}(\mathbf{U}_q^{\perp})$, any solution $\mathbf{Q}_{q,\infty}^{\star}$ to (1.103) will be orthogonal to $\mathbf{U}_q$, whatever the strategies $\mathbf{Q}_{-q,\infty}^{\star}$ of the other secondary users are. Thus, all the Nash equilibria in (1.103) satisfy the null constraints (1.5).

At this point, however, one may ask: What is the physical meaning of a solution to (1.103)? Does it still correspond to a waterfilling over a real MIMO channel and thus to the maximization of mutual information? The interpretation of game $\mathcal{G}_{\infty}$ and its solutions passes through game $\mathcal{G}_{\alpha}$ : we indeed prove next that the solutions to (1.103) can be reached as Nash equilibria of game $\mathcal{G}_{\alpha}$ for sufficiently large $\alpha > 0$.

**Relationship between game $\mathcal{G}_{\alpha}$ and $\mathcal{G}_{\infty}$:** The asymptotic behaviour of the Nash equilibria of $\mathcal{G}_{\alpha}$ as $\alpha \to +\infty$, is given in the following (the proof can be found in [63]).

**Theorem 1.14.** *Consider games $\mathcal{G}_{\alpha}$ and $\mathcal{G}_{\infty}$, with $r_{\hat{U}_q} < r_{H_{qq}} (= n_{T_q} = n_{R_q})$ for all $q \in \Omega$, and suppose that condition (C7) in Theorem 1.12 is satisfied. Then, the following hold:*

**(a)** *$\mathcal{G}_{\alpha}$ and $\mathcal{G}_{\infty}$ admit a unique NE, denoted by $\mathbf{Q}_{\alpha}^{\star}$ and $\mathbf{Q}_{\infty}^{\star}$, respectively;*

**(b)** *the two games are asymptotically equivalent, in the sense that*

$$\lim_{\alpha \to +\infty} \mathbf{Q}_{\alpha}^{\star} = \mathbf{Q}_{\infty}^{\star}. \tag{1.104}$$

Invoking Theorem 1.14 and Corollary 1.2 we obtained the following desired property of game $\mathcal{G}_{\alpha}$: Under condition (C7) of Theorem 1.12, the (unique) NE of $\mathcal{G}_{\alpha}$ tends to satisfy the null constraints (1.5) for sufficiently large $\alpha$ (see (1.103) and (1.104)), provided that the virtual interference matrices $\{\hat{\mathbf{U}}_q\}_{q \in \Omega}$ in (1.94) are chosen according to Corollary 1.2. This approach provides an alternative way to impose the null constraints (1.5).

#### 1.4.2.2 Distributed algorithms

To reach the Nash equilibria of game $\mathcal{G}_{\alpha}$ while satisfying the null constraints (1.5) (for sufficiently large $\alpha$), one can use the asynchronous IWFA as given in Algorithm 2, where the best-response $\mathsf{T}_q(\mathbf{Q}_{-q})$ in (1.82) is replaced by

$$\mathsf{T}_{q,\alpha}(\mathbf{Q}_{-q}) \triangleq \mathsf{WF}_q \left( \mathbf{H}_{qq}^{H} \mathbf{R}_{-q,\alpha}^{-1}(\mathbf{Q}_{-q},) \mathbf{H}_{qq} \right), \tag{1.105}$$

where the MIMO waterfilling operator $\mathsf{WF}_q$ is defined in (1.24). Observe that such an algorithm has the same nice properties of the algorithm proposed to reach the Nash equilibria of game $\mathcal{G}_{\text{null}}$ in (1.83). In particular, the best-response of each player $q$ can be still efficiently and locally computed via a MIMO waterfilling-like solution, provided that the virtual interference covariance matrix $\alpha \mathbf{U}_q \mathbf{U}_q^H$ is added to the MUI covariance matrix $\mathbf{R}_{-q}(\mathbf{Q}_{-q})$ measured at the $q$-th receiver. The convergence properties of the algorithm are given in the following.

**Theorem 1.15.** *Consider games $\mathcal{G}_\alpha$ and $\mathcal{G}_\infty$, with $r_{\hat{U}_q} < r_{H_{qq}} (= n_{T_q} = n_{R_q})$ for all $q \in \Omega$, and suppose that condition (C7) of Theorem 1.12 is satisfied. Then, the following hold:*

**(a)** *as $n \to \infty$, the asynchronous MIMO IWFA, described in Algorithm 2 and based on mapping in (1.105), converges* uniformly *with respect to $\alpha \in \mathbb{R}_+$ to the* unique *NE of game $\mathscr{G}_\alpha$, for any set of feasible initial conditions, and updating schedule satisfying* (A1)-(A3)*;*

**(b)** *the sequence $\mathbf{Q}_\alpha^{(n)} = \left(\mathbf{Q}_{q,\alpha}^{(n)}\right)_{q\in\Omega}$ generated by the algorithm satisfies:*

$$\lim_{n\to+\infty} \lim_{\alpha\to+\infty} \mathbf{Q}_\alpha^{(n)} = \lim_{\alpha\to+\infty} \lim_{n\to+\infty} \mathbf{Q}_\alpha^{(n)} = \mathbf{Q}_\infty^\star, \tag{1.106}$$

*where $\mathbf{Q}_\infty^\star$ is the (unique) NE of game $\mathcal{G}_\infty$.*

**Remark 8 - On the convergence/uniqueness conditions.** Condition (C7) guaranteeing the global convergence of the asynchronous IWFA to the unique NE of $\mathcal{G}_\alpha$ (for *any* $\alpha > 0$) has the desired property of being independent of both the interference generated by the primary users and the power budgets of the secondary users, which is the main difference with the uniqueness and convergence condition (C6) associated to game $\mathcal{G}_{\text{null}}$ in (1.83).

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**Example 1.1: Comparison of uniqueness/convergence conditions.** Since the uniqueness/convergence conditions given so far depend on the channel matrices $\{\mathbf{H}_{rq}\}_{r,q\in\Omega}$, there is a nonzero probability that they will not be satisfied for a given channel realization drawn from a given probability space. To quantify the adequacy of our conditions, we tested them over a set of random channel matrices whose elements are generated as circularly symmetric complex Gaussian random variables with variance equal to the inverse of the square distance between the associated transmitter-receiver links (flat-fading channel model). We consider a hierarchical CR network as depicted in Figure 1.2(a), composed of 3 secondary user MIMO links and one primary user (the base station BS), sharing the same band. To preserve the QoS of the primary users, null constraints are imposed on the secondary users in the direction of the receiver of the primary user. In Figure 1.2(b), we plot the probability that conditions (C6) and (C7) are satisfied versus the intra-pair distance $d \in (0; 1)$ (normalized by the cell's side) (see Figure 1.2(a)) between each secondary transmitter and the corresponding receiver (assumed for the simplicity of representation to be equal for all the secondary links), for different values of the transmit/receive antennas. Since condition (C6) depends on the interference generated by the primary user and the power budgets of the secondary users, we considered two different values of the SNR at the receivers of the secondary users, namely $\text{snr}_q \triangleq P_q/\sigma^2_{q,\text{tot}} = 0\text{dB}$ and $\text{snr}_q = 8\text{dB}$, for all $q \in \Omega$, where $\sigma^2_{q,\text{tot}}$ is the variance of thermal noise plus the interference generated by the primary user over all the substreams.

As expected, the probability of the uniqueness of the NE of both games $\mathcal{G}_{\text{null}}$ and $\mathcal{G}_{\alpha}$ and convergence of the IWFAs increases as each secondary transmitter approaches his receiver, corresponding to a decrease of the overall MUI. Moreover, condition (C6) is confirmed to be stronger than (C7) whatever the number of transmit/receive antennas, the intra-pair distance $d$, and the SNR value are, implying that game $\mathcal{G}_{\alpha}$ admits weaker (more desirable) uniqueness/convergence conditions than those of the original game $\mathcal{G}_{\text{null}}$.

---

**Example 1.2: Performance of $\mathcal{G}_{\text{null}}$ and $\mathcal{G}_{\infty}$.** As an example, in Figure 1.3, we compare games $\mathcal{G}_{\text{null}}$ and $\mathcal{G}_{\infty}$ in terms of sum-rate. All the Nash equilibria are computed using Algorithm 2 with mapping in (1.81) for game $\mathcal{G}_{\text{null}}$ and (1.103) for game $\mathcal{G}_{\infty}$. Specifically, in Figure 1.3(a), we plot the sum-rate at the (unique) NE of the games $\mathcal{G}_{\text{null}}$ and $\mathcal{G}_{\infty}$ for the CR network depicted in Figure 1.2(a) as a function of the intra-pair distance $d \in (0, 1)$ among the links, for different numbers of transmit/receive antennas. In Figure 1.3(b), we plot the outage sum-rate for the same systems as in Figure 1.3(a) and $d = 0.5$. For each secondary user, a null constraint in the direction of the receiver of the primary user is imposed. From the figures one infers that games $\mathcal{G}_{\text{null}}$ and $\mathcal{G}_{\infty}$ have almost the same performance in terms of sum-rate at the NE; even if in the game $\mathcal{G}_{\infty}$, given the strategies of the others, each player does not maximize his own rate, as happens in the game $\mathcal{G}_{\text{null}}$. This is due to the fact that the Nash equilibria of game $\mathcal{G}_{\text{null}}$ are in general not Pareto efficient.

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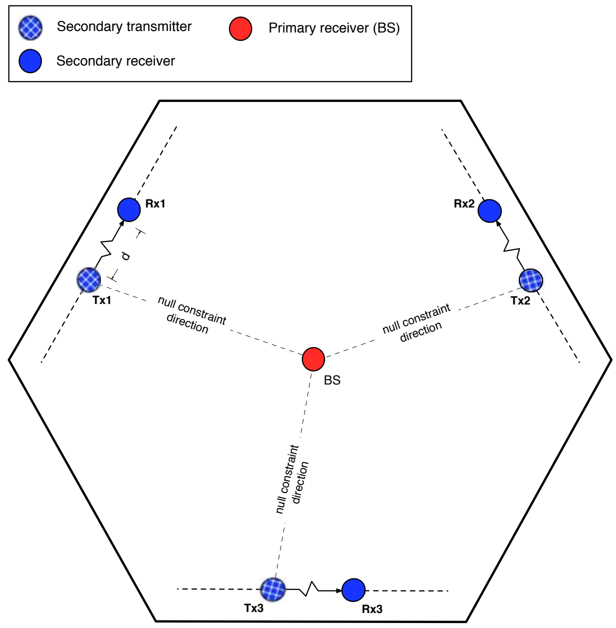
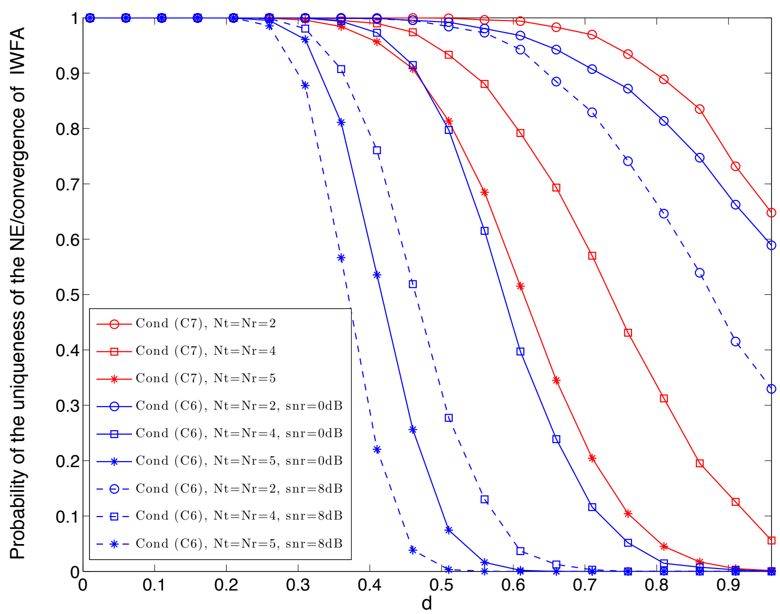
In conclusion, the above results indicate that game $\mathcal{G}_{\alpha}$, with sufficiently large $\alpha$, may be a valid alternative to game $\mathcal{G}_{\text{null}}$ to impose the null constraints (1.5), with more relaxed conditions for convergence.

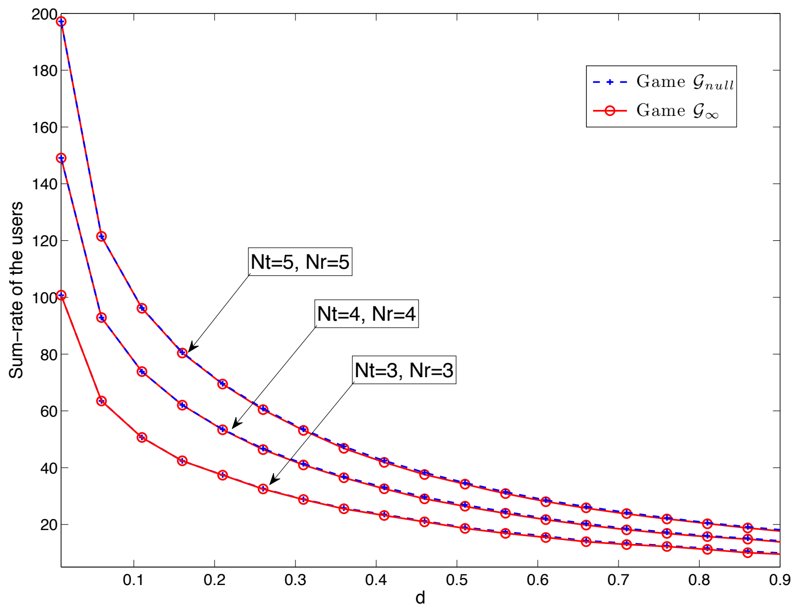



subplot (a)

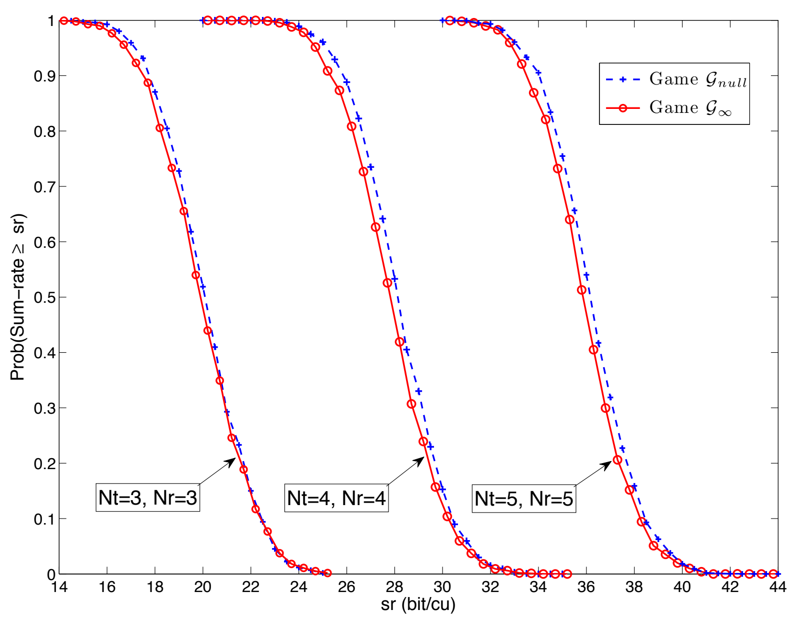



subplot (b)

**Figure 1.2** (a) CR MIMO system; (b) Probability of the uniqueness of the NE of games $\mathcal{G}_{\rm null}$ and $\mathcal{G}_{\alpha}$ and convergence of the asynchronous IWFA as a function of the normalized intra-pair distance $d \in (0, 1)$.




subplot (a)




subplot (b)

**Figure 1.3** Performance of games $\mathcal{G}_{\text{null}}$ and $\mathcal{G}_{\infty}$ in terms of Nash equilibria for the CR MIMO system given in Figure 1.2(a): (a) Average sum-rate at the NE versus the normalized intra-pair distance $d \in (0, 1)$ for $d = 0.5$; (b) Cumulative Distribution Function (CDF) of the sum-rate for the games $\mathcal{G}_{\text{null}}$ (plus-mark dashed-dot blue line curves) and $\mathcal{G}_{\infty}$ (circle-mark solid red line curves).

### 1.4.3 Game with null and soft constraints

We focus now on the rate maximization in the presence of both null and *soft shaping* constraints. The resulting game can be formulated as follows:

$$(\mathcal{G}_{\text{soft}}): \quad \begin{array}{ll} \underset{\mathbf{Q}_q \succeq \mathbf{0}}{\text{maximize}} & R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) \\ \text{subject to} & \mathsf{Tr}\left(\mathbf{G}_q^H \mathbf{Q}_q \mathbf{G}_q\right) \leq P_{\text{SU},q}^{\text{ave}} \\ & \lambda_{\max}\left(\mathbf{G}_q^H \mathbf{Q}_q \mathbf{G}_q\right) \leq P_{\text{SU},q}^{\text{peak}} \\ & \mathbf{U}_q^H \mathbf{Q}_q = \mathbf{0} \end{array} \quad \forall q \in \Omega, \tag{1.107}$$

where we have included both types of individual soft shaping constraints as well as null shaping constraints, and the transmit power constraint (1.2) has been absorbed into the trace soft constraint for convenience. For this, it is necessary that each $r_{G_q} \triangleq \mathsf{rank}\,(\mathbf{G}_q) = n_{T_q}$; otherwise there would be no power constraint along $\mathcal{N}(\mathbf{G}_q^H)$ (if user $q$ is allowed to transmit along $\mathcal{N}(\mathbf{G}_q^H)$, i.e., $\mathcal{N}(\mathbf{G}_q^H) \cap \mathcal{R}(\mathbf{U}_q)^\perp \neq \emptyset$). It is worth pointing out that, in practice, a transmit power constraint (1.2) in (1.107) will be dominated by the trace shaping constraint, which motivates the absence in (1.107) of an explicit power constraint as (1.2). More specifically, constraint (1.2) becomes redundant whenever $P_{\text{SU},q}^{\text{ave}} \leq P_q\, \lambda_{\min}(\mathbf{G}_q \mathbf{G}_q^H)$. On the other hand, if $P_{\text{SU},q}^{\text{ave}} \geq P_q\, \lambda_{\max}(\mathbf{G}_q \mathbf{G}_q^H)$, then constraint (1.6) can be removed without loss of optimality, and game $\mathcal{G}_{\text{soft}}$ reduces in the form of game $\mathcal{G}_{\text{null}}$. In the following, we then focus on the former case only.

#### 1.4.3.1 Nash equilibria: existence and uniqueness

Before studying game $\mathcal{G}_{\text{soft}}$, we need the following intermediate definitions. For any $q \in \Omega$, define the tall matrix $\overline{\mathbf{U}}_q \in \mathbb{C}^{n_{G_q} \times r_{U_q}}$ as $\overline{\mathbf{U}}_q \triangleq \mathbf{G}_q^\sharp \mathbf{U}_q$ (recall that $n_{G_q} \geq n_{T_q} > r_{U_q}$), and introduce: the semi-unitary matrix $\overline{\mathbf{U}}_q^\perp \in \mathbb{C}^{n_{G_q} \times r_{\overline{U}_q^\perp}}$ orthogonal to $\overline{\mathbf{U}}_q$, with $r_{\overline{U}_q^\perp} = n_{G_q} - r_{U_q} = \text{rank}(\overline{\mathbf{U}}_q^\perp)$, the set of modified channels $\overline{\mathbf{H}}_{rq} \in \mathbb{C}^{n_{R_q} \times r_{\overline{U}_r^\perp}}$, defined as

$$\overline{\mathbf{H}}_{rq} = \mathbf{H}_{rq} \mathbf{G}_r^{\sharp H} \overline{\mathbf{U}}_r^\perp, \quad \forall r, q \in \Omega, \tag{1.108}$$

the interference-plus-noise to noise ratios $\overline{\text{innr}}_q$'s, defined as

$$\overline{\text{innr}}_q \triangleq \frac{\rho\left(\mathbf{R}_{n_q} + \sum_{r \neq q} P_r \overline{\mathbf{H}}_{rq} \overline{\mathbf{H}}_{rq}^H\right)}{\lambda_{\min}(\mathbf{R}_{n_q})} \geq 1 \quad q \in \Omega, \tag{1.109}$$

and the nonnegative matrix $\mathbf{S}_{\text{soft}} \in \mathbb{R}_+^{Q \times Q}$ :

$$[\mathbf{S}_{\text{soft}}]_{qr} \triangleq \begin{cases} \overline{\text{innr}}_q \cdot \rho\left(\overline{\mathbf{H}}_{rq}^H \overline{\mathbf{H}}_{rq}\right) \rho\left(\overline{\mathbf{H}}_{qq}^{\sharp H} \overline{\mathbf{H}}_{qq}^\sharp\right), & \text{if } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \tag{1.110}$$

These definitions are useful to obtain sufficient conditions for the uniqueness of the NE of $\mathcal{G}_{\text{soft}}$. Finally, we introduce for any $q \in \Omega$ and given $n_q \in \{1, 2, \ldots, n_{T_q}\}$, the *modified* MIMO waterfilling operator $\overline{\mathsf{WF}}_q : \mathbb{S}_+^{n_q \times n_q} \ni \mathbf{X} \rightarrow$

$\mathbb{S}_{+}^{n_q \times n_q}$, defined as

$$\overline{\mathsf{WF}}_q(\mathbf{X}) \triangleq \mathbf{U}_X \left[\mu_{q,X}\mathbf{I}_{r_X} - \mathbf{D}_X^{-1}\right]_0^{P_q^{\text{peak}}} \mathbf{U}_X^H, \tag{1.111}$$

where $\mathbf{U}_X \in \mathbb{C}^{n_q \times r_X}$ and $\mathbf{D}_q \in \mathbb{R}_{++}^{r_X \times r_X}$ are defined as in (1.24) and $\mu_{q,X} > 0$ is the water-level chosen to satisfy $\text{Tr}\left\{\left[\mu_{q,X}\mathbf{I}_{r_X} - \mathbf{D}_X^{-1}\right]_0^{P_q^{\text{peak}}}\right\} = \min(P_q, r_X P_q^{\text{peak}})$ (see, e.g., [65] for practical algorithms to compute the water-level $\mu_{q,X}$ in (1.111)). Using the above definitions, we can now characterize the Nash equilibria of game $\mathcal{G}_{\text{soft}}$, as shown next.

**Theorem 1.16** (Existence and structure of the NE of $\mathcal{G}_{\text{soft}}$)**.** *Consider the game $\mathcal{G}_{\text{soft}}$ in (1.107), and suppose w.l.o.g. that $r_{G_q} = n_{T_q}$, for all $q \in \Omega$ (all matrices $\mathbf{G}_q$ are full row-rank). Then, the following hold:*

**(a)** *there always exists a NE, for any set of channel matrices and null/soft shaping constraints;*

**(b)** *if, in addition, $r_{U_q} < r_{H_{qq}}$ and $\text{rank}(\mathbf{H}_{qq}\mathbf{G}_q^{\sharp H}\overline{\mathbf{U}}_q^{\perp}) = r_{\overline{U}_q^{\perp}}$ for all $q \in \Omega$, all the Nash equilibria are the solutions to the following set of nonlinear matrix-value fixed-point equations:*

$$\mathbf{Q}_q^{\star} = \mathbf{G}_q^{\sharp H}\overline{\mathbf{U}}_q^{\perp}\overline{\mathsf{WF}}_q\left(\overline{\mathbf{H}}_{qq}^H\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}^{\star})\overline{\mathbf{H}}_{qq}\right)\overline{\mathbf{U}}_q^{\perp H}\mathbf{G}_q^{\sharp}, \quad \forall q \in \Omega, \tag{1.112}$$

*with $\overline{\mathsf{WF}}_q(\cdot)$ and $\mathbf{R}_{-q}(\mathbf{Q}_{-q})$ defined in (1.111) and (1.3), respectively.*

*Proof.* The proof of theorem is based on the following intermediate result.

**Lemma 1.5.** *Given $\mathbb{S}_{+}^{n_T} \ni \mathbf{R}_H = \mathbf{V}_H\mathbf{\Lambda}_H\mathbf{V}_H^H$, with $r_{R_H} = \text{rank}(\mathbf{R}_H)$, the solution to the following optimization problem*

$$\begin{array}{ll} \underset{\mathbf{Q}\succeq\mathbf{0}}{\text{maximize}} & \log\det(\mathbf{I} + \mathbf{R}_H\mathbf{Q}) \\ \text{subject to} & \mathsf{Tr}(\mathbf{Q}) \le P_T, \\ & \lambda_{\max}(\mathbf{Q}) \le P^{\text{peak}}, \end{array} \tag{1.113}$$

*with $P_T \le P^{\text{peak}} r_{R_H}$, is unique and it is given by*

$$\mathbf{Q}^{\star} = \mathbf{V}_{H,1}\left[\mu\mathbf{I}_{r_{R_H}} - \mathbf{\Lambda}_{H,1}^{-1}\right]_0^{P^{\text{peak}}}\mathbf{V}_{H,1}^H, \tag{1.114}$$

*where $\mathbf{V}_{H,1} \in \mathbb{C}^{n_T \times r_{R_H}}$ is the semi-unitary matrix of the eigenvectors of matrix $\mathbf{R}_H$ corresponding to the $r_{R_H}$ positive eigenvalues in the diagonal matrix $\mathbf{\Lambda}_{H,1}$, and $\mu > 0$ satisfies $\text{Tr}\left([\mu\mathbf{I}_{r_{R_H}} - \mathbf{\Lambda}_{H,1}^{-1}]_0^{P^{\text{peak}}}\right) = P_T$.*

Under $r_{G_q} = n_{T_q}$, for all $q \in \Omega$, game $\mathcal{G}_{\text{soft}}$ admits at least a NE, since it satisfies Theorem 1.2.

We prove now (1.112). To this end, we rewrite $\mathcal{G}_{\text{soft}}$ in (1.107) in a more convenient form. Introducing the transformation:

$$\overline{\mathbf{Q}}_q \triangleq \mathbf{G}_q^H\mathbf{Q}_q\mathbf{G}_q, \qquad \forall q \in \Omega \tag{1.115}$$

one can rewrite $\mathcal{G}_{\text{soft}}$ in terms of $\overline{\mathbf{Q}}_q$ as

$$\begin{array}{ll} \underset{\overline{\mathbf{Q}}_q \succeq \mathbf{0}}{\text{maximize}} & \log\det\left(\mathbf{I} + \mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\mathbf{G}_q^{\sharp}\mathbf{H}_{qq}^H\overline{\mathbf{R}}_{-q}^{-1}(\overline{\mathbf{Q}}_{-q})\mathbf{H}_{qq}\mathbf{G}_q^{\sharp H}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\overline{\mathbf{Q}}_q\right) \\ \text{subject to} & \mathsf{Tr}(\overline{\mathbf{Q}}_q) \le P_{\text{SU},q}^{\text{ave}} \\ & \lambda_{\max}(\overline{\mathbf{Q}}_q) \le P_{\text{SU},q}^{\text{peak}} \\ & \overline{\mathbf{Q}}_q = \mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\overline{\mathbf{Q}}_q\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})} \end{array} \quad \forall q \in \Omega, \tag{1.116}$$

where $\overline{\mathbf{R}}_{-q}(\overline{\mathbf{Q}}_{-q}) \triangleq \mathbf{R}_{n_q} + \sum_{r\neq q} \mathbf{H}_{rq}\mathbf{G}_r^{\sharp H}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_r^{\perp})}\overline{\mathbf{Q}}_r\mathbf{H}_{rq}^H\mathbf{G}_r^{\sharp}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_r^{\perp})}$. Observe now that the power constraint $\mathsf{Tr}(\overline{\mathbf{Q}}_q) \le P_{\text{SU},q}^{\text{ave}}$ in (1.116) can be replaced with $\mathsf{Tr}(\overline{\mathbf{Q}}_q) \le \overline{P}_{\text{SU},q}^{\text{ave}}$ w.l.o.g., where $\overline{P}_{\text{SU},q}^{\text{ave}} \triangleq \min(P_{\text{SU},q}^{\text{ave}}, r_{\overline{U}_q^{\perp}}P_{\text{SU},q}^{\text{peak}})$. Indeed, because of the null constraint, any solution $\overline{\mathbf{Q}}_q^{\star}$ to (1.116) will satisfy $\text{rank}(\overline{\mathbf{Q}}_q^{\star}) \le r_{\overline{U}_q^{\perp}}$, *whatever* the strategies $\overline{\mathbf{Q}}_{-q}$ of the others are, implying $\mathsf{Tr}(\overline{\mathbf{Q}}_q) = \sum_{k=1}^{r_{\overline{U}_q^{\perp}}} \lambda_k(\overline{\mathbf{Q}}_q) \le P_{\text{SU},q}^{\text{ave}}$ (the eigenvalues $\lambda_k(\overline{\mathbf{Q}}_q)$ are assumed to be arranged in decreasing order); which, together to $\lambda_{\max}(\overline{\mathbf{Q}}_q) \le P_{\text{SU},q}^{\text{peak}}$, leads to the desired equivalence. Using $\text{rank}(\mathbf{H}_{qq}\mathbf{G}_q^{\sharp H}\overline{\mathbf{U}}_q^{\perp}) = r_{\overline{U}_q^{\perp}}$ and invoking Lemma 1.5, game in (1.116) can be further simplified to

$$\begin{array}{ll} \underset{\overline{\mathbf{Q}}_q \succeq \mathbf{0}}{\text{maximize}} & \log\det\left(\mathbf{I} + \mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\mathbf{G}_q^{\sharp}\mathbf{H}_{qq}^H\overline{\mathbf{R}}_{-q}^{-1}(\overline{\mathbf{Q}}_{-q})\mathbf{H}_{qq}\mathbf{G}_q^{\sharp H}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\overline{\mathbf{Q}}_q\right) \\ \text{subject to} & \mathsf{Tr}(\overline{\mathbf{Q}}_q) \le \overline{P}_{\text{SU},q}^{\text{ave}} \\ & \lambda_{\max}(\overline{\mathbf{Q}}_q) \le P_{\text{SU},q}^{\text{peak}} \end{array} \quad \forall q \in \Omega. \tag{1.117}$$

Indeed, according to (1.114) in Lemma 1.5, any optimal solution $\overline{\mathbf{Q}}_q^{\star}$ to (1.117) will satisfy $\mathcal{R}(\overline{\mathbf{Q}}_q^{\star}) \subseteq \mathcal{R}(\overline{\mathbf{U}}_q^{\perp})$, implying that the null constraint $\overline{\mathbf{Q}}_q = \mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\overline{\mathbf{Q}}_q\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}$ in (1.116) is redundant.

Given the game in (1.117), all the Nash equilibria satisfy the following MIMO waterfilling-like equation (Lemma 1.5):

$$\overline{\mathbf{Q}}_q^{\star} = \overline{\mathsf{WF}}_q\left(\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\mathbf{G}_q^{\sharp}\mathbf{H}_{qq}^H\overline{\mathbf{R}}_{-q}^{-1}(\overline{\mathbf{Q}}_{-q}^{\star})\mathbf{H}_{qq}\mathbf{G}_q^{\sharp H}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})}\right) \tag{1.118}$$

$$= \overline{\mathbf{U}}_q^{\perp}\overline{\mathsf{WF}}_q\left(\overline{\mathbf{U}}_q^{\perp H}\mathbf{G}_q^{\sharp}\mathbf{H}_{qq}^H\overline{\mathbf{R}}_{-q}^{-1}(\overline{\mathbf{Q}}_{-q}^{\star})\mathbf{H}_{qq}\mathbf{G}_q^{\sharp H}\overline{\mathbf{U}}_q^{\perp}\right)\overline{\mathbf{U}}_q^{\perp H}, \ \forall q \in \Omega. \tag{1.119}$$

The structure of the Nash equilibria of game $\mathcal{G}_{\text{soft}}$ in (1.107) as given in (1.112) follows directly from (1.115) and (1.119). □

**Remark 9 - On the structure of the Nash equilibria**. The structure of the Nash equilibria in (1.112) states that the optimal transmission strategy of each user leads to a diagonalizing transmission with a proper power allocation, after pre/post multiplication by matrix $\mathbf{G}_q^{\sharp H}\overline{\mathbf{U}}_q^{\perp}$. Thus, even in the presence of soft constraints, the optimal transmission strategy of each user $q$, given the strategies

$\mathbf{Q}_{-q}$ of the others, can be efficiently computed via a MIMO waterfilling-like solution. Note that the Nash equilibria in (1.112) satisfy the null constrains in (1.5), since $\mathcal{R}(\overline{\mathbf{U}}_q^{\perp})^{\perp} = \mathcal{R}\left(\mathbf{G}_q^{\sharp}\mathbf{U}_q\right)$, implying $\mathbf{U}_q^H \mathbf{G}_q^{\sharp H} \overline{\mathbf{U}}_q^{\perp} = \mathbf{0}$ and thus $\mathcal{R}(\mathbf{Q}_q^{\star}) \perp \mathcal{R}\left(\mathbf{U}_q\right)$, for all $\mathbf{Q}_{-q} \succeq \mathbf{0}$ and $q \in \Omega$.

We provide now a more convenient expression for the Nash equilibria given in (1.112), that will be instrumental to derive conditions for the uniqueness of the equilibrium and the convergence of the distributed algorithms. Introducing the convex closed sets $\overline{\mathcal{Q}}_q$ defined as

$$\overline{\mathcal{Q}}_q \triangleq \left\{ \mathbf{X} \in \mathbb{S}_+^{n_{T_q}} \mid \operatorname{Tr}\{\mathbf{X}\} = \overline{P}_{\mathrm{SU},q}^{\mathrm{ave}}, \quad \lambda_{\max}(\mathbf{X}) \leq P_{\mathrm{SU},q}^{\mathrm{peak}} \right\}, \tag{1.120}$$

where $\overline{P}_{\mathrm{SU},q}^{\mathrm{ave}} \triangleq \min(P_{\mathrm{SU},q}^{\mathrm{ave}}, r_{\overline{U}_q^{\perp}} P_{\mathrm{SU},q}^{\mathrm{peak}})$, we have the following equivalent expression for the MIMO waterfilling solutions in (1.112), whose proof is similar to that of Lemma 1.2 and thus is omitted.

**Lemma 1.6** (NE as a projection)**.** *The set of nonlinear matrix-value fixed-point equations in (1.112) can be equivalently rewritten as*

$$\mathbf{Q}_q^{\star} = \mathbf{G}_q^{\sharp H} \overline{\mathbf{U}}_q^{\perp} \left[ -\left( \left( \overline{\mathbf{H}}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}^{\star}) \overline{\mathbf{H}}_{qq} \right)^{\sharp} + c_q \mathbf{P}_{\mathcal{N}(\overline{\mathbf{H}}_{qq})} \right) \right]_{\overline{\mathcal{Q}}_q} \overline{\mathbf{U}}_q^{\perp H} \mathbf{G}_q^{\sharp}, \qquad \forall q \in \Omega, \tag{1.121}$$

*where $c_q$ is a positive constant that can be chosen independent of $\mathbf{Q}_{-q}$ (c.f. [63]) and $\overline{\mathcal{Q}}_q$ is defined in (1.120).*

Using Lemma 1.6, we can study contraction properties of the multiuser MIMO waterfilling mapping $\overline{\mathsf{WF}}$ in (1.112) via (1.121) (following the same approach as in Theorem 1.7) and obtain sufficient conditions guaranteeing the uniqueness of the NE of game $\mathcal{G}_{\mathrm{soft}}$, as given next.

**Theorem 1.17** (Uniqueness of the NE)**.** *The solution to (1.121) is unique if*

$$\rho(\mathbf{S}_{\mathrm{soft}}) < 1, \tag{C8}$$

*where $\mathbf{S}_{\mathrm{soft}}$ is defined in (1.110).* □

Condition (C8) is also sufficient for the convergence of the distributed algorithms to the unique NE of $\mathcal{G}_{\mathrm{soft}}$, as detailed in the next section.

#### 1.4.3.2 Distributed algorithms

Similarly to games $\mathcal{G}_{\mathrm{null}}$ and $\mathcal{G}_{\alpha}$, the Nash equilibria of game $\mathcal{G}_{\mathrm{soft}}$ can be reached using the asynchronous IWFA algorithm given in Algorithm 2, based on the mapping

$$\overline{\mathsf{T}}_q(\mathbf{Q}_{-q}) \triangleq \mathbf{G}_q^{\sharp H} \overline{\mathbf{U}}_q^{\perp} \overline{\mathsf{WF}}_q \left( \overline{\mathbf{H}}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}) \overline{\mathbf{H}}_{qq} \right) \overline{\mathbf{U}}_q^{\perp H} \mathbf{G}_q^{\sharp}, \quad q \in \Omega, \tag{1.122}$$

where the MIMO waterfilling operator is defined in (1.111) and the modified channels $\overline{\mathbf{H}}_{qq}$'s are defined in (1.108). Observe that such an algorithm has the same nice properties of the algorithm proposed to reach the Nash equilibria of game $\mathcal{G}_{\text{null}}$ in (1.83) (see Remark 4 in Section 1.3.5), such as: low-complexity, distributed and asynchronous nature, fast convergence behaviour. Moreover, thanks to our game theoretical formulation including null and/or soft shaping constraints, the algorithm does not suffer of the main drawback of the classical sequential IWFA [20, 62, 64], i.e., the violation of the interference temperature limits [2]. The convergence properties of the algorithm are given in the following.

**Theorem 1.18.** *Suppose that condition* (*C8*) *in Theorem 1.17 is satisfied. Then, as* $n \to +\infty$, *the asynchronous MIMO IWFA, described in Algorithm 2 and based on the mapping in (1.122), converges to the unique solution to (1.121), for any set of feasible initial conditions, and updating schedule satisfying* (A1)-(A3).

## 1.5 Opportunistic Communications under Global Interference Constraints

We focus now the design of CR system in (1.1), including the global interference constraints in (1.7), instead of the conservative individual constraints considered so far. This problem has been formulated and studied in [66]. Because of the space limitation, here we provide only some basic results without proofs. For the sake of simplicity, we focus only on block transmissions over SISO frequency-selective channels. It is well-known that, in such a case, multicarrier transmission is capacity achieving for large block-length [11]. This allows the simplification of the system model in (1.1), since each channel matrix $\mathbf{H}_{rq}$ becomes a $N \times N$ Toeplitz circulant matrix with eigendecomposition $\mathbf{H}_{rq} = \mathbf{F}\mathbf{D}_{rq}\mathbf{F}^H$, where $\mathbf{F}$ is the normalized IFFT matrix, i.e., $[\mathbf{F}]_{ij} \triangleq e^{j2\pi(i-1)(j-1)/N}/\sqrt{N}$ for $i, j = 1, \ldots, N$, $N$ is the length of transmitted block, $\mathbf{D}_{rq} = \text{diag}(\{H_{rq}(k)\}_{k=1}^N)$ is the diagonal matrix whose $k$-th diagonal entry is the frequency-response of the channel between source $r$ and destination $q$ at carrier $k$, and $\mathbf{R}_{n_q} = \text{diag}(\{\sigma_q^2(k)\}_{k=1}^N)$.

Under this setup, the strategy of each secondary user $q$ becomes the power allocation $\mathbf{p}_q = \{p_q(k)\}_{k=1}^N$ over the $N$ carriers and the payoff function in (1.4) reduces to the information rate over the $N$ parallel channels

$$r_q(\mathbf{p}_q, \mathbf{p}_{-q}) = \sum_{k=1}^{N} \log\left(1 + \frac{|H_{qq}(k)|^2 \, p_q(k)}{\sigma_q^2(k) + \sum_{r \neq q} |H_{rq}(k)|^2 \, p_r(k)}\right). \tag{1.123}$$

Local power constraints and global interference constraints are imposed on the secondary users. The admissible strategy set of each player $q$ associated to local power constraints is then

$$\mathcal{P}_q \triangleq \left\{\mathbf{p} \,:\, \sum_{k=1}^{N} p(k) \leq P_q, \quad \mathbf{0} \leq \mathbf{p} \leq \mathbf{p}_q^{\max}\right\}, \tag{1.124}$$

where we also included possibly (local) spectral mask constraints $\mathbf{p}_q^{\max} = (p_q^{\max}(k))_{k=1}^N$. In the case of transmissions over frequency-selective channels, the global interference constraints in (1.7) impose an upper bound on the value of the per-carrier and total interference (the interference temperature limit [2]) that can be tolerated by each primary user $p = 1, \cdots, P$, and reduce to [66]

$$\begin{aligned}
&\text{(total interference)}: && \sum_{q=1}^{Q}\sum_{k=1}^{N} |H_{q,p}(k)|^2 \, p_q(k) \le P_{p,\text{tot}}^{\text{ave}} \\
&\text{(per-carrier interference)}: && \sum_{q=1}^{Q} |H_{q,p}(k)|^2 \, p_q(k) \le P_{p,k}^{\text{peak}}, \quad \forall k = 1, \cdots, N,
\end{aligned} \tag{1.125}$$

where $H_{q,p}(k)$ is the channel transfer function between the transmitter of the $q$-th secondary user and the receiver of the $p$-th primary user, and $P_{p,\text{tot}}^{\text{ave}}$ and $P_{p,k}^{\text{peak}}$ are the interference temperature limit and the maximum interference over subcarrier $k$ tolerable by the $p$-th primary user, respectively. These limits are chosen by each primary user, according to his QoS requirements.

The aim of each secondary user is to maximize his own rate $r_q(\mathbf{p}_q, \mathbf{p}_{-q})$ under the local power constraints in (1.124) and the global interference constraints in (1.125). Note that the interference constraints introduce a global coupling among the admissible power allocations of all the players. This means that now the secondary users are not allowed to choose their power allocations individually, since this would lead to an infeasible strategy profile, being the global interference constraints in general not satisfied. To keep the resource power allocation as decentralized as possible while imposing global interference constraints, the basic idea proposed in [66] is to introduce a proper pricing mechanism, controlled by the primary users, through a penalty in the payoff function of each player, so that the interference generated by all the secondary users will depend on these prices. The challenging goal is then to find the proper decentralized pricing mechanism that guarantees the global interference constraints be satisfied while the secondary users reaching an equilibrium. Stated in mathematical terms, we have the following NE problem [66]

$$\begin{aligned}
&(\mathcal{G}_{\text{VI}}): \\
&\underset{\mathbf{p}_q \ge \mathbf{0}}{\text{maximize}} \;\; r_q(\mathbf{p}_q, \mathbf{p}_{-q}) - \sum_{p=1}^{P}\sum_{k=1}^{N} \lambda_{p,k}^{\text{peak}} |H_{q,p}(k)|^2 \, p_q(k) - \sum_{p=1}^{P} \lambda_{p,\text{tot}} \sum_{k=1}^{N} |H_{q,p}(k)|^2 \, p_q(k) \\
&\text{subject to } \mathbf{p}_q \in \mathcal{P}_q \\
&\qquad 0 \le \lambda_{p,\text{tot}} \perp P_{p,\text{tot}}^{\text{ave}} - \sum_{q=1}^{Q}\sum_{k=1}^{N} |H_{q,p}(k)|^2 \, p_q(k) \ge 0, \quad \forall p, \\
&\qquad 0 \le \lambda_{p,k}^{\text{peak}} \perp P_{p,k}^{\text{peak}} - \sum_{q=1}^{Q} |H_{q,p}(k)|^2 \, p_q(k) \ge 0, \quad \forall p, k,
\end{aligned} \tag{1.126}$$

for all $q \in \Omega$, where $\lambda_{p,\text{tot}}$ and $\boldsymbol{\lambda}_p^{\text{peak}} = \{\lambda_{p,k}^{\text{peak}}\}_{k=1}^N$ are the prices used to keep the interference temperature limit and the per-carrier interference generated by the secondary users at the receiver of the $p$-th primary user under the thresholds $P_{p,\text{tot}}^{\text{ave}}$ and $\{P_{p,k}^{\text{peak}}\}_{k=1}^N$, respectively. The per-carrier/global interference constraints written as in (1.126) state that either the interference constraints are satisfied with equality and nonnegative associated prices or a price is zero if the associated interference constraint is strictly satisfied (no punishment is needed in this case).

### 1.5.1 Equilibrium solutions: existence and uniqueness

The coupling among the strategies of the players of $\mathcal{G}_{\text{VI}}$ due to the global interference constraints presents a new challenge for the analysis of this class of Nash games that cannot be addressed using results from game theory or game theoretical models proposed in the literature [6, 7, 8, 9, 21, 62]. For this purpose, we need the framework given by the more advanced theory of finite-dimensional VIs [32, 67] that provides a satisfactory resolution to the game $\mathcal{G}_{\text{VI}}$, as detailed next. We first introduce the following definitions. Define the joint admissible strategy set of game $\mathcal{G}_{\text{VI}}$ as

$$\mathcal{K} \triangleq \mathcal{P} \cap \left\{ \begin{array}{ll} \mathbf{p} : \displaystyle\sum_{q=1}^{Q}\sum_{k=1}^{N} |H_{q,p}(k)|^2\, p_q(k) \le P_{p,\text{tot}}^{\text{ave}}, \forall p = 1, \cdots, P & \\ \quad\displaystyle\sum_{q=1}^{Q} |H_{q,p}(k)|^2\, p_q(k) \le P_{p,k}^{\text{peak}}, \qquad \forall p = 1, \cdots, P,\ k = 1, \cdots, N \end{array} \right\}, \tag{1.127}$$

with $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_Q$, and the vector function $\mathbf{F} : \mathcal{K} \ni \mathbf{p} \mapsto \mathbf{F}(\mathbf{p}) \in \mathbb{R}_-^{QN}$

$$\mathbf{F}(\mathbf{p}) \triangleq \begin{pmatrix} \mathbf{F}_1(\mathbf{p}) \\ \vdots \\ \mathbf{F}_Q(\mathbf{p}) \end{pmatrix}, \text{ where each } \mathbf{F}_q(\mathbf{p}) \triangleq \left( -\frac{|H_{qq}(k)|^2}{\sigma_q^2(k) + \sum_r |H_{rq}(k)|^2\, p_r(k)} \right)_{k=1}^{N}. \tag{1.128}$$

Finally, to rewrite the solutions to $\mathcal{G}_{\text{VI}}$ in a convenient form, we introduce the interference-plus-noise to noise ratios $\mathsf{innr}_{rq}(k)$, defined as

$$\text{innr}_{rq}(k) \triangleq \frac{\sigma_r^2(k) + \sum_t |H_{tr}(k)|^2\, p_t^{\max}(k)}{\sigma_q^2(k)}, \tag{1.129}$$

and, for each $q$ and given $\mathbf{p}_{-q} \ge \mathbf{0}$ and $\boldsymbol{\lambda} \ge \mathbf{0}$, define the waterfilling-like mapping $\mathsf{wf}_q$ as

$$[\mathsf{wf}_q(\mathbf{p}_{-q};\boldsymbol{\lambda})]_k \triangleq \left[ \frac{1}{\mu_q + \gamma_q(k;\boldsymbol{\lambda})} - \frac{\sigma_q^2(k) + \sum_{r\neq q} |H_{rq}(k)|^2 p_r(k)}{|H_{qq}(k)|^2} \right]_0^{p_q^{\max}(k)}, \tag{1.130}$$

with $k = 1, \cdots, N$, where $\gamma_q(k;\boldsymbol{\lambda}) = \sum_{p=1}^{P} |H_{q,p}(k)|^2 \, (\lambda_{p,k}^{\text{peak}} + \lambda_{p,\text{tot}})$ and $\mu_q \geq 0$ is chosen to satisfy the power constraint $\sum_{k=1}^{N} [\mathsf{wf}_q(\mathbf{p}_{-q};\boldsymbol{\lambda})]_k \leq P_q$ ($\mu_q = 0$ if the inequality is strictly satisfied).

**Theorem 1.19** ([66]). *Consider the NE problem $\mathcal{G}_{\text{VI}}$ in (1.126), the following hold:*

**(a)** *$\mathcal{G}_{\text{VI}}$ is equivalent to the VI problem defined by the pair $(\mathcal{K}, \mathbf{F})$, which is to find a vector $\mathbf{p}^\star \in \mathcal{K}$ such that*

$$(\mathbf{p} - \mathbf{p}^\star)^T \, \mathbf{F}(\mathbf{p}^\star) \geq 0, \quad \forall \mathbf{p} \in \mathcal{K}, \tag{1.131}$$

*with $\mathcal{K}$ and $\mathbf{F}(\mathbf{p})$ defined in (1.127) and (1.128), respectively;*

**(b)** *there always exists a solution to the VI problem in (1.131), for any given set of channels, power budgets, and interference constraints;*

**(c)** *given the set of the optimal prices $\hat{\boldsymbol{\lambda}} = \{\hat{\boldsymbol{\lambda}}_p^{\text{peak}}, \hat{\boldsymbol{\lambda}}_{p,\text{tot}}\}_{p=1}^{P}$, the optimal power allocation vector $\mathbf{p}^\star(\hat{\boldsymbol{\lambda}}) = (\mathbf{p}_q^\star(\hat{\boldsymbol{\lambda}}))_{q=1}^{Q}$ of the secondary users at a NE of game $\mathcal{G}_{\text{VI}}$ is the solution to the following vector waterfilling-like fixed-point equation:*

$$\mathbf{p}_q^\star(\hat{\boldsymbol{\lambda}}) = \mathsf{wf}_q\left(\mathbf{p}_{-q}^\star(\hat{\boldsymbol{\lambda}}); \hat{\boldsymbol{\lambda}}\right), \quad \forall q \in \Omega, \tag{1.132}$$

*with $\mathsf{wf}_q$ defined in (1.130);*

**(d)** *the optimal power allocation vector $\mathbf{p}^\star$ of game $\mathcal{G}_{\text{VI}}$ is unique if the two following set of conditions are satisfied:*[9]

$$\begin{array}{ll} \text{Low received MUI:} & \displaystyle\sum_{r \neq q} \max_k \left\{ \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2} \cdot \text{innr}_{rq}(k) \right\} < 1, \quad \forall q \in \Omega, \\ \text{Low generated MUI:} & \displaystyle\sum_{q \neq r} \max_k \left\{ \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2} \cdot \text{innr}_{rq}(k) \right\} < 1, \quad \forall r \in \Omega, \end{array} \tag{C9}$$

*with $\text{innr}_{rq}(k)$ defined in (1.129).*

The equivalence between the game $\mathcal{G}_{\text{VI}}$ in (1.126) and the VI problem in (1.131), as stated in Theorem 1.19(a) is in the following sense: If $\mathbf{p}^\star$ is a solution of the VI$(\mathcal{K}, \mathbf{F})$, then there exists a set of prices $\boldsymbol{\lambda}^\star = (\boldsymbol{\lambda}_p^\star, \boldsymbol{\lambda}_{p,\text{tot}}^\star)_{p=1}^{P} \geq \mathbf{0}$ such that $(\mathbf{p}^\star, \boldsymbol{\lambda}^\star)$ is an equilibrium pair of $\mathcal{G}_{\text{VI}}$; conversely if $(\mathbf{p}^\star, \boldsymbol{\lambda}^\star)$ is an equilibrium of $\mathcal{G}_{\text{VI}}$, then $\mathbf{p}^\star$ is a solution of the VI$(\mathcal{K}, \mathbf{F})$. Finally, observe that condition (C9) has the same nice interpretations of those obtained for the games introduced so far: The uniqueness of the NE of $\mathcal{G}_{\text{VI}}$ is guaranteed if the interference among the secondary users is not too high, in the sense specified by (C9).

[9] Milder conditions are given in [66]

### 1.5.2 Distributed algorithms

To obtain efficient algorithms that distributively compute both the optimal power allocations of the secondary users and prices, we can borrow from the wide literature of solutions methods for VIs [32, 67]. Many alternative algorithms have been proposed in [66] to solve game $\mathcal{G}_{\text{VI}}$ that differ in: i) the signaling among primary and secondary users needed to be implemented; ii) the computational effort; iii) the convergence speed; and iv) the convergence analysis. Because of the space limitation, here we focus only on one of them, based on the *Projection Algorithm with variable steps* (for the sake of simplicity, here we use a constant step size) [67, Alg. 12.1.4] and formally described in Algorithm 3, where the waterfilling mapping $\mathsf{wf}_q$ is defined in (1.130).

**Algorithm 3: Projection algorithm with constant step size**

1 : Set $n = 0$, initialize $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(0)} \geq \mathbf{0}$, and choose the step size $\tau > 0$

2 : `repeat`

3 : Given $\boldsymbol{\lambda}^{(n)}$, compute $\mathbf{p}^{\star}(\boldsymbol{\lambda}^{(n)})$ as the solution to the fixed-point equation

4 :
$$\mathbf{p}_q^{\star}(\boldsymbol{\lambda}^{(n)}) = \mathsf{wf}_q\left(\mathbf{p}_{-q}^{\star}(\boldsymbol{\lambda}^{(n)}); \boldsymbol{\lambda}^{(n)}\right), \quad \forall q \in \Omega \tag{1.133}$$

5 : Update the price vectors: for all $p = 1, \cdots, P$, compute

6 :
$$\lambda_{p,\text{tot}}^{(n+1)} = \left[\lambda_{p,\text{tot}}^{(n)} - \tau\left(P_{p,\text{tot}}^{\text{ave}} - \sum_{q=1}^{Q}\sum_{k=1}^{N} |H_{q,p}(k)|^2\, p_q^{\star}(k; \boldsymbol{\lambda}^{(n)})\right)\right]^{+} \tag{1.134}$$

7 :
$$\lambda_{p,k}^{(n+1)} = \left[\lambda_{p,k}^{(n)} - \tau\left(P_{p,k}^{\text{peak}} - \sum_{q=1}^{Q} |H_{q,p}(k)|^2\, p_q^{\star}(k; \boldsymbol{\lambda}^{(n)})\right)\right]^{+}, \ \forall k = 1, \cdots, N \tag{1.135}$$

8 : `until the prescribed convergence criterion is satisfied`

The algorithm can be interpreted as follows. In the main loop, at the $n$-th iteration, each primary user $p$ measures the received interference generated by the secondary users and, locally and independently from the other primary users, adjusts his own set of prices $\boldsymbol{\lambda}_p^{(n)}$ accordingly, via a simple projection scheme (see (1.134) and (1.135)). The primary users broadcast their own prices $\boldsymbol{\lambda}_p^{(n)}$'s to the secondary users, who then play the game in (1.126) keeping fixed the prices to the value $\boldsymbol{\lambda}^{(n)}$. The Nash equilibria of such a game are the fixed-points of mapping $\mathsf{wf} = (\mathsf{wf}_q)_{q\in\Omega}$ as given in (1.132), with $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^{(n)}$. Interestingly, the secondary users can reach these solutions using any algorithm falling within the class of asynchronous IWFA as described in Algorithm 2 (e.g., simultaneous or sequential) and based on mapping $\mathsf{wf} = (\mathsf{wf}_q)_{q\in\Omega}$ in (1.132). Convergence properties of Algorithm 3 are given in the following.

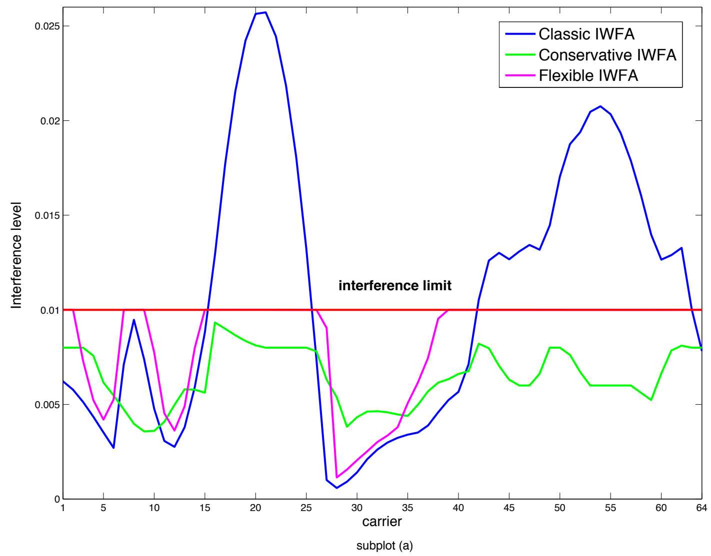
**Theorem 1.20** ([66]). *Suppose that condition (C9) in Theorem 1.19 is satisfied. Then, there exists some* $\tau_0 > 0$[10] *such that, as* $n \to +\infty$, *Algorithm 3 converges to a solution to* $\mathcal{G}_{\mathrm{VI}}$ *in (1.126), for any set of feasible initial conditions and* $\tau \in (0, \tau_0)$.

**Remark 10 - Features of Algorithm** 3. Even though the per-carrier and global interference constraints impose a coupling among the feasible power allocation strategies of the secondary users, the equilibrium of game $\mathcal{G}_{\mathrm{VI}}$ can be reached using iterative algorithms that are fairly distributed with a minimum signaling from the primary to the secondary users. In fact, in Algorithm 3, the primary users, to update their prices, only need to measure the interference generated by the secondary users, which can be performed locally and independently from the other primary users. Regarding the secondary users (see (1.130)), once $\gamma_q(k; \boldsymbol{\lambda})$'s are given, the optimal power allocation can be computed locally by each secondary user, since only the measure of the received MUI over the $N$ subcarriers is needed. However, the computation of $\gamma_q(k; \boldsymbol{\lambda})$'s requires a signaling among the primary and secondary users: At each iteration, the primary users have to broadcast the new values of the prices and the secondary users estimate the $\gamma_q(k; \boldsymbol{\lambda})$'s, which requires the estimate from each secondary user of the (cross-)channel transfer functions between his transmitter and the primary receivers. This estimate can be performed once at the beginning of the transmission and updated at the rate of the coherence time of the channel.

---

**Example 1.3: Comparison of proposed algorithms.** As a numerical example, in Figure 1.4, we compare some of algorithms proposed in this chapter in terms of interference generated against the primary users. We consider a CR system composed of 6 secondary links randomly distributed within an hexagonal cell and one primary user (the BS at the center of the cell). In Figure 1.4(a) we plot the power spectral density (PSD) of the interference due to the secondary users at the receiver of the primary user, generated using the classical IWFA [20, 64, 62], the IWFA with individual interference constraints (i.e., a special case of Algorithm 2 applied to game $\mathcal{G}_{\mathrm{soft}}$) that we call *conservative* IWFA, and the IWFA with global interference constraints (based on Algorithm 3) that we call *flexible* IWFA. For the sake of simplicity, we consider only a constant interference threshold over the whole spectrum occupied by the primary user, i.e., $P_{p,k}^{\mathrm{peak}} = 0.01$ for all $k = 1, \cdots, N$. We clearly see from the picture that while classical IWFA violates the interference constraints, both conservative and flexible IWFAs satisfy them, but the global interference constraints impose less stringent conditions on the transmit power of the secondary users that those imposed by the individual interference constraints. However, this comes at the price of more signaling from

[10] An expression for $\tau_0$ is given in [66].



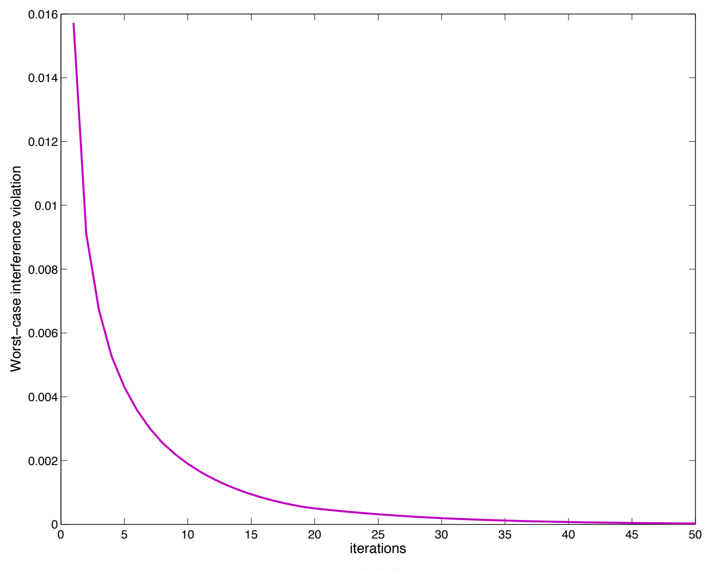




**Figure 1.4** Comparison of different algorithms: (a) Power spectral density of the interference profile at the primary user's receiver generated by the secondary users; (b) worst-case violation of the interference constraint achieved by Algorithm 3 (flexible IWFA).

the primary to the secondary users. Interestingly, for the example considered in the figure, Algorithm 3 converges quite fast, as shown in Figure 1.4(b), where we plot the worst-case violation of the interference constraint achieved by the algorithm versus the number of iterations of the outer loop.

Finally, in Figure 1.5, we compare the conservative IWFA and the flexible IWFA in terms of achievable sum-rate as a function of the maximum tolerable interference at the primary receiver, within the same setup described above (we considered the same interference threshold $P^{\text{peak}}$ for all the subcarriers). As expected, the flexible IWFA exhibits a much better performance, thanks to less stringent constraints on the transmit powers of the secondary users.

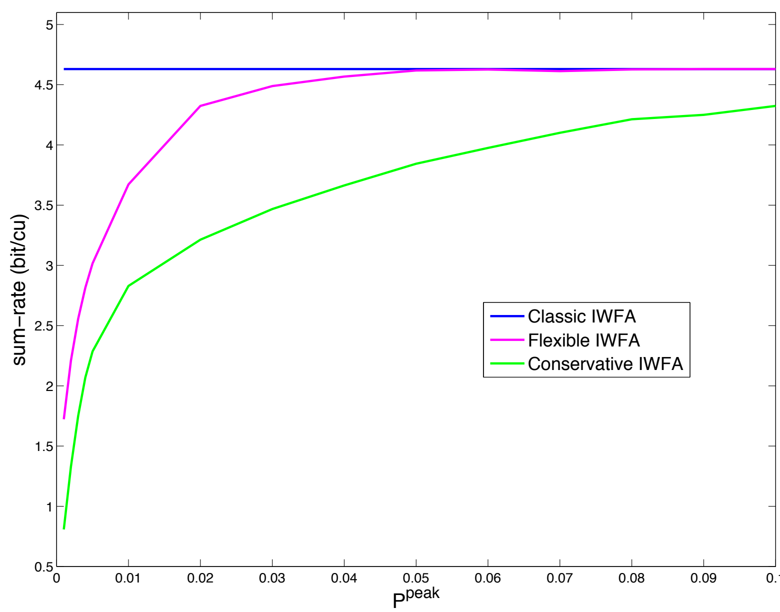



**Figure 1.5** Conservative IWFA versus flexible IWFA: achievable sum-rate as a function of the maximum tolerable interference at the primary receiver.

## 1.6 Conclusions

In this chapter we have proposed different equilibrium models to formulate and solve resource allocation problems in CR systems, using a competitive optimality principle based on the NE concept. We have seen how game theory and the more general VI theory provide the natural framework to address and solve some of the challenging issues in CR, namely: 1) the establishment of conditions guaranteeing that the dynamical interaction among cognitive nodes, under different constraints on the transmit spectral mask and on interference induced to primary users, admits a (possibly unique) equilibrium; and 2) the design of decentralized algorithms able to reach the equilibrium points, with minimal coordination among the nodes. The proposed algorithms differ in the trade-off between performance (in terms of information rate) achievable by the secondary users and the amount of information to be exchanged between the primary and the secondary users. Thus the algorithms are valid candidate to be applied to both main paradigms having emerged for CR systems, namely the common model and the spectral leasing approach. Results proposed in this chapter are based on recent works [6, 7, 8, 10, 24, 63, 66, 68].

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# References


[1] FCC Spectrum Policy Task Force, "Report of the spectrum efficiency working group," Nov. 2002. [Online]. Available: http://www.fcc.gov/sptf/files/SEWGFinalReport1.pdf

[2] S. Haykin, "Cognitive radio: Brain-empowered wireless communications," *IEEE Jour. on Selected Areas in Communications*, vol. 23, no. 2, pp. 201–220, Feb. 2005.

[3] Q. Zhao and B. Sadler, "A survey of dynamic spectrum access," *IEEE Communications Magazine*, vol. 24, no. 3, pp. 79–89, May 2009.

[4] A. Goldsmith, S. A. Jafar, I. Maric, and S. Srinivasa, "Breaking spectrum gridlock with cognitive radios: An information theoretic perspective," *Proceedings of the IEEE*, 2009.

[5] J. Mitola, "Cognitive radio for flexible mobile multimedia communication," in *IEEE International Workshop on Mobile Multimedia IEEE 1999 International Workshop on Mobile Multimedia Communications (MoMuC 1999)*, Sandiego, California, Usa, Nov. 15-17 1999, pp. 3–10.

[6] G. Scutari, D. P. Palomar, and S. Barbarossa, "Optimal linear precoding strategies for wideband noncooperative systems based on game theory—part I: Nash equilibria," *IEEE Trans. on Signal Processing*, vol. 56, no. 3, pp. 1230–1249, March 2008.

[7] ——, "Optimal linear precoding strategies for wideband noncooperative systems based on game theory—part II: Algorithms," *IEEE Trans. on Signal Processing*, vol. 56, no. 3, pp. 1250–1267, March 2008.

[8] ——, "Asynchronous iterative water-filling for Gaussian frequency-selective interference channels," *IEEE Trans. on Information Theory*, vol. 54, no. 7, pp. 2868–2878, July 2008.

[9] J.-S. Pang, G. Scutari, F. Facchinei, and C. Wang, "Distributed power allocation with rate constraints in Gaussian parallel interference channels," *IEEE Trans. on Information Theory*, vol. 54, no. 8, pp. 3471–3489, Aug. 2008.

[10] G. Scutari, D. P. Palomar, and S. Barbarossa, "Competitive design of multiuser MIMO systems based on game theory: A unified view," *IEEE Jour. on Selected Areas in Communications*, vol. 26, no. 7, pp. 1089–1103, Sept. 2008.

[11] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* John Wiley & Sons, 1991.

[12] N. Devroye, P. Mitran, and V.Tarokh, "Limits on communications in a cognitive radio channel," *IEEE Communications Magazine*, vol. 44, no. 6, pp. 44–49, June 2006.

[13] *Working Group on Wireless Regional Area Networks.* IEEE Std. 802.22. [Online]. Available: http://www.ieee802.org/22/

[14] N. Devroye, M. Vu, and V. Tarokh, "Cognitive radio networks: Highlights of information theoretic limits, models, and design," *IEEE Signal Processing Magazine*, vol. 25, no. 6, pp. 12–23, Nov. 2008.

[15] B. Wild and K. Ramchandran, "Detecting primary receivers for cognitive radio applications," in *Proc. of the IEEE 2005 Symp. New Frontiers Dynamic Spectrum Access Networks (DYSPAN 2005)*, Baltimore, Maryland, USA, pp. 124–130.

[16] Z. Quan, S. Cui, H. V. Poor, and A. H. Sayed, "Collaborative wideband sensing for cognitive radios: An overview of challenges and solutions," *IEEE Signal Processing Magazine*, vol. 25, no. 6, pp. 60–73, Nov. 2008.

[17] Z.-Q. Luo and S. Zhang, "Spectrum management: Complexity and duality," *Jour. of Selected Topics in Signal Processing*, vol. 2, no. 1, pp. 57–72, Feb. 2008.

[18] W. Yu and R. Lui, "Dual methods for nonconvex spectrum optimization of multicarrier systems," *IEEE Trans. on Communications*, vol. 54, no. 7, pp. 1310–1322, July 2006.

[19] S. Ye and R. S. Blum, "Optimized signaling for MIMO interference systems with feedback," *IEEE Trans. on Signal Processing*, vol. 51, no. 11, pp. 2839–2848, Nov. 2003.

[20] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE Jour. on Selected Areas in Communications*, vol. 20, no. 5, pp. 1105–1115, June 2002.

[21] Z.-Q. Luo and J.-S. Pang, "Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines," *EURASIP Jour. on Applied Signal Processing*, vol. 2006, pp. 1–10, May 2006.

[22] G. Arslan, M. F. Demirkol, and Y. Song, "Equilibrium efficiency improvement in MIMO interference systems: a decentralized stream control approach," *IEEE Trans. on Wireless Communications*, vol. 6, no. 8, pp. 2984–2993, Aug. 2007.

[23] E. Larsson and E. Jorswieck, "Competition versus collaboration on the MISO interference channel," *IEEE Jour. on Selected Areas in Communications*, vol. 26, no. 7, pp. 1059–1069, Sept. 2008.

[24] G. Scutari, D. P. Palomar, and S. Barbarossa, "The MIMO iterative waterfilling algorithm," *IEEE Trans. on Signal Processing*, vol. 57, no. 5, May 2009.

[25] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*. Dover Publications, 1991.

[26] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[27] J.-P. Aubin, *Mathematical Method for Game and Economic Theory*. Dover Publications, Nov. 2007.

[28] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory*. 2nd ed. Academic Press, London, 1989.

[29] E. van Damme, *Stability and perfection of Nash equilibria*. 2nd edn, Springer-Verlag, Berlin, 1996.

[30] D. Fudenberg and J. Tirole, *Game theory*. MIT Press, Cambridge, 1991.

[31] C. B. Garcia and W. I. Zangwill, *Pathways to solutions, fixed points, and equilibria*. Prentice-Hall, New Jersey, 1981.

[32] I. V. Konnov, *Equilibrium Models and Variational Inequalities*. Elsevier B.V., Amsterdam, 2007.

[33] F. Facchinei and C. Kanzow, "Generalized Nash equilibrium problems," *A Quarterly Journal of Operations Research (4OR)*, vol. 5, no. 3, pp. 173–210, Sept. 2007.

[34] D. Monderer and L. S. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, no. 1, pp. 124–143, May 1996.

[35] D. Topkis, *Supermodularity and Complementarity*. Princeton University Press, 1998.

[36] E. Altman and L. Wynter, "Equilibrium, games, and pricing in transportation and telecommunications networks," *Networks and Spacial Economics*, vol. 4, no. 1, pp. 7–21, March 2004.

[37] E. Altman, T. Boulogne, R. E. Azouzi, T. Jimenez, and L. Wynter, "A survey on networking games," *Computers and Operations Research*, vol. 33, pp. 286–311, 2006.

[38] A. B. MacKenzie and S. B. Wicker, "Game theory and the design of self-configuring, adaptive wireless networks," *IEEE Communication Magazine*, vol. 39, no. 11, pp. 126–131, Nov. 2001.

[39] A. Chinchuluun, P. M. Pardalos, A. Migdalas, and L. Pitsoulis, *Pareto Optimality, Game Theory and Equilibria*, Series: Springer Optimization and Its Applications, vol. 17, ed. Springer-Verlag, New York, 2008.

[40] M. J. Osborne and A. Rubinstein, *A Course in Game Theory*. MIT Press, July 2004.

[41] J. Nash, "Equilibrium points in $n$-person game," in *National Academy of Science*, vol. 36, 1950, pp. 48–49.

[42] F. Facchinei and J.-S. Pang, "Nash equilibria: The variational approach," in *Convex Optimization in Signal Processing and Communications*, D. P. Palomar and Y. C. Eldar, Eds. London: Cambridge University Press, 2009.

[43] F. Szidarovszky, "Nonlinear games," in *Pareto Optimality, Game Theory And Equilibria*, A. Chinchuluun, P. M. Pardalos, A. Migdalas, and L. Pit-

soulis, Eds. New York: Springer, 2008, vol. 17, ch. 16, pp. 95–117.

[44] I. L. Glicksburg, “A social equilibrium existence theorem,” in *American Mathematical Society*, vol. 38, 1952, pp. 170–174.

[45] G. Debreu, “A social equilibrium existence theorem,” in *National Academy of Science*, vol. 38, no. 2, Oct. 1952, pp. 886–893.

[46] J. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” *Econometrica*, vol. 33, no. 3, pp. 520–534, July 1965.

[47] K. Fan, “A generalization of Tychonoff's fixed point theorem,” *Mathematische Annalen*, vol. 42, pp. 305–310, 1961.

[48] S. Uryasev and R. Y. Rubinstein, “On relaxation algorithms in computation of noncooperative equilibria,” *IEEE Trans. on Automatic Control*, vol. 39, no. 6, pp. 1263–1267, 1994.

[49] D. Gabay and H. Moulin, “On the uniqueness and stability of Nash-equilibria in noncooperative games,” in *Applied Stochastic Control in Econometrics and Management Science.* Edited by A. Bensoussan et al., North-Holland Publishing Company, pp. 271–293, 1980.

[50] A. Simsek, A. Ozdaglar, and D. Acemoglu, “Uniqueness of generalized equilibrium for box constrained problems and applications,” in *43rd Allerton Conf. on Communications, Control, and Computing*, Allerton House, Monticello, Illinois, USA, Sept. 28-30, 2005.

[51] ——, “Generalized Poincare-Hopf theorem for compact nonsmooth regions,” *Mathematics of Operations Research*, vol. 32, no. 1, pp. 193–214, Feb.

[52] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*, 2nd ed. Athena Scientific Press, 1989.

[53] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables.* Society for Industrial Mathematics (SIAM), Phila, USA, 1987.

[54] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem.* Cambridge Academic Press, 1992.

[55] G. Scutari, “Competition and cooperation in wireless communication networks,” Ph.D. dissertation, INFOCOM Dept., University of Rome, ”La Sapienza”, Nov. 2004.

[56] B. Zheng and Z. Xiong, “A new equivalent condition of the reverse order law for $g$-inverses of multiple matrix products,” *Electronic Journal of Linear Algebra*, vol. 17, no. 3, pp. 1–8, Jan. 2008.

[57] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus, with Applications in Statistics and Econometrics.* John Wiley & Sons, 1999.

[58] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill Publishing Co, 1976.

[59] R. M. Mcleod, “Mean value theorems for vector valued functions,” *Edinburgh Mathematical Society*, vol. 14, pp. 197–209, Series II 1965.

[60] M. Furi and M. Martelli, “On the mean value theorem, inequality, and inclusion,” *The American Mathematical Monthly*, vol. 9, pp. 840–846, Nov. 1991.

[61] A. Hjørungnes and D. Gesbert, "Complex-valued matrix differentiation: Techniques and key results," *IEEE Trans. on Signal Processing*, vol. 55, no. 4, pp. 2740–2746, June 2007.

[62] J. Huang, R. Cendrillon, M. Chiang, and M. Moonen, "Autonomous spectrum balancing for digital subscriber lines," *IEEE Trans. on Signal Processing*, vol. 55, no. 8, pp. 4241–4257, Aug. 2007.

[63] G. Scutari and D. P. Palomar, "MIMO cognitive radio: A game theoretical approach," *IEEE Trans. on Signal Processing,* (under review).

[64] R. Etkin, A. Parekh, and D. Tse, "Spectrum sharing for unlicensed bands," *IEEE Jour. on Selected Areas in Communications*, vol. 25, no. 3, pp. 517–528, April 2007.

[65] D. P. Palomar and J. Fonollosa, "Practical algorithms for a family of water-filling solutions," *IEEE Trans. on Signal Processing*, vol. 53, no. 2, pp. 686–695, Feb. 2005.

[66] J.-S. Pang, G. Scutari, D. P. Palomar, and F. Facchinei, "Cognitive radio wireless systems: Flexible designs via variational inequality theory," *IEEE Trans. on Signal Processing,* (under review).

[67] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problem.* Springer-Verlag, New York, 2003.

[68] G. Scutari, D. P. Palomar, and S. Barbarossa, "Cognitive MIMO radio: Competitive optimality design based on subspace projections," *IEEE Signal Processing Magazine*, vol. 25, no. 6, pp. 46–59, Nov. 2008.

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