Rank-Constrained Separable Semidefinite Programming With Applications to Optimal Beamforming

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***Abstract*—Consider a downlink communication system where multiantenna base stations transmit independent data streams to decentralized single-antenna users over a common frequency band. The goal of the base stations is to jointly adjust the beamforming vectors to minimize the transmission powers while ensuring the signal-to-interference-noise ratio requirement of each user within the system. At the same time, it may be necessary to keep the interference generated on other coexisting systems under a certain tolerable level. In addition, one may want to include general individual shaping constraints on the beamforming vectors. This beamforming problem is a separable homogeneous quadratically constrained quadratic program, and it is difficult to solve in general. In this paper, we give conditions under which strong duality holds and propose efficient algorithms for the optimal beamforming problem. First, we study rank-constrained solutions of general separable semidefinite programs (SDPs) and propose rank reduction procedures to achieve a lower rank solution. Then we show that the SDP relaxation of three classes of optimal beamforming problem always has a rank-one solution, which can be obtained by invoking the rank reduction procedures.**

***Index Terms*—Downlink beamforming, individual shaping constraints, rank reduction procedure, semidefinite program (SDP) relaxation, separable homogeneous quadratically constrained quadratic program (QCQP), soft-shaping interference constraints.**

I. Introduction

TRANSMIT beamforming design has been an intensive research topic in the past decade because the multiple antennas create a dimension for multiple access, i.e., spatial-division multiple access (see [1] and [2]). The signals for the different users are weighted and transmitted in a common channel, by which higher spectral efficiency is gained. The design problem is the optimization of the beamforming vectors.

Optimal downlink beamforming is typically formulated in two different ways: one is the minimization of the total transmission power subject to quality of service (QoS) constraints for each user and the other is the maximization of the minimal QoS, subject to a power constraint. Both problems are essentially equivalent; see [3] for the detailed relation between the two formulations. An elegant approach to solve the optimal beamforming problem was proposed in [1] (see also [2]) by using a semidefinite program (SDP) relaxation technique and the Perron–Frobenius theory for matrices with nonnegative entries. In [3], the authors extended the problem by considering the optimal beamforming design problem with indefinite shaping constraints, termed individual shaping constraint herein, in addition to the QoS constraints. There are other different approaches to solve the optimal beamforming problem with QoS constraints; see [4] and [5], and references therein. For optimal downlink beamforming in a multicast scenario (i.e., sending the same information to several users), we refer to [7] and references therein.

In a downlink beamforming problem, multiuser interference can be presubtracted at the base stations by dirty-paper precoding technique (see [8]) as well as be treated as interference (as in the aforementioned works). For the implementation and effect of dirty-paper precoding on optimal beamforming and power control, one may refer to [9] and references therein.

In this paper, we consider an additional type of constraint termed soft-shaping interference constraint (see [10]) besides the signal-to-interference-noise ratio (SINR) constraints (i.e., the QoS constraints) and the individual shaping constraints. In modern communications, there are often several coexisting wireless systems in the region of interest, and these systems may operate on a common frequency band due to limited availability of frequency bands. The setup may include multiple access, peer-to-peer link, or other broadcast channels. Reasonably, when designing downlink beamforming vectors, we may want to take into consideration that the interference level generated to the users of coexisting systems should be kept under the required thresholds. This motivates the introduction of the soft-shaping constraints on the beamforming vectors.

The beamforming problem formulation based on the minimization transmission power subject to these constraints belongs to the class of separable homogeneous quadratically constrained quadratic program (QCQP) (see [7]), and it may not have the strong duality property; or, equivalently, its SDP relaxation may have only optimal solutions of rank higher than one. Nevertheless, it is possible to find some instances of a separable QCQP that can still be solved efficiently (they are termed "hidden convex" in some optimization literature). In this paper, we study rank-constrained solutions of general separable SDPs and propose rank reduction procedures to achieve a rank-constrained optimal solution. Based on the study, we show that three

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classes of the optimal beamforming problem have strong duality by arguing that the corresponding SDP relaxation has a rank-one optimal solution. Particularly, the three main contributions of the paper include i) conditions under which rank-constrained solutions can be efficiently obtained (for which strong duality holds); ii) practical rank-reduction algorithms to obtain such solutions in practice; and iii) additional rank-one results for the SDP relaxation of three classes of the optimal beamforming problem.

This paper is organized as follows. In Section II, we introduce the system model and formulate the optimal beamforming problem, which minimizes the transmission power subject to SINR constraints, soft-shaping interference constraints, and individual shaping constraints. In Sections III and IV, we study rank-constrained solutions of general separable SDPs and propose the rank-reduction procedures. Based on these, we show that the optimal beamforming problem is solvable under some mild and different conditions. In Section V, we present some numerical results for simulated scenarios of the optimal beamforming problem. Section VI draws some concluding remarks.

Notation: We adopt the notation of using boldface for vectors $\boldsymbol{a}$ (lower case) and matrices $\boldsymbol{A}$ (upper case). The transpose operator and the complex conjugate transpose operator are denoted by the symbols $(\cdot)^T$ and $(\cdot)^H$, respectively. $\mathrm{tr}(\cdot)$ is the trace of the square matrix argument. $\boldsymbol{I}$ and $\boldsymbol{0}$ denote, respectively, the identity matrix and the matrix with zero entries (their size is determined from context). The letter j represents the imaginary unit (i.e., $j=\sqrt{-1}$), while the letter i often serves as index in this paper. For any complex number x, we use $|x|$ to represent the modulus of x. The Euclidean norm of the vector $\boldsymbol{x}$ is denoted by $\|\boldsymbol{x}\|$. The curled inequality symbol $\succeq$ (its strict form $\succ$ and reverse form $\preceq$) is used to denote generalized inequality: $\boldsymbol{A}\succeq\boldsymbol{B}$ means that $\boldsymbol{A}-\boldsymbol{B}$ is an Hermitian positive semidefinite matrix ($\boldsymbol{A}\succ\boldsymbol{B}$ for positive definiteness and $\boldsymbol{A}\preceq\boldsymbol{B}$ for negative semidefiniteness). The notation $\mathcal{H}^K$ stands for the linear space containing all $K\times K$ Hermitian matrices.

II. Motivation and Contribution: Downlink Beamforming Problem via Semidefinite Relaxation

A. System Model for the Downlink Beamforming Problem

Consider a wireless system where several base stations serve a number of single-antenna users over a common frequency band. Let the number of base stations be N, each with an array of K antenna elements,[1] and let the number of users of the system be L. Each user m is assigned to a base station and receives an independent data stream $s_m(t)$ from the base station. Let $\kappa_m\in\{1,\ldots,N\}$ be the base station assigned to user m. It is assumed that the scalar-valued data streams $s_m(t), m=1,\ldots,L$, are temporally white with zero mean and unit variance. The transmitted signal by the kth base station is

$$\boldsymbol{x}_k(t)=\sum_{m\in\mathcal{I}_k}\boldsymbol{w}_m s_m(t) \tag{1}$$

where $\boldsymbol{w}_m\in\mathbb{C}^K$ is the transmit beamforming vector for user m and the index set $\mathcal{I}_k\subseteq\{1,\ldots,L\}$ represents the set of users assigned to base station k.

The signal received by user m is expressed with the baseband signal model

$$r_m(t)=\boldsymbol{h}_{m,\kappa_m}^H(t)\boldsymbol{x}_{\kappa_m}(t)+\sum_{k=1,k\neq\kappa_m}^{N}\boldsymbol{h}_{m,k}^H(t)\boldsymbol{x}_k(t)+n_m(t) \tag{2}$$

where $\boldsymbol{h}_{m,k}(t)\in\mathbb{C}^K$ is the time-varying channel vector between base station k and user m and $n_m(t)$ is a zero-mean complex Gaussian noise with variance σ_m^2. The downlink channel correlation matrices are defined as

$$\boldsymbol{R}_{ml}=\mathrm{E}[\boldsymbol{h}_{m,\kappa_l}(t)\boldsymbol{h}_{m,\kappa_l}^H(t)], m,l=1,\ldots,L.$$

It is assumed that the matrices $\boldsymbol{R}_{ml}, m,l=1,\ldots,L$, are known at the base stations. The SINR of user m is given by

$$\mathrm{SINR}_m=\frac{\boldsymbol{w}_m^H\boldsymbol{R}_{mm}\boldsymbol{w}_m}{\sum_{l=1,l\neq m}^{L}\boldsymbol{w}_l^H\boldsymbol{R}_{ml}\boldsymbol{w}_l+\sigma_m^2}. \tag{3}$$

We highlight that this is a long-term SINR (as opposed to the instantaneous SINR). If we consider the use of dirty-paper precoding instead of treating the other signals as interference, the SINR becomes

$$\mathrm{SINR}_m=\frac{\boldsymbol{w}_m^H\boldsymbol{R}_{mm}\boldsymbol{w}_m}{\sum_{l\in\mathcal{I}_{\kappa_m},l>m}\boldsymbol{w}_l^H\boldsymbol{R}_{mm}\boldsymbol{w}_l+\sum_{l\notin\mathcal{I}_{\kappa_m}}\boldsymbol{w}_l^H\boldsymbol{R}_{ml}\boldsymbol{w}_l+\sigma_m^2}$$

with some prefixed encoding order for users assigned to every base station (see [9]).

A typical objective of downlink beamforming design is to ensure that each user can retrieve the signal of interest with the desired QoS, which is usually described by $\mathrm{SINR}_m\geq\rho_m$ with a given threshold ρ_m for user m. This corresponds to the SINR constraint in the classical optimal beamforming problem (for instance, see [1]). Furthermore, additional constraints may be of interest, as described next.

1) Soft-Shaping Interference Constraints: In many applications, it is necessary to control the amount of interference generated along some particular directions, e.g., to protect coexisting systems (see [10]), defined as follows. Let $\boldsymbol{h}_{m,k}$ be the channel between base station k and coexisting system's user m, where the amount of interference received has to be limited to τ_m. Note that we reserve the indexes $m=1,\ldots,L$, for users within the system and use the indexes $m=L+1,\ldots,M$ for users of coexisting systems. Then, the interference constraint is

$$\sum_{k=1}^{N}\mathrm{E}[|\boldsymbol{h}_{m,k}^H\boldsymbol{x}_k(t)|^2]=\sum_{k=1}^{N}\boldsymbol{h}_{m,k}^H\mathrm{E}[\boldsymbol{x}_k(t)\boldsymbol{x}_k^H(t)]\boldsymbol{h}_{m,k}\leq\tau_m, m=L+1,\ldots,M. \tag{4}$$

[1]Discussion for base stations equipped with different numbers of antenna elements is trivially the same.

Note that the expectation in (4) is over the signal (rather than the channel), and thus it is an instantaneous constraint, although average constraints can be easily considered as well. Evidently, (4) is tantamount to

$$\sum_{k=1}^{N} \boldsymbol{h}_{m,k}^H \left(\sum_{l \in \mathcal{I}_k} \boldsymbol{w}_l \boldsymbol{w}_l^H \right) \boldsymbol{h}_{m,k} \leq \tau_m, m = L+1, \ldots, M \quad (5)$$

which in turn can be rewritten as

$$\sum_{l=1}^{L} \boldsymbol{w}_l^H (\boldsymbol{h}_{m,\kappa_l} \boldsymbol{h}_{m,\kappa_l}^H) \boldsymbol{w}_l \leq \tau_m, m = L+1, \ldots, M. \quad (6)$$

Using the notation

$$\boldsymbol{S}_{ml} = \boldsymbol{h}_{m,\kappa_l} \boldsymbol{h}_{m,\kappa_l}^H, m = L+1, \ldots, M, l = 1, \ldots, L \quad (7)$$

we can rewrite (6) as

$$\sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq \tau_m, m = L+1, \ldots, M. \quad (8)$$

This constraint is called soft-shaping interference constraint (see [10]). To consider average constraints, it suffices to take the expectation over the channel in (7).

We remark that in the formulation of (8), the matrices $\boldsymbol{S}_{ml}$, $m = L+1, \ldots, M, l = 1, \ldots, L$ are positive semidefinite and of rank one. Alternatively, we could define $\boldsymbol{S}_{ml} = \boldsymbol{G}_{ml} \boldsymbol{G}_{ml}^H$ with an arbitrary rank, where the matrix $\boldsymbol{G}_{ml}$ is such that the range space identifies the subspaces where the interference level should be kept under the required threshold.

Interestingly, the soft-shaping interference constraint has another useful interpretation in per-base station power constraint. If one sets $\boldsymbol{S}_{ml} = \boldsymbol{I}$ for $\forall l \in \mathcal{I}_{k_0}$ and $\boldsymbol{S}_{ml} = \boldsymbol{0}, \forall l \notin \mathcal{I}_{k_0}$, then the soft-shaping interference constraints corresponds to the total power transmitted from base station k_0 to the coexisting user m.

In particular, if τ_m is set to be zero in (8) for some m, then we guarantee no interference generated at that location; this type of constraint is termed null-shaping interference constraint or, in short, null interference constraint (see [10]).

2) Individual Shaping Constraints: In addition to the SINR constraints and the soft-shaping interference constraints, we consider the following two groups of individual shaping constraints on the beamforming vectors (see [3]):

$$\boldsymbol{w}_l^H \boldsymbol{B}_l \boldsymbol{w}_l = 0, \forall l \in \mathcal{E}_1, \boldsymbol{w}_l^H \boldsymbol{B}_l \boldsymbol{w}_l \geq 0, \forall l \in \bar{\mathcal{E}}_1 \quad (9)$$

$$\boldsymbol{w}_l^H \boldsymbol{D}_l \boldsymbol{w}_l = 0, \forall l \in \mathcal{E}_2, \boldsymbol{w}_l^H \boldsymbol{D}_l \boldsymbol{w}_l \geq 0, \forall l \in \bar{\mathcal{E}}_2 \quad (10)$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are subsets of the index set $\{1, \ldots, L\}$ of users within the system and $\bar{\mathcal{E}}_1$ and $\bar{\mathcal{E}}_2$ are the complements of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Also, in (9) and (10), the parameters $\boldsymbol{B}_l$ and $\boldsymbol{D}_l$ can be any Hermitian matrices. By properly selecting $\boldsymbol{B}_l$ and $\boldsymbol{D}_l$, one can formulate different kinds of constraints on the beamforming vectors (for details, see [3] and references therein).

Observe that the two main differences between the soft-shaping interference constraints (excluding null interference constraints) and the individual shaping constraints are i) the former affect the beamforming vectors jointly while the latter is on an individual basis and ii) the matrices $\boldsymbol{B}_l$ and $\boldsymbol{D}_l$ need not be definite, whereas the matrices in the interference constraints are positive semidefinite by definition. As a matter of fact, we point out that a null-shaping interference constraint is equivalent to a particular case of individual shaping constraint as well, since $\sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq 0$ with $\boldsymbol{S}_{ml} \succeq \boldsymbol{0}\ \forall l$, if and only if $\boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l = 0$ for all $l = 1, \ldots, L$.

In addition to the aforementioned SINR and global/individual shaping constraints, one can also consider other types of constraints such as similarity constraints that impose the beamvector $\boldsymbol{w}_l$ to be not too different from a desired beamvector $\boldsymbol{w}_{0l}$: $\|\boldsymbol{w}_l - \boldsymbol{w}_{0l}\|^2 \leq \epsilon$ (see [6]).

B. Formulation of the Optimal Beamforming Problem

We consider a beamforming formulation to find the beamforming vectors $\boldsymbol{w}_l$, $l = 1, \ldots, L$, that minimize the total transmit power at the base stations

$$\sum_{l=1}^{L} \|\boldsymbol{w}_l\|^2 \quad (11)$$

while ensuring a desired QoS for each user, as well as the additional constraints on the beamforming vectors previously described. Specifically, we consider the optimization problem shown in (12) at the bottom of the page. By rearranging the first L constraints, we can recast the optimal beamforming problem (OBP) into a separable homogeneous QCQP (see [7]) as shown in (13) at the bottom of the next page. This is a nonconvex problem. Furthermore, it was shown in [7] that a separable homogeneous QCQP problem is NP-hard in general, and the SDP relaxation for it may not be tight. Nonetheless, it is possible that some instances of a separable homogeneous QCQP have strong duality (see [1]–[3]). Note also that (OBP)

$$\text{(OBP)} \begin{cases} \underset{\boldsymbol{w}_l \in \mathbb{C}^K, l=1,\ldots,L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{w}_l \\ \text{subject to} & \frac{\boldsymbol{w}_m^H \boldsymbol{R}_{mm} \boldsymbol{w}_m}{\sum_{l \neq m} \boldsymbol{w}_l^H \boldsymbol{R}_{ml} \boldsymbol{w}_l + \sigma_m^2} \geq \rho_m, m = 1, \ldots, L \\ & \sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq \tau_m, m = L+1, \ldots, M \\ & \boldsymbol{w}_l^H \boldsymbol{B}_l \boldsymbol{w}_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & \boldsymbol{w}_l^H \boldsymbol{D}_l \boldsymbol{w}_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \end{cases} \quad (12)$$

attains its optimality with the first L constraints being equality, i.e., being active.[2]

C. Semidefinite Program Relaxation

A common approach to deal with nonconvex problems in practice is to relax the nonconvex constraints to obtain a convex problem that may approximate the original one (see [11]). Let us denote by $\boldsymbol{A} \bullet \boldsymbol{B}$ the inner product $\operatorname{tr}(\boldsymbol{AB})$ of Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Clearly, $\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A} \bullet \boldsymbol{x}\boldsymbol{x}^H$. The SDP relaxation of (13) is shown in (14) at the bottom of the page.

It is seen that adding the rank-one constraints $\operatorname{rank}(\boldsymbol{X}_l) = 1$, $\forall l$, into (SDR) yields the original problem (OBP), and that if problem (SDR) has a rank-one optimal solution, say, $\boldsymbol{X}_l = \boldsymbol{w}_l\boldsymbol{w}_l^H$, $l = 1, \ldots, L$, then $(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_L)$, is optimal for the optimal beamforming problem (OBP) [or, in other words, strong duality holds for (OBP)]. Thus solving (OBP) amounts to finding a rank-one optimal solution of (SDR). An SDP is convex and can be solved by interior-point methods in polynomial time (for example, see [11]), and there are several easy-to-use solvers available.

D. Solvable Optimal Beamforming Problem

In this paper, we will study a rank-constrained solution of a separable SDP in more general form than (SDR). Based on these, we show that several classes of (SDR) have a rank-one optimal solution. Thus, the corresponding classes of (OBP) are solvable. In particular, the classes of solvable (OBP) we identify are with the following parameters.

I) $M = L+2$, and $\boldsymbol{B}_l = \boldsymbol{D}_l = \boldsymbol{0}, \forall l$. In this case, (OBP) has L SINR constraints, two additional soft-shaping interference constraints, and no individual shaping constraint.

II) $M = L$. In this case, (OBP) has L SINR constraints, two groups of individual shaping constraints, and no soft-shaping interference constraints.

III) $M = L+2$, and the parameters $\boldsymbol{B}_l$ and $\boldsymbol{D}_l$, $l = 1, \ldots, L$, are semidefinite matrices (either positive semidefinite or negative semidefinite). In this case, (OBP) has L SINR constraints, two additional soft-shaping interference constraints, and a slightly stricter version of individual shaping constraints.

[2]If one of the first L inequality constraints, say, the first constraint, is inactive for the optimal solution $(\boldsymbol{w}_1^\star, \ldots, \boldsymbol{w}_L^\star)$, then we can always reduce the norm of $\boldsymbol{w}_1^\star$ to get another feasible solution with better objective function value; however, this contradicts the fact that $(\boldsymbol{w}_1^\star, \ldots, \boldsymbol{w}_L^\star)$ is optimal.

III. Rank-Constrained Solution To Separable Semidefinite Program

In this section, we study the separable SDP (SDR) with SINR constraints and soft-shaping interference constraints only. This problem can be written in a more general form:

$$
\text{(P0)} \begin{cases} \underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l \trianglerighteq_m b_m, m = 1, \ldots, M \\ & \boldsymbol{X}_l \succeq 0, l = 1, \ldots, L \end{cases} \tag{15}
$$

where $\boldsymbol{C}_l, \boldsymbol{A}_{ml} \in \mathcal{H}^K$, $l = 1, \ldots, L$, $m = 1, \ldots, M$, i.e., they are Hermitian matrices (not necessarily positive semidefinite), $b_m \in \mathbb{R}$, $\trianglerighteq_m \in \{\geq, =, \leq\}$, $m = 1, \ldots, M$, and the design variables $\boldsymbol{X}_l$, $l = 1, \ldots, L$, are Hermitian matrices. Suppose that $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ is an optimal solution of (P0) with rank profile $R_l = \operatorname{rank}(\boldsymbol{X}_l)$, $l = 1, \ldots, L$. In general, there is no control on the rank profile of an optimal solution. In this section, we study how to generate another solution with constrained rank from the solution with arbitrary rank profile. In addition, we investigate a sufficient condition for a rank-one solution of the separable SDP, and thus find a class of solvable optimal beamforming problem.

A. Rank-Constrained Solution of Separable SDP

To simplify the analysis, we first consider the problem (P0) with only equality constraints and then extend conclusions to (P0) with inequality and/or equality constraints. Thus we consider

$$
\text{(P1)} \begin{cases} \underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l = b_m, m = 1, \ldots, M \\ & \boldsymbol{X}_l \succeq 0, l = 1, \ldots, L \end{cases} \tag{16}
$$

$$
\begin{cases} \underset{\boldsymbol{w}_l \in \mathbb{C}^K, l=1,\ldots,L}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{w}_l^H \boldsymbol{w}_l \\ \text{subject to} & \boldsymbol{w}_m^H \boldsymbol{R}_{mm} \boldsymbol{w}_m - \rho_m \sum_{l \neq m} \boldsymbol{w}_l^H \boldsymbol{R}_{ml} \boldsymbol{w}_l \geq \rho_m \sigma_m^2, m = 1, \ldots, L \\ & \sum_{l=1}^L \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq \tau_m, m = L+1, \ldots, M \\ & \boldsymbol{w}_l^H \boldsymbol{B}_l \boldsymbol{w}_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & \boldsymbol{w}_l^H \boldsymbol{D}_l \boldsymbol{w}_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \end{cases} \tag{13}
$$

$$
\text{(SDR)} \begin{cases} \underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{I} \bullet \boldsymbol{X}_l \\ \text{subject to} & \boldsymbol{R}_{mm} \bullet \boldsymbol{X}_m - \rho_m \sum_{l \neq m} \boldsymbol{R}_{ml} \bullet \boldsymbol{X}_l \geq \rho_m \sigma_m^2, m = 1, \ldots, L \\ & \sum_{l=1}^L \boldsymbol{S}_{ml} \bullet \boldsymbol{X}_l \leq \tau_m, m = L+1, \ldots, M \\ & \boldsymbol{B}_l \bullet \boldsymbol{X}_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & \boldsymbol{D}_l \bullet \boldsymbol{X}_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \\ & \boldsymbol{X}_l \succeq 0, l = 1, \ldots, L \end{cases} \tag{14}
$$

The corresponding dual problem is the following SDP:

$$(\mathrm{D1})\begin{cases}\underset{y_1,\ldots,y_M}{\text{maximize}} & \sum_{m=1}^M y_m b_m \\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^M y_m \boldsymbol{A}_{ml} \succeq 0, \\ & \quad l = 1,\ldots,L.\end{cases} \tag{17}$$

Suppose that (P1) and (D1) are solvable[3] and let $(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L)$, $(y_1,\ldots,y_M,\boldsymbol{Z}_1,\ldots,\boldsymbol{Z}_L)$ be optimal solutions of the problems (P1) and (D1), respectively (existence is ensured by assuming that both the primal and dual SDPs have interior points in their feasible regions, respectively). Notice that they comply with the complementarity conditions[4]

$$\boldsymbol{X}_l\boldsymbol{Z}_l = \boldsymbol{0}, l = 1,\ldots,L \tag{18}$$

which are equivalent to (due to the positive semidefiniteness of $\boldsymbol{X}_l$ and $\boldsymbol{Z}_l$)

$$\boldsymbol{X}_l \bullet \boldsymbol{Z}_l = 0, l = 1,\ldots,L. \tag{19}$$

Lemma 3.1: Suppose that the separable SDP (P1) and its dual (D1) are solvable. Then, the problem (P1) has always an optimal solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ such that

$$\sum_{l=1}^L \operatorname{rank}^2(\boldsymbol{X}_l^\star) \le M.$$

Proof: See Appendix A. ■

We now extend Lemma 3.1 to the separable SDP (P0) with inequality and/or equality constraints. The dual problem of (P0) is

$$(\mathrm{D0})\begin{cases}\underset{y_1,\ldots,y_M}{\text{maximize}} & \sum_{m=1}^M y_m b_m \\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^M y_m \boldsymbol{A}_{ml} \succeq 0, \\ & \quad l = 1,\ldots,L \\ & y_m \trianglerighteq_m^* 0, m = 1,\ldots,M\end{cases} \tag{20}$$

where

$$\trianglerighteq_m^* \text{ is } \begin{cases}\ge, & \text{if } \trianglerighteq_m \text{ is } \ge \\ \text{unrestricted}, & \text{if } \trianglerighteq_m \text{ is } =, m = 1,\ldots,M. \\ \le, & \text{if } \trianglerighteq_m \text{ is } \le\end{cases} \tag{21}$$

Suppose that $(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L)$ and $(y_1,\ldots,y_M,\boldsymbol{Z}_1,\ldots,\boldsymbol{Z}_L)$ are optimal solutions of (P0) and (D0), respectively, and

$$a_m = \sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l, m = 1,\ldots,M. \tag{22}$$

(Observe that $a_m = b_m$ for those m, with $\trianglerighteq_m$ being "=") Then they satisfy the complementary conditions of the SDPs (P0) and (D0), which are (18) and

$$y_m(a_m - b_m) = 0, m = 1,\ldots,M. \tag{23}$$

We verify if the current solution $(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L)$ fulfills the rank constraint $\sum_{l=1}^L \operatorname{rank}^2(\boldsymbol{X}_l) \le M$. If the answer is yes, then we stop; otherwise, by applying the rank reduction procedure proposed in the proof of Lemma 3.1, we can find another solution of (P0), say, $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$, such that $\sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l^\star = a_m$, $m = 1,\ldots,M$, $\boldsymbol{X}_l^\star \succeq 0$, $l = 1,\ldots,L$ (i.e., primal feasibility), $\boldsymbol{X}_l^\star \bullet \boldsymbol{Z}_l = 0$, $l = 1,\ldots,L$, (23) (i.e., the complementary conditions), and the rank constraint $\sum_{l=1}^L \operatorname{rank}^2(\boldsymbol{X}_l^\star) \le M$, are all satisfied. In other words, the solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ is obtained using (P1) (and Lemma 3.1) with the equality constraint sets matching the optimal solution to (P0). This leads to the next theorem.

Theorem 3.2: Suppose that the separable SDP (P0) and its dual (D0) are solvable. Then, the problem (P0) has always an optimal solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ such that

$$\sum_{l=1}^L \operatorname{rank}^2(\boldsymbol{X}_l^\star) \le M. \tag{24}$$

Algorithm 1 summarizes the procedure to produce a rank-constrained solution of the separable SDP (P0) according to Theorem 3.2.

At each iteration of Algorithm 1, an undetermined system of linear equations must be solved. Since the system of linear equations has M equations and $U = \sum_{l=1}^L R_l^2$ real-valued unknowns, it can be equivalently rewritten into the form $\boldsymbol{Ax} = \boldsymbol{0}$, where $\boldsymbol{A} \in \mathbb{R}^{M\times U}$, $\boldsymbol{x} \in \mathbb{R}^U$. Finding a nonzero solution (for $U > M$) amounts to finding the null space of matrix A. For a detailed approach, one may refer to [12], while in Matlab simulations, one may use the function `null (A)`.

It is known from Pataki [13] (see also [7]) that the separable SDP (P0), either real-valued case or complex-valued case,[5] has an optimal solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ satisfying

$$\sum_{l=1}^L \frac{\operatorname{rank}(\boldsymbol{X}_l^\star)(\operatorname{rank}(\boldsymbol{X}_l^\star)+1)}{2} \le M. \tag{25}$$

However, this rank bound can be improved to (24) for the complex-valued SDP (P0). We remark that the key to improve the bound (25) of an optimal solution lies in the fact that the number of unknowns of the system of linear equations (48) (see the proof in Appendix A) is $\sum_{l=1}^L (R_l)^2$, which in turn is due to the complex-valued Hermitian matrix variables $\boldsymbol{\Delta}_l$. Conse-

[3]By "solvable," we mean that the problem is feasible and bounded, and the optimal value is attained; see [11, p. 1].

[4]The KKT conditions for SDP (or general linear conic programming) consist of complementary conditions and primal and dual feasibility only. One may further refer to [11, Th. 1.7.1, 4)] for strong duality theorem of SDP.

[5]We say SDP with real-valued parameters and design variables to be real-valued SDP; in contrast, we say SDP with complex-valued parameters and design variables to be complex-valued SDP.

quently, the usual purification technique[6] is applicable, and the basis of Algorithm 1 is an extension of it (see Appendix A for details).

Algorithm 1: Rank-constrained solution procedure (I) for separable SDP

Input: $\boldsymbol{C}_l, \boldsymbol{A}_{ml}, b_m, \trianglerighteq_m, l = 1, \ldots, L, m = 1, \ldots, M$;

Output: an optimal solution $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ with $\sum_{l=1}^{L} \operatorname{rank}^2(\boldsymbol{X}_l) \leq M$;

solve the separable SDP (P0) finding $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L$, with arbitrary ranks;

evaluate $R_l = \operatorname{rank}(\boldsymbol{X}_l)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^{L} R_l^2$;

while $U > M$ **do**

  decompose $\boldsymbol{X}_l = \boldsymbol{V}_l \boldsymbol{V}_l^H, l = 1, \ldots, L$;

  find a nonzero solution $(\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_L)$ of the system of linear equations:

$$\sum_{l=1}^{L} \boldsymbol{V}_l^H \boldsymbol{A}_{ml} \boldsymbol{V}_l \bullet \boldsymbol{\Delta}_l = 0, m = 1, \ldots, M,$$

  where $\boldsymbol{\Delta}_l$ is a $R_l \times R_l$ Hermitian matrix for all l;

  evaluate the eigenvalues $\delta_{l1}, \ldots, \delta_{lR_l}$ of $\boldsymbol{\Delta}_l$ for $l = 1, \ldots, L$;

  determine l_0 and k_0 such that

$$|\delta_{l_0 k_0}| = \max\{|\delta_{lk}| : 1 \leq k \leq R_l, 1 \leq l \leq L\}.$$

  compute $\boldsymbol{X}_l = \boldsymbol{V}_l(\boldsymbol{I}_{R_l} - (1/\delta_{l_0 k_0})\boldsymbol{\Delta}_l)\boldsymbol{V}_l^H$, $l = 1, \ldots, L$;

  evaluate $R_l = \operatorname{rank}(\boldsymbol{X}_l)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^{L} R_l^2$;

end while

Observe that whenever problem (P0) is nonseparable, i.e., $L = 1$, the rank bound (24) of an optimal solution of the problem (P0) becomes

$$\operatorname{rank}^2(\boldsymbol{X}^\star) \leq M$$

which coincides with the result in [16, Th. 5.1)] (see also a related result in [17, Lemma 1]), while the rank bound (25) of an optimal solution of the problem (P0) becomes

$$\frac{\operatorname{rank}(\boldsymbol{X}^\star)(\operatorname{rank}(\boldsymbol{X}^\star) + 1)}{2} \leq M$$

which has been well known from [13] as well. This means that to achieve the same rank of optimal solutions, complex-valued SDP allows more constraints than real-valued SDP does, as illustrated in Fig. 1.

It follows from Theorem 3.2 that strong duality holds for the considered rank-constrained SDP, as stated next.

[6]For purification process of a solution of linear programming, one may refer to [14, pp. 465–466]; and for purification technique of a solution of nonseparable SDP, one may refer to [13] (real-valued SDP) or [16] (complex-valued SDP).

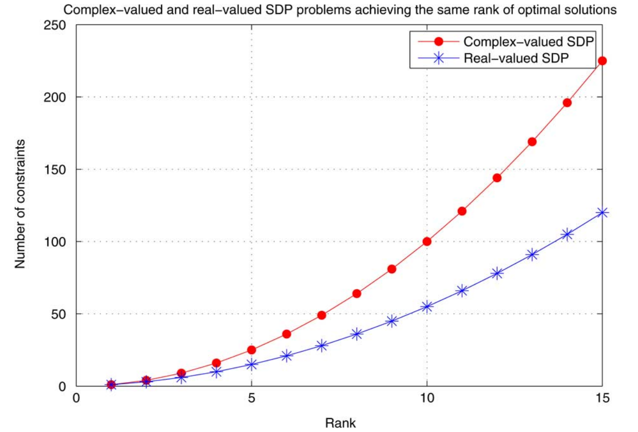

Fig. 1. The numbers of constraints for complex-valued and real-valued SDPs when they achieve the same rank of optimal solutions.

Corollary 3.3: Suppose that the separable SDP (P0) and its dual (D0) are solvable. Then, strong duality holds for the following rank-constrained SDP:

$$\begin{cases} \underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l \trianglerighteq_m b_m, m = 1, \ldots, M \\ & \sum_{l=1}^{L} \operatorname{rank}^2(\boldsymbol{X}_l) \leq M \\ & \boldsymbol{X}_l \succeq 0, l = 1, \ldots, L \end{cases} \tag{26}$$

i.e., there is no duality gap between problem (26) and its dual (D0).[7]

It is noteworthy that for $L = 1$ and $M \leq 3$, Corollary 3.3 simplifies to the following compact corollary.

Corollary 3.4: If the SDP relaxation of the following QCQP and its dual are solvable, and $M \leq 3$, then

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{x}^H \boldsymbol{C} \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^H \boldsymbol{A}_m \boldsymbol{x} \trianglerighteq_m b_m, m = 1, \ldots, M \end{cases} \tag{27}$$

has no duality gap, where $\boldsymbol{C}$, $\boldsymbol{A}_m$, can be any Hermitian matrices.

Observe that if Pataki's bound (25) is applied, then one can conclude that the QCQP (27) with only two constraints (i.e., $M \leq 2$) is solvable polynomially. Note also that the problem solved in [15] belongs to a subclass of (27) by setting

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{Q}_0 & \boldsymbol{b}_0 \\ \boldsymbol{b}_0^H & c_0 \end{bmatrix}, \boldsymbol{A}_m = \begin{bmatrix} \boldsymbol{Q}_m & \boldsymbol{b}_m \\ \boldsymbol{b}_m^H & c_m \end{bmatrix}, m = 1, 2$$

and

$$\boldsymbol{A}_3 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

and $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, $\trianglerighteq_1 \in \{\geq\}$, $\trianglerighteq_2 \in \{=\}$, and $\trianglerighteq_3 \in \{=\}$.

[7]The dual of (26) can be derived by computing the Lagrange dual function and using the definition of Lagrange dual problem (for example, see [11]) by treating the rank operator as a function.

B. Rank-One Solution of Separable SDP

Theorem 3.2 provides a condition for the rank profile of an optimal solution of problem (P0) but gives no specific rank profile of an optimal solution. Interestingly, it turns out that for some cases where the constraints of the separable SDP are not "too much," there is only one rank profile satisfying (24), for example, the case of a rank-one optimal solution,[8] as illustrated in the proposition below.

Proposition 3.5: Suppose that the primal problem (P0) and the dual problem (D0) are solvable. Suppose also that any optimal solution of the problem (P0) has no zero matrix component.[9] If $M \leq L+2$, then (P0) has an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ with each $\boldsymbol{X}_l^\star$ of rank one.

Proof: See Appendix B. ■

Observe that if Pataki's rank bound (25) is applied in the proof of Proposition 3.5, we get that (P0) has a rank-one solution when $M \leq L+1$ only.

The assumption in Proposition 3.5 that any optimal solution has no zero matrix component is important. Sufficient conditions that guarantee this are the following (which in fact guarantee that any feasible point of problem (P0) has no zero matrix component):[10]

$$L \leq M \tag{28}$$

$$-\boldsymbol{A}_{ml} \succeq 0, \forall l \neq m, m, l = 1, \ldots, L \tag{29}$$

$$\trianglerighteq_m \in \{\geq, =\}, m = 1, \ldots, L, \tag{30}$$

$$b_m > 0, m = 1, \ldots, L. \tag{31}$$

Thus, from Proposition 3.5, (P0) has a rank-one optimal solution if the parameters of (P0) satisfy the conditions (28)–(31) and $M \leq L+2$.

Corollary 3.6: Suppose (28)–(31) are satisfied. Then, the following separable QCQP can be solved polynomially (e.g., with Algorithm 1):

$$\begin{cases} \underset{\boldsymbol{x}_l \in \mathbb{C}^K}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{x}_l^H \boldsymbol{C}_l \boldsymbol{x}_l \\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{x}_l^H \boldsymbol{A}_{ml} \boldsymbol{x}_l \trianglerighteq_m b_m, m = 1, \ldots, M \end{cases} \tag{32}$$

[8]Rank-one solution of a separable SDP means an optimal solution of the problem with each matrix component of rank-one.

[9]Throughout this paper, by saying that a solution $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ has no zero matrix component, we mean each $\boldsymbol{X}_l$ is not equal to the zero matrix for any $l = 1, \ldots, L$.

[10]Suppose that the parameters of the constraints of problem (P0) satisfy (28)–(31). If there is a feasible point $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ such that $\boldsymbol{X}_1 = \boldsymbol{0}$, then the left-hand side of the first constraint is nonpositive while the right-hand side $b_1 > 0$, and thus a contradiction arises.

where $M \leq L+2$, provided that its SDP relaxation and its dual are solvable.

It is easily verified that the optimal beamforming problem (OBP) described in Section II, with $M = L+2$ and $\boldsymbol{B}_l = \boldsymbol{D}_l = \boldsymbol{0}, \forall l$, belongs to (32) with the conditions (28)–(31) all fulfilled. Therefore such optimal beamforming problem is solvable, and an optimal solution is obtained by solving its SDP relaxation (SDR) and calling the rank reduction procedure described in Algorithm 1. In this case, (OBP) has L SINR constraints, two additional soft-shaping interference constraints, and no individual shaping constraints.

It is also noteworthy that the optimal beamforming problem is solvable with L SINR constraints and one similarity constraint imposing the beamvector for internal user l to be not too far from a desired beamvector $\boldsymbol{w}_{0l}$: $\|\boldsymbol{w}_l - \boldsymbol{w}_{0l}\|^2 \leq \epsilon$ (see [6]). In particular, such a similarity constraint can be rewritten equivalently as two homogeneous quadratic constraints, and thus the optimal beamforming problem is a special case of (32).

IV. Rank-Constrained Solution to Separable SDP With Individual Shaping Constraints

In this section, we consider the SDP (P0) with additional individual shaping constraints as shown in (33) at the bottom of the page, where $\boldsymbol{C}_l, \boldsymbol{A}_{ml} \in \mathcal{H}^K$, $b_m \in \mathbb{R}$, $\trianglerighteq_m \in \{\geq, =, \leq\}$, $l = 1, \ldots, L$, $m = 1, \ldots, M$, and the matrix parameters in the individual shaping constraints are the same as those in (9) and (10). Problem (P2) is clearly more general than the SDP relaxation (SDR). We will investigate procedure of producing a solution of (P2) satisfying some rank constraint and then present a sufficient condition for the existence of a rank-one solution of (P2). The optimization tools to be used in the section include modified procedures of rank reduction (developed in Section III) and a specific rank-one decomposition technique (see [16]).

A. Rank-Constrained Solution Procedures

For the ease of analysis, we again consider problem (P3) with the first M constraints being equality constraints, as shown in (34) at the bottom of the next page. The dual problem of (P3) is shown in (35) at the bottom of the next page. Let $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$, $(y_1, \ldots, y_M, \mu_1, \ldots, \mu_L, \lambda_1, \ldots, \lambda_L, \boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_L)$ be optimal solutions of the problems (P3) and (D3), respectively. Thus, they satisfy the complementary conditions

$$\boldsymbol{X}_l \bullet \boldsymbol{Z}_l = 0, l = 1, \ldots, L, \tag{36}$$

$$\mu_l \boldsymbol{B}_l \bullet \boldsymbol{X}_l = 0, \forall l \in \bar{\mathcal{E}}_1, \tag{37}$$

$$\lambda_l \boldsymbol{D}_l \bullet \boldsymbol{X}_l = 0, \forall l \in \bar{\mathcal{E}}_2. \tag{38}$$

$$\text{(P2)} \begin{cases} \underset{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l \trianglerighteq_m b_m, m = 1, \ldots, M \\ & \boldsymbol{B}_l \bullet \boldsymbol{X}_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & \boldsymbol{D}_l \bullet \boldsymbol{X}_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \\ & \boldsymbol{X}_l \succeq 0, l = 1, \ldots, L \end{cases} \tag{33}$$

It is clear that by invoking Theorem 3.2, we can get an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ of (P3) with

$$\sum_{l=1}^{L} \text{rank}^2(\boldsymbol{X}_l^\star) \leq M + 2L. \tag{39}$$

Now, we derive tighter upper bounds for the rank of an optimal solution, resorting to the rank-one decomposition technique [16, Th. 2.1]. As shall be seen in the next section, the new upper bounds lead to rank-one solution arguments of the separable SDP (P2). To proceed, let us quote the rank-one decomposition theorem in [16] as the following lemma.

Lemma 4.1: Suppose that $\boldsymbol{X}$ is an $N \times N$ complex Hermitian positive semidefinite matrix of rank R and $\boldsymbol{A}_1, \boldsymbol{A}_2$ are two $N \times N$ given Hermitian matrices. Then, there is a rank-one decomposition of $\boldsymbol{X}$ (synthetically denoted as $\mathcal{D}(\boldsymbol{X}, \boldsymbol{A}_1, \boldsymbol{A}_2)$), $\boldsymbol{X} = \sum_{r=1}^{R} \boldsymbol{x}_r \boldsymbol{x}_r^H$, such that

$$\boldsymbol{x}_r^H \boldsymbol{A}_1 \boldsymbol{x}_r = \frac{\boldsymbol{X} \bullet \boldsymbol{A}_1}{R} \text{ and } \boldsymbol{x}_r^H \boldsymbol{A}_2 \boldsymbol{x}_r = \frac{\boldsymbol{X} \bullet \boldsymbol{A}_2}{R}$$

$r = 1, \ldots, R$.

Given a general rank optimal solution of (P3), we can obtain another optimal solution with a constrained rank, as stated in the next two lemmas, and the construction of such desired rank-constrained solution can be found in the proofs.

Lemma 4.2: Suppose that the separable SDP (P3) and its dual (D3) are solvable. Then, problem (P3) has a solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ such that

$$\sum_{l=1}^{L} \text{rank}(\boldsymbol{X}_l^\star) \leq M. \tag{40}$$

Proof: See Appendix C. ■

Observe that, as compared to Theorem 3.2 (and Lemma 3.1), the expression in (40) does not contain the square in the ranks. Now, we consider a slightly stricter version of the individual shaping constraints for the case when there are L' pairs of parameters $(\boldsymbol{B}_l, \boldsymbol{D}_l)$, for some $L' \in \{0, 1, \ldots, L\}$, having the property that $\boldsymbol{B}_l$ and $\boldsymbol{D}_l$ are semidefinite for all l (meaning that each $\boldsymbol{B}_l$ is either positive semidefinite or negative semidefinite, and each $\boldsymbol{D}_l$ is either positive semidefinite or negative semidefinite). In particular, the condition that $L' = 0$ means no additional assumptions on any $(\boldsymbol{B}_l, \boldsymbol{D}_l)$.

Lemma 4.3: Suppose that the separable SDP (P3) and its dual (D3) are solvable, and suppose that $(\boldsymbol{B}_l, \boldsymbol{D}_l)$, for $l = 1, \ldots, L'$, for some $L' \in \{0, 1, \ldots, L\}$, are semidefinite. Then, the problem (P3) has a solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ such that

$$\sum_{l=1}^{L'} \text{rank}^2(\boldsymbol{X}_l^\star) + \sum_{l=L'+1}^{L} \text{rank}(\boldsymbol{X}_l^\star) \leq M.$$

Proof: See Appendix D. ■

From Lemma 4.3, one can see that individual constraints defined by semidefinite matrices do not contribute to the loss of the square in the rank, which is very desirable. Note also that when $L' = 0$, Lemma 4.3 coincides with Lemma 4.2.

Lemma 4.3 can be extended to the general separable SDP (P2). Observe that the dual problem of (P2) is as shown in (41) at the bottom of the page, where each $\trianglerighteq_m^*$ is defined in the same way as (21).

Theorem 4.4: Suppose that the separable SDP (P2) and its dual (D2) are solvable and that $(\boldsymbol{B}_l, \boldsymbol{D}_l)$, for $l = 1, \ldots, L'$, for some $L' \in \{0, 1, \ldots, L\}$, are semidefinite. Then, problem (P2) has a solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ such that

$$\sum_{l=1}^{L'} \text{rank}^2(\boldsymbol{X}_l^\star) + \sum_{l=L'+1}^{L} \text{rank}(\boldsymbol{X}_l^\star) \leq M. \tag{42}$$

$$(\text{P3}) \begin{cases} \underset{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l = b_m, m = 1, \ldots, M \\ & \boldsymbol{B}_l \bullet \boldsymbol{X}_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & \boldsymbol{D}_l \bullet \boldsymbol{X}_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \\ & \boldsymbol{X}_l \succeq 0, l = 1, \ldots, L \end{cases} \tag{34}$$

$$(\text{D3}) \begin{cases} \underset{y_m, \mu_l, \lambda_l}{\text{maximize}} & \sum_{m=1}^{M} y_m b_m \\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{M} y_m \boldsymbol{A}_{ml} - \mu_l \boldsymbol{B}_l - \lambda_l \boldsymbol{D}_l \succeq 0, l = 1, \ldots, L \\ & \mu_l \geq 0, \forall l \in \bar{\mathcal{E}}_1, \lambda_l \geq 0, \forall l \in \bar{\mathcal{E}}_2. \end{cases} \tag{35}$$

$$(\text{D2}) \begin{cases} \underset{y_m, \mu_l, \lambda_l}{\text{maximize}} & \sum_{m=1}^{M} y_m b_m \\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{M} y_m \boldsymbol{A}_{ml} - \mu_l \boldsymbol{B}_l - \lambda_l \boldsymbol{D}_l \succeq 0, l = 1, \ldots, L \\ & y_m \trianglerighteq_m^* 0, m = 1, \ldots, M \\ & \mu_l \geq 0, \forall l \in \bar{\mathcal{E}}_1, \lambda_l \geq 0, \forall l \in \bar{\mathcal{E}}_2 \end{cases} \tag{41}$$

Algorithm 2 contains the procedure for generating a rank-constrained solution of (P2).

Algorithm 2: Rank-constrained solution procedure (II) for separable SDP with additional individual shaping constraints

`Input:` C_l, A_{ml}, $\trianglerighteq_m$, b_m, B_l, D_l, $l = 1, \ldots, L$, $m = 1, \ldots, M$, $L' \in \{0, 1, \ldots, L\}$;

`Output:` an optimal solution $(X_1, \ldots, X_L)$ with $\sum_{l=1}^{L'} \text{rank}^2(X_l) + \sum_{l=L'+1}^{L} \text{rank}(X_l) \leq M$;

1: solve the separable SDP with shaping constraints (P2) finding $X_1, \ldots, X_L$, with arbitrary rank;
2: evaluate $R_l = \text{rank}(X_l)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^{L'} R_l^2 + \sum_{l=L'+1}^{L} R_l$;
3: **while** $U > M$ **do**
4: perform decomposition $X_l = V_l V_l^H$, for $l = 1, \ldots, L'$, and call the rank-one decomposition $\mathcal{D}(X_l, B_l, D_l)$ and output $X_l = V_l V_l^H$, for $l = L'+1, \ldots, L$;
5: find a nonzero solution $(\Delta_1, \ldots, \Delta_L)$ of the system of linear equations:

$$\sum_{l=1}^{L} V_l^H A_{ml} V_l \bullet \Delta_l = 0, m = 1, \ldots, M,$$

where Δ_l is an $R_l \times R_l$ Hermitian matrix for $l = 1, \ldots, L'$, and Δ_l is an $R_l \times R_l$ real-valued diagonal matrix for $l = L'+1, \ldots, L$; let δ_{lk} be eigenvalues of Δ_l for $l = 1, \ldots, L$, $1 \leq k \leq R_l$.
6: determine l_0 and k_0 such that

$$|\delta_{l_0 k_0}| = \max\{|\delta_{lk}| : 1 \leq k \leq R_l, 1 \leq l \leq L\}.$$

7: compute $X_l = V_l(I_{R_l} - (1/\delta_{l_0 k_0})\Delta_l)V_l^H$, $l = 1, \ldots, L$;
8: evaluate $R_l = \text{rank}(X_l)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^{L'} R_l^2 + \sum_{l=L'+1}^{L} R_l$;
9: **end while**

We remark that in the above rank-reduction procedure, a key fact leading to the new rank bound (42) is related to the fact that each $R_l \times R_l$ real-valued diagonal matrix has R_l free variables and each $R_l \times R_l$ Hermitian matrix has R_l^2 real-valued free variables. Whenever the optimization variables are real-valued, the reduction matrices Δ_l, $l = 1, \ldots, L'$, are changed from being Hermitian to symmetric. This leads to the rank bound

$$\sum_{l=1}^{L'} \frac{\text{rank}(X_l^\star)(\text{rank}(X_l^\star) + 1)}{2} + \sum_{l=L'+1}^{L} \text{rank}(X_l^\star) \leq M$$

for Theorem 4.4 in the setting of real-valued design variables, since each symmetric $R_l \times R_l$ reduction matrix has $R_l(R_l+1)/2$ free variables. Again, when $L = 1$, it reduces to nonseparable case or coupled case.

In a similar vein to Corollary 3.3, we claim strong duality for a rank-constrained separable SDP with individual shaping constraints as the following corollary.

Corollary 4.6: Suppose that the separable SDP (P2) and its dual (D2) are solvable, and suppose that (B_l, D_l), for $l = 1, \ldots, L'$, for some $L' \in \{0, 1, \ldots, L\}$, are semidefinite. Then, the rank-constrained separable SDP shown in the equation at the bottom of the page has zero duality gap with the dual problem (D2).

In particular, Corollary 4.5 means that a class of homogeneous nonseparable QCQP is solvable polynomially, as stated in the next corollary.

Corollary 4.6: Suppose that the SDPs (P2) and (D2) are solvable, with $L = 1$ and $M \leq 3$. If B and D are semidefinite, then the following QCQP is solvable polynomially:

$$\begin{cases} \underset{x}{\text{minimize}} & x^H C x \\ \text{subject to} & x^H A_m x \trianglerighteq_m b_m, m = 1, \ldots, M \\ & x^H B x \trianglerighteq_{M+1} 0 \\ & x^H D x \trianglerighteq_{M+2} 0 \end{cases}$$

where $\trianglerighteq_m \in \{\geq, =, \leq\}$, $m = 1, \ldots, M+2$.

B. Rank-One Solution of General Separable SDP

If the assumption of Proposition 3.5 is valid, i.e., if any optimal solution $(X_l, \ldots, X_L)$ of (P2) has no zero matrix component (or, equivalently, $\text{rank}(X_l) \geq 1$, $\forall l$), then, recalling (39), we can find an optimal solution $(X_1^\star, \ldots, X_L^\star)$ of (P2) with

$$L \leq \sum_{l=1}^{L} \text{rank}^2(X_l^\star) \leq M + 2L.$$

A rank-one solution can then be guaranteed for some choices of M and L, for example, $L = 2, M = 0$, or $L = 1, M = 1$. Nonetheless, capitalizing on Theorem 4.4, we claim that the

$$\begin{cases} \underset{X_1, \ldots, X_L}{\text{minimize}} & \sum_{l=1}^{L} C_l \bullet X_l \\ \text{subject to} & \sum_{l=1}^{L} A_{ml} \bullet X_l \trianglerighteq_m b_m, m = 1, \ldots, M \\ & B_l \bullet X_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & D_l \bullet X_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \\ & \sum_{l=1}^{L'} \text{rank}^2(X_l) + \sum_{l=L'+1}^{L} \text{rank}(X_l) \leq M \\ & X_l \succeq 0, l = 1, \ldots, L \end{cases}$$

separable SDP (P2) does have a rank-one solution if some mild assumptions are imposed.

Proposition 4.7: Suppose that problems (P2) and (D2) are solvable. Suppose also that any optimal solution $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ of problem (P2) has no zero matrix component. Then, (P2) has a rank-one solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ if one of two assumptions holds:

- $M = L$;
- $M \leq L+2$ and $(\boldsymbol{B}_l, \boldsymbol{D}_l)$, for all $l = 1, \ldots, L$, are semidefinite.

Proof: See Appendix E. ■

Recall that conditions (28)–(31) ensure that every feasible point of (P2) has no zero matrix component, and this leads to the next corollary.

Corollary 4.8: Suppose (28)–(31) are satisfied. Then, the separable QCQP as shown in (43) at the bottom of the page can be solved polynomially (e.g., with Algorithm 2), where either

- $M = L$;
- $M \leq L+2$ and $(\boldsymbol{B}_l, \boldsymbol{D}_l)$, for all $l = 1, \ldots, L$, are semidefinite;

as long as its SDP relaxation and its dual are solvable.

Accordingly, the optimal beamforming problem (OBP) of Section II, with $M = L$, is solvable, provided that the corresponding SDP relaxation (SDR) and its dual are solvable; an optimal solution of (OBP) can be returned by solving (SDR) and calling Algorithm 2 with $L' = 0$. In this case, (OBP) has L SINR constraints, two groups of individual shaping constraints, and no soft-shaping interference constraints.

Likewise, if the specific (SDR) and its dual are solvable, then (OBP) is solvable, with $M \leq L+2$ and the parameters $\boldsymbol{B}_l$ and $\boldsymbol{D}_l$, $l = 1, \ldots, L$, being semidefinite matrices. An optimal solution (OBP) can be output by solving the SDP (SDR) and calling Algorithm 2 with $L' = L$. In this case, (OBP) has L SINR constraints, two soft-shaping interference constraints, and a slightly stricter version of individual shaping constraints. In particular, the SDP [a special case of (SDR)] shown in (44) at the bottom of the page has a rank-one solution, with $M = L+4$ and $\tau_{L+3} = \tau_{L+4} = 0$, provided that both the problem and its dual are solvable. In fact, the last two constraints are individual shaping constraints.

V. Numerical Results

In this section, we provide some numerical examples illustrating the performance of the algorithms for the optimal downlink beamforming problem.

For simplicity, we simulate a scenario with a single base station serving three users, i.e., $L = 3$, in problem (OBP). Each user is equipped with a single antenna, and the base station has $K = 8$ antenna elements. The users are located at $\theta_1 = 10°$, $\theta_2 = 25°$, and $\theta_3 = -5°$ relative to the array broadside. The channel covariance matrix for users $l = 1, 2, 3$ is generated according to (see [2])

$$[\boldsymbol{R}_l(\theta_l)]_{pq} = e^{j\pi(p-q)\sin\theta_l} e^{-\frac{(-\pi(p-q)\sigma_\theta \cos\theta_l)^2}{2}} \tag{45}$$

$p, q \in \{1, \ldots, K\}$, where $\sigma_\theta = 2°$ is the angular spread of local scatterers surrounding user l. The noise variance is set to $\sigma_l^2 = 0.1$ for users $l = 1, 2, 3$. The SINR threshold values are set to $\rho_l = 1$, $l = 1, 2, 3$. In addition, we consider two external users from another coexisting wireless system, located at $\tilde{\theta}_1 = 30°$ and $\tilde{\theta}_2 = 50°$, relative to the array broadside of the base station. The channel between the base station and the two external users is given by (assuming a uniformly spaced array at the base station)

$$\boldsymbol{h}_m(\tilde{\theta}_m) = [1\, e^{j\phi_m} \ldots e^{j(K-1)\phi_m}]^T, m = 1, 2 \tag{46}$$

where $\phi_m = 2\pi d \sin(\tilde{\theta}_m)/\lambda$ and d is the antenna element separation (we set $d/\lambda = 1/2$). This gives two additional soft-shaping interference constraints in the optimal beamforming problem (OBP).

In order to illustrate the effect of these two additional soft-shaping interference constraints (or null-shaping interference constraints), we evaluate the power radiation pattern of the base station, for $\theta \in [-90°, 90°]$, according to

$$P(\theta) = \boldsymbol{h}(\theta)\boldsymbol{h}(\theta)^H \bullet \left(\boldsymbol{w}_1^\star \boldsymbol{w}_1^{\star H} + \boldsymbol{w}_2^\star \boldsymbol{w}_2^{\star H} + \boldsymbol{w}_3^\star \boldsymbol{w}_3^{\star H}\right)$$

where $(\boldsymbol{w}_1^\star, \boldsymbol{w}_2^\star, \boldsymbol{w}_3^\star)$ is an optimal solution of the problem (OBP) and $\boldsymbol{h}(\theta)$ is defined in (46). We make use of the optimization solver CVX (see [18]) to solve the SDPs.

$$\begin{cases} \underset{\boldsymbol{x}_l \in \mathbb{C}^K}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{x}_l^H \boldsymbol{C}_l \boldsymbol{x}_l \\ \text{subject to} & \sum_{l=1}^L \boldsymbol{x}_l^H \boldsymbol{A}_{ml} \boldsymbol{x}_l \trianglerighteq_m b_m, m = 1, \ldots, M \\ & \boldsymbol{w}_l^H \boldsymbol{B}_l \boldsymbol{w}_l = (\geq) 0, \forall l \in \mathcal{E}_1(\bar{\mathcal{E}}_1) \\ & \boldsymbol{w}_l^H \boldsymbol{D}_l \boldsymbol{w}_l = (\geq) 0, \forall l \in \mathcal{E}_2(\bar{\mathcal{E}}_2) \end{cases} \tag{43}$$

$$\text{(SDR1)} \begin{cases} \underset{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{I} \bullet \boldsymbol{X}_l \\ \text{subject to} & \boldsymbol{R}_{mm} \bullet \boldsymbol{X}_m - \rho_m \sum_{l \neq m} \boldsymbol{R}_{ml} \bullet \boldsymbol{X}_l \geq \rho_m \sigma_m^2, m = 1, \ldots, L \\ & \sum_{l=1}^L \boldsymbol{S}_{ml} \bullet \boldsymbol{X}_l \leq \tau_m, m = L+1, \ldots, M \end{cases} \tag{44}$$

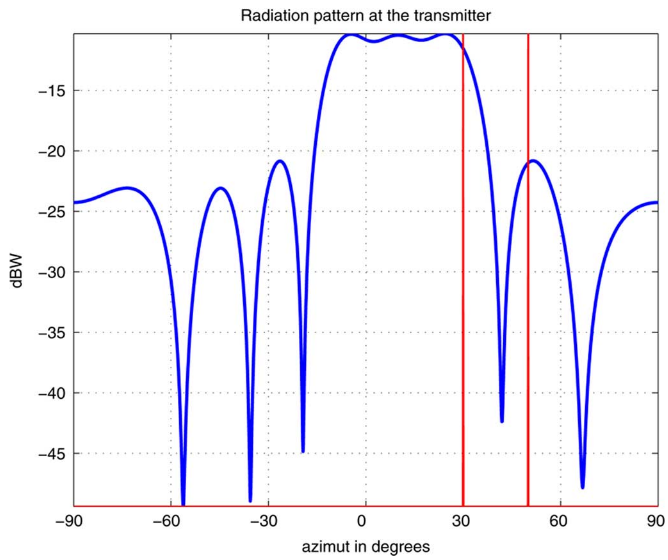

Fig. 2. Radiation pattern of the transmitter for the problem with SINR constraints only. The required transmit power is 16.10 dBm.

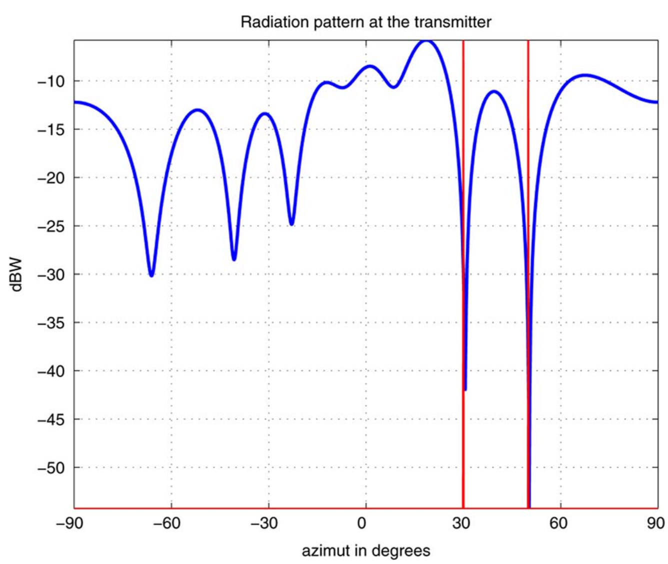

Fig. 3. Radiation pattern of the transmitter for the problem with two additional soft-shaping interference constraints $\tilde{\theta}_1 = 30^\circ$, $\tilde{\theta}_2 = 50^\circ$, $\tau_1 = -30$ dBW, and $\tau_2 = -40$ dBW. The required transmit power is 19.05 dBm.

Example 1: In this example, we set the tolerable values for the two external users are $\tau_1 = 0.001$ and $\tau_2 = 0.0001$, respectively, that is, $M = L + 2 = 5$, $\boldsymbol{B}_l = \boldsymbol{D}_l = \boldsymbol{0}$, $\forall l$, in problem (OBP).

Fig. 2 shows the optimal radiation pattern of the base station in the case that (OBP) in (12) has only the SINR constraints without the soft-shaping interference constraints (the minimal required transmit power is 16.10 dBm). As can be seen, the interference generated at the locations of the two external users ($\tilde{\theta}_1 = 30^\circ$ and $\tilde{\theta}_2 = 50^\circ$) is high. Fig. 3 shows the radiation pattern when soft-shaping interference constraints are included in (OBP) to protect the two external users, in addition to the SINR constraints. In this case, as expected, the radiation power in the directions of the two external users is below the prescribed values τ_1 and τ_2, at the cost of an increase of the minimal

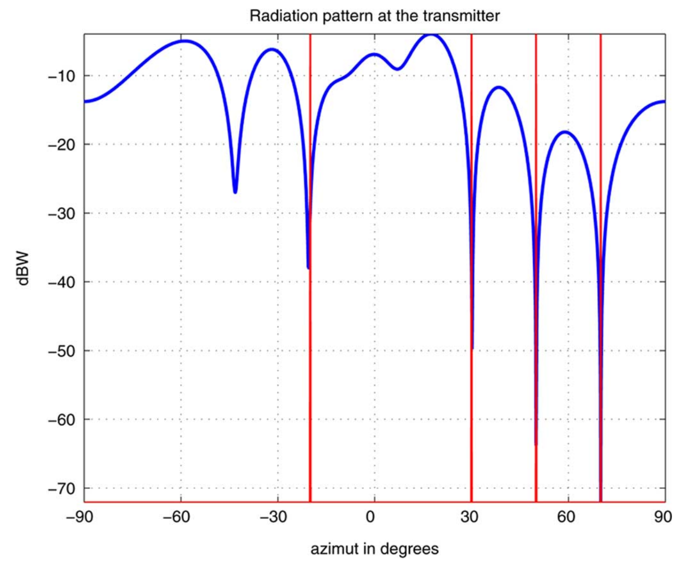

Fig. 4. Radiation pattern of the transmitter for the problem with two additional null-shaping interference constraints and two additional soft-shaping interference constraints $\tilde{\theta}_1 = -20^\circ$, $\tilde{\theta}_2 = 30^\circ$, $\tilde{\theta}_3 = 50^\circ$, $\tilde{\theta}_4 = 70^\circ$, $\tau_1 = -30$ dBW, $\tau_2 = -40$ dBW, $\tau_3 = -\mathrm{Inf}$ dBW, $\tau_4 = -\mathrm{Inf}$ dBW. The required transmit power is 20.81 dBm.

transmit power from 16.10 to 19.05 dBm. In other words, in the former case (as shown in Fig. 2), the minimal transmit power is lower but with no control on the radiation power towards the two external users; in the latter case (as shown in Fig. 3), the minimal transmit power is higher but with being able to keep the radiation power towards the two external users under the given tolerable values, respectively. The tradeoff is evident from optimization standpoint: the feasible region in the former case is larger than the one in the latter case, and thus the minimal value in the former case is less than the minimal value in the latter case.

Example 2: In this case, we consider four external users from other wireless systems, located at $\tilde{\theta}_1 = -20^\circ$, $\tilde{\theta}_2 = 30^\circ$, $\tilde{\theta}_3 = 50^\circ$, and $\tilde{\theta}_4 = 70^\circ$, with tolerable values $\tau_1 = 0.001$, $\tau_2 = 0.0001$, and $\tau_3 = \tau_4 = 0$, respectively. The SINR threshold values for the three internal users are the same as those used in Example 1. Therefore, the problem considered in the example is tantamount to (OBP)'s having three SINR constraints, two soft-shaping interference constraints (i.e., $M = 5$), and two null-shaping interference constraints (or, equivalently, two group of individual shaping constraints with $\boldsymbol{B}_l = \boldsymbol{h}(\tilde{\theta}_3)\boldsymbol{h}(\tilde{\theta}_3)^H$ and $\boldsymbol{D}_l = \boldsymbol{h}(\tilde{\theta}_4)\boldsymbol{h}(\tilde{\theta}_4)^H$, $\forall l \in \mathcal{E}_1 = \mathcal{E}_2 = \{1, 2, 3\}$).

Fig. 4 shows the optimal radiation pattern of the base station with all types of constraints (the minimal required transmit power is 20.81 dBm).

Example 3: In this example, we compare the minimal required transmit power transmitted at the base station for different SINR threshold values, which we set to $\rho_1 = \rho_2 = \rho_3 = \rho$. We consider two external users from other wireless systems, located at $\tilde{\theta}_1 = 30^\circ$ and $\tilde{\theta}_2 = 50^\circ$, respectively, with tolerable values set to $\tau_1 = \tau_2 = 0.01$. In other words, $M = 5$, $\boldsymbol{B}_l = \boldsymbol{D}_l = \boldsymbol{0}$, $\forall l$ in problem (OBP). Fig. 5 displays the minimal required transmit power versus ρ for the cases of null interference constraints, soft-shaping interference constraints,

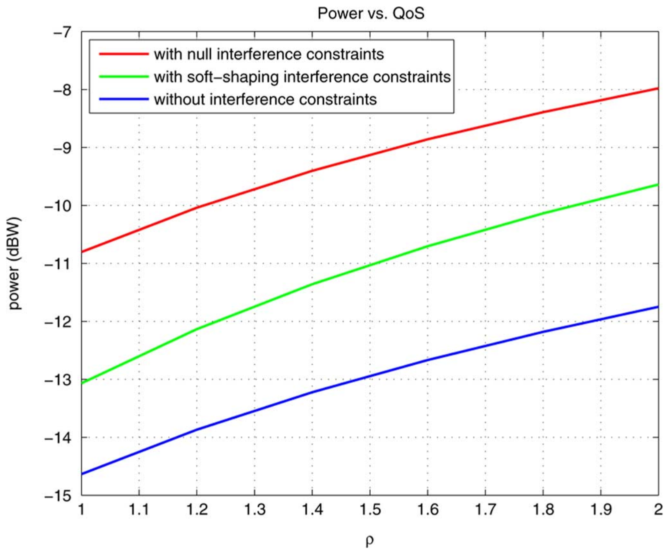

Fig. 5. Minimal transmission power versus the threshold value of QoS.

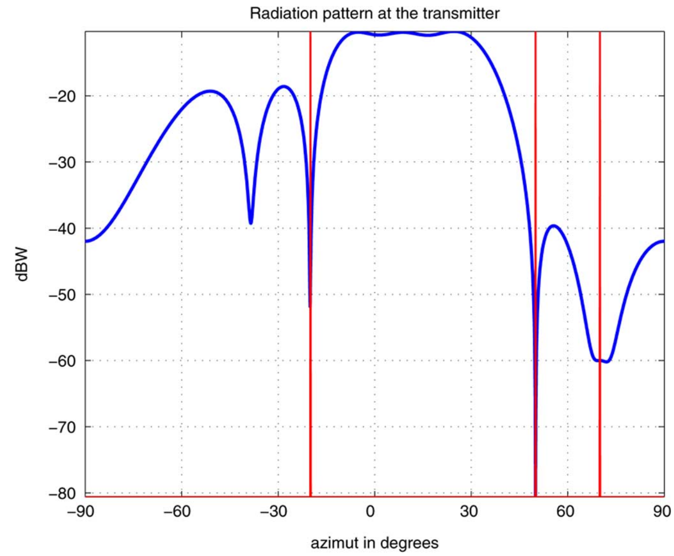

Fig. 6. Radiation pattern of the transmitter for the problem with $\tilde{\theta}_1 = -20°$, $\tilde{\theta}_2 = 50°$, $\tilde{\theta}_3 = 70°$, $\tau_1 = -50$ dBW, $\tau_2 = -\mathrm{Inf}$ dBW, $\tau_3 = -60$ dBW, and $\tau_3' = -\mathrm{Inf}$ dBW. The required transmit power is 16.38 dBm.

and no interference constraints. It can be observed that the required power is larger with null-shaping interference constraints than with soft-shaping interference constraints (as the null constraints are more stringent than the soft ones).

Example 4: This example considers interference constraints robust to uncertainty in the direction of an external user. This can be easily achieved by inducing an interference constraint not just along the estimated channel $\boldsymbol{h}_m(\tilde{\theta}_m)$ but also along its derivative (see [19])

$$\frac{d\boldsymbol{h}_m(\tilde{\theta}_m)}{d\tilde{\theta}_m} = [0 \ j\psi_m e^{j\phi_m} \cdots j(K-1)\psi_m e^{j(K-1)\phi_m}]^T$$

where $\psi_m = 2\pi d \cos(\tilde{\theta}_m)/\lambda$ and ϕ_m is the same as the one in (46). In particular, the derivative shaping constraint is

$$\frac{d\boldsymbol{h}_m(\tilde{\theta}_m)}{d\tilde{\theta}_m}\left(\frac{d\boldsymbol{h}_m(\tilde{\theta}_m)}{d\tilde{\theta}_m}\right)^H \bullet \big(\boldsymbol{w}_1\boldsymbol{w}_1^H + \boldsymbol{w}_2\boldsymbol{w}_2^H + \boldsymbol{w}_3\boldsymbol{w}_3^H\big) \leq \tau_m'. \quad (47)$$

In addition to the three internal users, we consider in this example three external users from other wireless systems, located at $\tilde{\theta}_1 = -20°$, $\tilde{\theta}_2 = 50°$, $\tilde{\theta}_3 = 70°$, with tolerable values $\tau_1 = 0.00001$, $\tau_2 = 0$, $\tau_3 = 0.000001$, respectively. For external user 3, we add the derivative shaping constraint (47) with $\tau_3' = 0$. This problem is equivalent to (OBP) with $M = 5$, $\boldsymbol{B}_l = \boldsymbol{h}(\tilde{\theta}_2)\boldsymbol{h}(\tilde{\theta}_2)^H$, $\boldsymbol{D}_l = (d\boldsymbol{h}_m(\tilde{\theta}_m))/(d\tilde{\theta}_m)((d\boldsymbol{h}_m(\tilde{\theta}_m))/(d\tilde{\theta}_m))^H|_{\tilde{\theta}_m=\tilde{\theta}_3}$, $\forall l$. Fig. 6 exhibits the optimal radiation pattern of the base station (the required transmit power is 16.38 dBm). As can be seen, the power radiated around $\tilde{\theta}_3 = 70°$ is maintained below the tolerable value.

VI. Conclusion

In this paper, we have considered the optimal downlink beamforming problem that minimizes the transmission power subject to SINR constraints for users within the system, as well as soft-shaping interference constraints to protect other users from coexisting systems, and individual shaping constraints. The problem belongs to the class of separable homogeneous QCQP. For this reason, we have studied rank-constrained solutions of general separable SDPs and proposed efficient rank reduction procedures. Based on these, we have shown that three classes of the optimal beamforming problem (OBP) are solvable by arguing that the corresponding SDP relaxation has always a rank-one optimal solution. The solution can be generated by solving the SDP relaxation and invoking the rank reduction procedure. Numerical examples have been conducted to illustrate the flexibility of the proposed framework of rank-constrained SDP in the context of downlink beamforming.

Appendix

A. Proof of Lemma 3.1

Proof: Suppose $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ and $(y_1, \ldots, y_M, \boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_M)$ are optimal solutions of the problems (P1) and (D1), respectively. Let $R_l = \mathrm{rank}(\boldsymbol{X}_l)$, $l = 1, \ldots, L$. By decomposing $\boldsymbol{X}_l = \boldsymbol{V}_l\boldsymbol{V}_l^H$, $\boldsymbol{V}_l \in \mathbb{C}^{K\times R_l}$, we get

$$\sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l = \sum_{l=1}^{L} \boldsymbol{V}_l^H \boldsymbol{A}_{ml}\boldsymbol{V}_l \bullet \boldsymbol{I}_{R_l} = b_m, m = 1, \ldots, M.$$

We consider the following system of linear equations:

$$\sum_{l=1}^{L} \boldsymbol{V}_l^H \boldsymbol{A}_{ml}\boldsymbol{V}_l \bullet \boldsymbol{\Delta}_l = 0, m = 1, \ldots, M \quad (48)$$

where $\boldsymbol{\Delta}_l$ is an R_l-by-R_l Hermitian matrix. Note that there are $(R_l)^2$ real-valued unknowns[11] in the entries of the complex-valued $\boldsymbol{\Delta}_l$; therefore the system (48) has M equations and $U = \sum_{l=1}^{L}(R_l)^2$ unknowns.

[11]The number of unknowns for a real part of $\boldsymbol{\Delta}_l$ is $R_l(R_l+1)/2$, and the number of unknowns for imaginary part of $\boldsymbol{\Delta}_l$ is $(R_l-1)R_l/2$.

If $U > M$, then there is a nonzero solution of the system of linear (48), say, $(\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_L)$. Let δ_{lk}, $k = 1, \ldots, R_l$, be eigenvalues of $\boldsymbol{\Delta}_l$, $l = 1, \ldots, L$. Let l_0 and k_0 be such that

$$|\delta_{l_0k_0}| = \max\{|\delta_{lk}| : 1 \le k \le R_l, 1 \le l \le L\}.$$

Thus it is easily seen that the matrices

$$\boldsymbol{I}_{R_l} - \frac{1}{\delta_{l_0k_0}}\boldsymbol{\Delta}_l \succeq 0, l = 1, \ldots, L.$$

Let $\boldsymbol{X}'_l = \boldsymbol{V}_l(\boldsymbol{I}_{R_l} - (1/\delta_{l_0k_0})\boldsymbol{\Delta}_l)\boldsymbol{V}_l^H$, $l = 1, \ldots, L$. Then we claim that the new $\boldsymbol{X}'_l$, $l = 1, \ldots, L$, have the following properties:

- rank reduced of at least one: $\sum_{l=1}^L R'_l \le (\sum_l^L R_l) - 1$, where $R'_l = \text{rank}(\boldsymbol{X}'_l)$, $\forall l$;
- primal feasibility: $\sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}'_l = \sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l = b_m$, $m = 1, \ldots, M$, and $\boldsymbol{X}'_l \succeq 0$, $\forall l$;
- complementarity (optimality): $\boldsymbol{X}'_l\boldsymbol{Z}_l = \boldsymbol{V}_l(\boldsymbol{I}_{R_l} - (1/\delta_{l_0k_0})\boldsymbol{\Delta}_l)\boldsymbol{V}_l^H\boldsymbol{Z}_l = \boldsymbol{0}$, $l = 1, \ldots, L$, due to the fact that $\boldsymbol{X}_l \bullet \boldsymbol{Z}_l = \boldsymbol{I}_{R_l} \bullet \boldsymbol{V}_l^H\boldsymbol{Z}_l\boldsymbol{V}_l = 0 \Longleftrightarrow \boldsymbol{V}_l^H\boldsymbol{Z}_l\boldsymbol{V}_l = \boldsymbol{0}$.

In other words, the feasible solution $(\boldsymbol{X}'_1, \ldots, \boldsymbol{X}'_L)$ is optimal for problem (P1).

Now check if $\sum_{l=1}^L (R'_l)^2 > M$. If it is the case, repeat the above rank-deduction procedure; else, then stop, and we have $\sum_{l=1}^L (R'_l)^2 \le M$. This proves the theorem. ■

B. Proof of Proposition 3.5

Proof: By assumption, any given optimal solution $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ of problem (P0) has no zero matrix component, i.e., $\text{rank}(\boldsymbol{X}_l) \ge 1$. It follows by Theorem 3.2 that problem (P0) has an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ such that

$$L \le \sum_{l=1}^L \text{rank}^2(\boldsymbol{X}_l^\star) \le M \le L + 2.$$

This implies $\text{rank}(\boldsymbol{X}_l^\star) = 1$, for $l = 1, \ldots, L$. ■

C. Proof of Lemma 4.2

Proof: Suppose $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ and $(y_1, \ldots, y_M, \mu_1, \ldots, \mu_L, \lambda_1, \ldots, \lambda_L, \boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_M)$ are optimal solutions of the problems (P3) and (D3), respectively. Let $\beta_l = \boldsymbol{B}_l \bullet \boldsymbol{X}_l$, $\gamma_l = \boldsymbol{D}_l \bullet \boldsymbol{X}_l$, $R_l = \text{rank}(\boldsymbol{X}_l)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^L R_l$. Assume that $U > M$; and without loss of generality, we suppose that $R_l \ge 1, \forall l$ (if some $R_{l_0} = 0$, then keep the component $\boldsymbol{X}_{l_0} = \boldsymbol{0}$ always). It follows by the rank-one decomposition in Lemma 4.1 that for each l, there is a rank-one decomposition of $\boldsymbol{X}_l$

$$\begin{aligned} &\boldsymbol{X}_l = \boldsymbol{V}_l\boldsymbol{V}_l^H, \\ &\boldsymbol{V}_l = [\boldsymbol{v}_{l1}, \ldots, \boldsymbol{v}_{lR_l}], \boldsymbol{v}_{lk} \in \mathbb{C}^K, k = 1, \ldots, R_l \end{aligned}$$

such that

$$\boldsymbol{v}_{lk}^H\boldsymbol{B}_l\boldsymbol{v}_{lk} = \frac{\beta_l}{R_l}, \boldsymbol{v}_{lk}^H\boldsymbol{D}_l\boldsymbol{v}_{lk} = \frac{\gamma_l}{R_l}, k = 1, \ldots, R_l.$$

Applying the same rank reduction procedure as that in the proof of Lemma 3.1, we let

$$\boldsymbol{\Delta}_l = \begin{bmatrix} \delta_{l1} & & \\ & \ddots & \\ & & \delta_{lR_l} \end{bmatrix}, l = 1, \ldots, L$$

be real-valued diagonal matrices (instead of Hermitian matrices), which satisfy the system of linear equation

$$\sum_{l=1}^L \boldsymbol{V}_l^H\boldsymbol{A}_{ml}\boldsymbol{V}_l \bullet \boldsymbol{\Delta}_l = 0, m = 1, \ldots, M$$

and let l_0 and k_0 be such that

$$|\delta_{l_0k_0}| = \max\{|\delta_{lk}| : 1 \le k \le R_l, 1 \le l \le L\}.$$

Now we verify that the solution

$$\boldsymbol{X}'_l = \boldsymbol{V}_l\left(\boldsymbol{I}_{R_l} - \frac{1}{\delta_{l_0k_0}}\boldsymbol{\Delta}_l\right)\boldsymbol{V}_l^H, l = 1, \ldots, L$$

has the following properties.

- Rank reduced of at least one: $\sum_{l=1}^L R'_l \le (\sum_l^L R_l) - 1$, where $R'_l = \text{rank}(\boldsymbol{X}'_l)$, $\forall l$.
- Complementarity (optimality) conditions: since $\boldsymbol{V}_l^H\boldsymbol{Z}_l\boldsymbol{V}_l = \boldsymbol{0}$, $l = 1, \ldots, L$, then $\boldsymbol{Z}_l \bullet \boldsymbol{X}'_l = 0$, $l = 1, \ldots, L$; further, we have

$$\mu_l\boldsymbol{B}_l \bullet \boldsymbol{X}'_l = 0, \forall l \in \bar{\mathcal{E}}_1, \lambda_l\boldsymbol{D}_l \bullet \boldsymbol{X}'_l = 0, \forall l \in \bar{\mathcal{E}}_2.$$

Indeed, since

$$\begin{aligned} \mu_l\beta_l &= \mu_l\boldsymbol{B}_l \bullet \boldsymbol{X}_l = 0, \forall l \in \bar{\mathcal{E}}_1 \\ \lambda_l\gamma_l &= \lambda_l\boldsymbol{D}_l \bullet \boldsymbol{X}_l = 0, \forall l \in \bar{\mathcal{E}}_2 \end{aligned}$$

then

$$\begin{aligned} &\mu_l\boldsymbol{B}_l \bullet \boldsymbol{X}'_l \\ &\quad = \mu_l\boldsymbol{V}_l^H\boldsymbol{B}_l\boldsymbol{V}_l \bullet \left(\boldsymbol{I}_{R_l} - \frac{1}{\delta_{l_0k_0}}\boldsymbol{\Delta}_l\right) \\ &= \frac{\mu_l\beta_l}{R_l}\left(R_l - \frac{\text{tr}\boldsymbol{\Delta}_l}{\delta_{l_0k_0}}\right) = 0, \forall l \in \bar{\mathcal{E}}_1 \\ &\lambda_l\boldsymbol{D}_l \bullet \boldsymbol{X}'_l \\ &\quad = \lambda_l\boldsymbol{V}_l^H\boldsymbol{D}_l\boldsymbol{V}_l \bullet \left(\boldsymbol{I}_l - \frac{1}{\delta_{l_0k_0}}\boldsymbol{\Delta}_l\right) \\ &= \frac{\lambda_l\gamma_l}{R_l}\left(R_l - \frac{\text{tr}\boldsymbol{\Delta}_l}{\delta_{l_0k_0}}\right) = 0, \forall l \in \bar{\mathcal{E}}_2. \end{aligned}$$

- Primal feasibility: $\sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}'_l = \sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l = b_m$, $m = 1, \ldots, M$, and $\boldsymbol{X}'_l \succeq 0$, $\forall l$; also

$$\begin{aligned} \boldsymbol{B}_l \bullet \boldsymbol{X}'_l &= \frac{\beta_l}{R_l}\left(R_l - \frac{\text{tr}\boldsymbol{\Delta}_l}{\delta_{l_0k_0}}\right) \\ &\begin{cases} = 0 & \text{if } l \in \mathcal{E}_1 \\ = 0 & \text{if } l \in \bar{\mathcal{E}}_1, \beta_l = 0 \\ \ge 0 & \text{if } l \in \bar{\mathcal{E}}_1, \beta_l > 0 \end{cases} \end{aligned} \tag{49}$$

$$\boldsymbol{D}_l \bullet \boldsymbol{X}'_l = \frac{\gamma_l}{R_l}\left(R_l - \frac{\mathrm{tr}\boldsymbol{\Delta}_l}{\delta_{l_0 k_0}}\right) \begin{cases} = 0, & \text{if } l \in \mathcal{E}_2 \\ = 0, & \text{if } l \in \bar{\mathcal{E}}_2, \gamma_l = 0 \\ \geq 0, & \text{if } l \in \bar{\mathcal{E}}_2, \gamma_l > 0. \end{cases} \tag{50}$$

As a matter of fact, if $l \in \bar{\mathcal{E}}_1, \beta_l > 0$, then $(\beta_l/R_l)(R_l - (\mathrm{tr}\boldsymbol{\Delta}_l)/(\delta_{l_0 k_0})) \geq 0$, due to $R_l \geq (\mathrm{tr}\boldsymbol{\Delta}_l)/(\delta_{l_0 k_0})$; and similarly, $\boldsymbol{D}_l \bullet \boldsymbol{X}'_l = (\gamma_l/R_l)(R_l - (\mathrm{tr}\boldsymbol{\Delta}_l)/(\delta_{l_0 k_0})) \geq 0$, if $l \in \bar{\mathcal{E}}_2, \gamma_l > 0$.

In other words, the feasible solution $(\boldsymbol{X}'_1, \ldots, \boldsymbol{X}'_L)$ is optimal for problem (P3) and is of rank reduced at least by one. Repeat the above procedure, starting from employing the rank-one decomposition lemma, if $\sum_{l=1}^{L} R'_l > M$. Then, at the end, we get an optimal solution $\boldsymbol{X}^{\star}_l, l = 1, \ldots, L$, of problem (P3), with $\sum_{l=1}^{L} \mathrm{rank}(\boldsymbol{X}^{\star}_l) \leq M$.

D. Proof of Lemma 4.3

Proof: If $L' = 0$, the lemma has been proven by Lemma 4.2. Now, we only show the case of $L' = 1$, and for $L' \geq 2$, the proof is completely similar. Suppose $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L)$ and $(y_1, \ldots, y_M, \mu_1, \ldots, \mu_L, \lambda_1, \ldots, \lambda_L, \boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_M)$ are optimal solutions of problems (P3) and (D3), respectively. Let $\beta_l = \boldsymbol{B}_l \bullet \boldsymbol{X}_l$, $\gamma_l = \boldsymbol{D}_l \bullet \boldsymbol{X}_l$, $R_l = \mathrm{rank}(\boldsymbol{X}_l)$, $l = 1, \ldots, L$, and $U = R_1^2 + \sum_{l=2}^{L} R_l$. Assume that $U > M$; and $R_l \geq 1, \forall l$.

Like the proof of Lemma 4.2, apply the rank-one decomposition in Lemma 4.1 for $l = 2, \ldots, L$, and find

$$\boldsymbol{X}_l = \boldsymbol{V}_l \boldsymbol{V}_l^H$$
$$\boldsymbol{V}_l = [\boldsymbol{v}_{l1}, \ldots, \boldsymbol{v}_{lR_l}], \boldsymbol{v}_{lk} \in \mathbb{C}^K, k = 1, \ldots, R_l$$

such that

$$\boldsymbol{v}_{lk}^H \boldsymbol{B}_l \boldsymbol{v}_{lk} = \frac{\beta_l}{R_l}, \boldsymbol{v}_{lk}^H \boldsymbol{D}_l \boldsymbol{v}_{lk} = \frac{\gamma_l}{R_l}, k = 1, \ldots, R_l.$$

Compute any decomposition $\boldsymbol{X}_1 = \boldsymbol{V}_1 \boldsymbol{V}_1^H$. We let $\boldsymbol{\Delta}_1$ be an $R_1 \times R_1$ Hermitian matrix with eigenvalues $\delta_{11}, \ldots, \delta_{1R_1}$ and $\boldsymbol{\Delta}_l$ be a diagonal matrix with real-valued elements $\delta_{l1}, \ldots, \delta_{lR_l}, l = 2, \ldots, L$. Assume that they satisfy the linear equation system

$$\sum_{l=1}^{L} \boldsymbol{V}_l^H \boldsymbol{A}_{ml} \boldsymbol{V}_l \bullet \boldsymbol{\Delta}_l = 0, m = 1, \ldots, M$$

and let l_0 and k_0 be such that

$$|\delta_{l_0 k_0}| = \max\{|\delta_{lk}| : 1 \leq k \leq R_l, 1 \leq l \leq L\}.$$

Set

$$\boldsymbol{X}'_l = \boldsymbol{V}_l \left(\boldsymbol{I}_{R_l} - \frac{1}{\delta_{l_0 k_0}} \boldsymbol{\Delta}_l\right) \boldsymbol{V}_l^H, l = 1, \ldots, L.$$

We verify that the new solution $\boldsymbol{X}'_l$ has the following properties.

- Rank reduced at least by one.
- Complementary conditions: clearly, $\boldsymbol{Z}_l \bullet \boldsymbol{X}'_l = 0$, $l = 1, \ldots, L$, and $\mu_l \boldsymbol{B}_l \bullet \boldsymbol{X}'_l = 0$, $l \in \bar{\mathcal{E}}_1$, $l \neq 1$, and $\lambda_l \boldsymbol{D}_l \bullet \boldsymbol{X}'_l = 0$, $l \in \bar{\mathcal{E}}_2$, $l \neq 1$ (as shown in the proof of Lemma 4.2). Now suppose $l = 1 \in \bar{\mathcal{E}}_1$ and then we have to check $\mu_1 \boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 = 0$ (note that there is no need to check it if $l = 1 \in \mathcal{E}_1$). If $\boldsymbol{B}_1 \preceq 0$, then $\boldsymbol{B}_1 \bullet \boldsymbol{X}_1 = 0$, and this implies $\boldsymbol{V}_1^H \boldsymbol{B}_1 \boldsymbol{V}_1 = \boldsymbol{0}$, which in turn means that $\boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 = 0$ and $\mu_1 \boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 = 0$. If $\boldsymbol{B}_1 \succeq 0$, we have either $\boldsymbol{B}_1 \bullet \boldsymbol{X}_1 = 0$ or $\boldsymbol{B}_1 \bullet \boldsymbol{X}_1 > 0$; in the former case, it follows that $\boldsymbol{V}_1^H \boldsymbol{B}_1 \boldsymbol{V}_1 = \boldsymbol{0}$, and then $\boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 = 0$; in the latter case, $\mu_1 = 0$ due to $\mu_1 \boldsymbol{B}_1 \bullet \boldsymbol{X}_1 = 0$, and thus $\mu_1 \boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 = 0$. Similarly, we have $\lambda_1 \boldsymbol{D}_1 \bullet \boldsymbol{X}_1 = 0$ if $l = 1 \in \bar{\mathcal{E}}_2$.
- Primal feasibility: evidently, $\sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}'_l = b_m$, $m = 1, \ldots, M$, and $\boldsymbol{X}'_l \succeq 0, \forall l$. From the proof of Lemma 4.2 [see (49) and (50)], we can see that $\boldsymbol{X}'_l, l = 2, \ldots, L$, fulfill the individual shaping constraints. Now we need to verify $\boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 \trianglerighteq_{11} 0$, where $\trianglerighteq_{11} \in \{=, \geq\}$. If $\boldsymbol{B}_1 \bullet \boldsymbol{X}_1 = 0$, then $\boldsymbol{V}_1^H \boldsymbol{B}_1 \boldsymbol{V}_1 = \boldsymbol{0}$, and then $\boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 = 0$; if $\boldsymbol{B}_1 \bullet \boldsymbol{X}_1 \geq 0$, then $\boldsymbol{B}_1 \bullet \boldsymbol{X}'_1 \geq 0$ in case of $\boldsymbol{B}_1 \succeq 0$, and $\boldsymbol{B}_1 \bullet \boldsymbol{X}' = 0$ in case of $\boldsymbol{B}_1 \preceq 0$. Also $\boldsymbol{D}_1 \bullet \boldsymbol{X}'_1 \trianglerighteq_{21} 0$ is confirmed by an equal argument.

Consequently, the solution $(\boldsymbol{X}'_1, \ldots, \boldsymbol{X}'_L)$ is optimal for (P3) and is of rank reduced by one at least. Repeat the above rank-reduction procedure; at the end, we get an optimal solution $(\boldsymbol{X}^{\star}_1, \ldots, \boldsymbol{X}^{\star}_L)$ with $\mathrm{rank}^2(\boldsymbol{X}^{\star}_1) + \sum_{l=2}^{L} \mathrm{rank}(\boldsymbol{X}^{\star}_l) \leq M$.

E. Proof of Proposition 4.7

Proof: By assumption, any optimal solution has $\mathrm{rank}(\boldsymbol{X}_l) \geq 1$, $\forall l = 1, \ldots, L$. If $M = L$, then applying Theorem 4.4 with $L' = 0$ gives that there is an optimal solution $(\boldsymbol{X}^{\star}_1, \ldots, \boldsymbol{X}^{\star}_L)$ such that

$$L \leq \sum_{l=1}^{L} \mathrm{rank}(\boldsymbol{X}^{\star}_l) \leq M = L.$$

This means that $\mathrm{rank}(\boldsymbol{X}^{\star}_l) = 1, \forall l$.

If $M \leq L+2$ and $(\boldsymbol{B}_l, \boldsymbol{D}_l)$, for all $l = 1, \ldots, L$, are semidefinite, then applying Theorem 4.4 with $L' = L$ leads to the fact that there is an optimal solution $(\boldsymbol{X}^{\star}_1, \ldots, \boldsymbol{X}^{\star}_L)$ such that

$$L \leq \sum_{l=1}^{L} \mathrm{rank}^2(\boldsymbol{X}^{\star}_l) \leq M \leq L+2.$$

This also means that $\mathrm{rank}(\boldsymbol{X}^{\star}_l) = 1, \forall l$. ■

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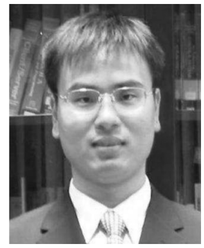

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