Monotone Games for Cognitive Radio Systems

Gesualdo Scutari, Daniel P. Palomar, Francisco Facchinei, and Jong-Shi Pang[1]

Noncooperative game theory is a branch of game theory for the resolution of conflicts among interacting decision-makers (called players), each behaving selfishly to optimize one's own well-being. In recent years, there has been a growing interest in using noncooperative games to model many communications and networking problems, where the interaction among several agents is by no means negligible and centralized approaches are not suitable. Examples are power control and resource sharing in wireless/wired and peer-to-peer networks (e.g., [39, 20, 34, 6, 32, 26, 33, 36]), cognitive radio systems (e.g., [31, 35, 27]), and distributed routing, flow and congestion control in communication networks (e.g., [1] and references therein). Two recent special issues on the subject are [17, 18]. A more general framework suitable for investigating and solving various optimization problems and equilibrium models, even when classical game theory may fail, is known to be the variational inequality (VI) problem that constitutes a very general class of problems in nonlinear analysis [11].

Building on the VI framework, in this chapter, we present a brief treatment of two classes of Nash problems and their application to the design of cognitive radio (CR) systems. The first is the class of Nash Equilibrium Problems (NEP) where the interactions among players take place at the level of objective functions only. The second is the class of Generalized Nash Equilibrium Problems (GNEP) where in addition we have that the choices available to each player also depend by the actions taken by his rivals. We focus on the existence and global uniqueness of equilibria, and on distributed algorithms based on the best-response mapping. The results discussed in this chapter are based on [12, 35, 30], to which we refer the interested reader for a more comprehensive treatment of the subject.

The chapter is organized as follows. The first part−Section 0.1 for NEPs and Section 0.2 for GNEPs−is devoted to the development of general results. The second part of the chapter−Section 0.3−applies those results to the design of CR systems.

[1] Gesualdo Scutari
State University of New York (SUNY) at Buffalo, Buffalo, NY, USA. Email: gesualdo@buffalo.edu.

Daniel P. Palomar
Hong Kong University of Science and Technology, Hong Kong. Email: palomar@ust.hk.

Francisco Facchinei
University of Rome La Sapienza, Rome, Italy. Email: facchinei@dis.uniroma1.it.

Jong-Shi Pang
University of Illinois at Urbana-Champaign, Urbana, IL, USA. Email: jspang@uiuc.edu.

0.1 Nash Equilibrium Problems (NEP)

In a general noncooperative game, there are Q players each of whom has a cost function and a strategy set that may depend on the other players' actions. In a NEP, player i's strategy set $\mathscr{Q}_i \subseteq \mathbb{R}^{n_i}$ is independent of the other players' strategies; player i's cost function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ depends on all players' strategies which are described by a vector $\mathbf{x} \triangleq (\mathbf{x}_1, \ldots, \mathbf{x}_Q)$, where $\mathbf{x}_i$ is the action of the player i and $\mathbf{x}_{-i} \triangleq (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_Q)$ denote the vector of all players' strategies variables except that of player i. The joint strategy set of the NEP is given by $\mathscr{Q} = \prod_{i=1}^{Q} \mathscr{Q}_i$, whereas $\mathscr{Q}_{-i} \triangleq \prod_{j \neq i} \mathscr{Q}_j$. The NEP is formally defined by the tuple $\mathscr{G} = \langle \mathscr{Q}, \mathbf{f} \rangle$, with $\mathbf{f} \triangleq (f_i)_{i=1}^{Q}$: The aim of player i, given the other players' strategies $\mathbf{x}_{-i}$, is to choose an $\mathbf{x}_i \in \mathscr{Q}_i$ that minimizes his cost function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$, i.e.,

$$\begin{array}{l} \underset{\mathbf{x}_i}{\text{minimize}} \;\; f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to } \mathbf{x}_i \in \mathscr{Q}_i. \end{array} \tag{0.1}$$

Roughly speaking, a NEP is a set of *coupled* optimization problems.

Definition 1. A pure-strategy Nash Equilibrium (NE), or simply a solution of the NEP, is a feasible point $\mathbf{x}^\star$ such that

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \qquad \forall \mathbf{x}_i \in \mathscr{Q}_i \tag{0.2}$$

holds for each player $i = 1, \ldots, Q$. □

In words, a NE is a feasible strategy profile $\mathbf{x}^\star$ with the property that no *single* player can benefit from a *unilateral* deviation from $\mathbf{x}_i^\star$. A useful way to see a NE is as a fixed-point of the best-response mapping for each player. Let $\mathscr{B}_i(\mathbf{x}_{-i})$ be the (possibly empty) set of optimal solutions of the i-th optimization problem (0.1) and set $\mathscr{B}(\mathbf{x}) \triangleq \mathscr{B}_1(\mathbf{x}_{-1}) \times \mathscr{B}_2(\mathbf{x}_{-2}) \times \cdots \times \mathscr{B}_Q(\mathbf{x}_{-Q})$. It is clear that a point $\mathbf{x}^\star$ is a NE if and only if it is a fixed-point of $\mathscr{B}(\mathbf{x})$, i.e., if and only if $\mathbf{x}^\star \in \mathscr{B}(\mathbf{x}^\star)$. This observation is the key to the standard approach to the study of NEPs: the so called fixed-point approach, which is based on the use of the well-developed machinery of fixed-point theory. This approach is adopted in the analysis of several games proposed in the signal processing and communication literature (see, e.g., [39, 8, 34, 6, 26, 33, 32, 36] and [31, 35]). However, the applicability of the fixed-point based analysis as used in the aforementioned papers requires the ability to compute the best-response mapping $\mathscr{B}(\mathbf{x})$ in closed form; a fact that strongly limits the applicability of this methodology.

In this chapter, we overcome this limitation by studying NEPs through their reduction to a VI problem. This approach is pursued also in [25, 24, 12, 30, 38] and, resting on the well developed theory of VIs, has the advantage of permitting an easy derivation of many results about existence, uniqueness, and stability of the solutions. But its main benefit is probably that of leading quite naturally to the derivation of implementable solution algorithms along with their convergence properties.

0.1.1 Connection to Variational Inequalities

At the basis of the VI approach to NEPs there is an easy equivalence between a NEP and a suitably defined partitioned VI. The (partitioned) VI problem is defined next. Given $\mathscr{Q}$ defined as in Section 0.1, let $\mathbf{F} : \Omega \supset \mathscr{Q} \to \mathbb{R}^n$ be a continuous function on $\Omega \triangleq \prod_{i=1}^{Q} \Omega_i$ with each Ω_i being an open subset of $\mathbb{R}^{n_i}$ containing $\mathscr{Q}_i$, and $n \triangleq \sum_{i=1}^{Q} n_i$. We write $\mathbf{F}(\mathbf{x}) = (\mathbf{F}_i(\mathbf{x}))_{i=1}^{Q}$ where $\mathbf{F}_i : \Omega \to \mathbb{R}^{n_i}$ is the i-th component block function of $\mathbf{F}$.

Definition 2. The variational inequality, denoted by $\mathrm{VI}(\mathscr{Q}, \mathbf{F})$, is to find a vector $\mathbf{x}^\star \in \mathscr{Q}$ such that [11, Def. 1.1.1]

$$(\mathbf{x} - \mathbf{x}^\star)^T \mathbf{F}(\mathbf{x}^\star) \geq 0, \qquad \forall \mathbf{x} \in \mathscr{Q}. \tag{0.3}$$

The set of solutions to this problem is denoted $\mathrm{SOL}(\mathscr{Q}, \mathbf{F})$.

Several standard problems in nonlinear programming, game theory, and nonlinear analysis can be naturally formulated as a VI problem; many examples can be found in [11, Ch. 1], [19, 37]. In particular, the equivalence between NEPs and VIs is given in the following proposition, whose proof follows readily from the minimum principle for convex problems and the Cartesian structure of the joint strategy set $\mathscr{Q}$ [11, Prop. 1.4.2].

Proposition 1. *Given the NEP $\mathscr{G} = \langle \mathscr{Q}, \mathbf{f} \rangle$, suppose that for each player i the following hold:*

i) The (nonempty) strategy set $\mathscr{Q}_i$ is closed and convex;
ii) For every fixed $\mathbf{x}_{-i} \in \mathscr{Q}_{-i}$, the payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is convex and continuously differentiable in $\mathbf{x}_i \in \Omega_i \supset \mathscr{Q}_i$.

Then, the game $\mathscr{G}$ is equivalent to the $\mathrm{VI}(\mathscr{Q}, \mathbf{F})$*, where* $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$.

Building on the VI reformulation above and the well-developed framework of VIs, we focus on the main properties of the NEP, namely: the existence and uniqueness of the solution and the design of distributed algorithms along with their convergence properties. To this end, throughout the chapter, we make the following convexity/smoothness assumption.

Assumption 1. For each $i = 1, \ldots, Q$, the set $\mathscr{Q}_i \subset \Omega_i$ is a nonempty, closed and convex subset of $\mathbb{R}^{n_i}$ and the function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is convex in $\mathbf{x}_i \in \mathscr{Q}_i$ for every fixed $\mathbf{x}_{-i} \in \mathscr{Q}_{-i}$ and twice continuously differentiable in $\mathbf{x} \in \Omega \supset \mathscr{Q} = \prod_i \mathscr{Q}_i$ with bounded second derivatives on $\mathscr{Q}$.

Remark 1 (On Assumption 1). For the purpose of this chapter, it is enough to focus only on games that satisfy Assumption 1 (all the games we study in the second part of the chapter satisfy this condition indeed). However, Assumption 1 can be relaxed, following the techniques developed in [12].

0.1.2 Solution Analysis of the NEP

The solution analysis of a NEP can be carried out in several ways. Here, we address this issue by using the equivalence between the NEP $\mathscr{G}$ and the VI$(\mathscr{Q},\mathbf{F})$ illustrated in Proposition 1. Some of the results in this section (especially those related to existence) could be obtained under weaker assumptions; however our derivation based on the VI reformulation is interesting in its own right and furthermore prepares the ground for the algorithmic developments of next sections. We begin our analysis by introducing some basic definitions.

Definition 3. A mapping $\mathbf{F} = (\mathbf{F}_i(\mathbf{x}))_{i=1}^{Q} : \Omega \supset \mathscr{Q} \ni \mathbf{x} \to \mathbb{R}^n$ is

(i) *monotone* on $\mathscr{Q}$ if for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathscr{Q}$,

$$(\mathbf{x}-\mathbf{y})^T(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) \geq 0; \tag{0.4}$$

(ii) *strictly monotone* on $\mathscr{Q}$ if for all $\mathbf{x} \neq \mathbf{y}$ in $\mathscr{Q}$ the inequality in (0.4) is strict;

(iii) *strongly monotone* on $\mathscr{Q}$ if a constant $c_{\rm sm} > 0$ exists such that for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathscr{Q}$,

$$(\mathbf{x}-\mathbf{y})^T(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) \geq c_{\rm sm}\|\mathbf{x}-\mathbf{y}\|^2. \tag{0.5}$$

The constant $c_{\rm sm}$ is called the strong monotonicity constant of $\mathbf{F}$;

(iv) a *uniform P-function* on $\mathscr{Q} = \prod_i \mathscr{Q}_i$ if a constant $c_{\rm uP} > 0$ exists such that for all $\mathbf{x} = (\mathbf{x}_i)_{i=1}^{Q}$ and $\mathbf{y} = (\mathbf{y}_i)_{i=1}^{Q}$ in $\mathscr{Q}$,

$$\max_{1\leq i\leq Q}(\mathbf{x}_i-\mathbf{y}_i)^T(\mathbf{F}_i(\mathbf{x})-\mathbf{F}_i(\mathbf{y})) \geq c_{\rm uP}\|\mathbf{x}-\mathbf{y}\|^2. \tag{0.6}$$

Among the above monotonicity properties, the following relations hold:

$$\boxed{\text{strongly monotone} \Rightarrow \text{uniform P} \Rightarrow \text{strictly monotone} \Rightarrow \text{monotone.}} \tag{0.7}$$

Monotonicity properties play in the VI realm the same role that convex functions play in optimization. In fact, we recall that a differentiable function f is convex, strictly convex, strongly convex on a convex set $\mathscr{Q}$ if and only if its gradient is monotone, strictly monotone, strongly monotone on $\mathscr{Q}$. The next theorem collects some rather standard results on solution properties of a VI [11]; thanks to Proposition 1, these results readily extend to NEPs.

Theorem 1. *Given the NEP $\mathscr{G} = <\mathscr{Q}, \mathbf{f}>$, suppose that $\mathscr{G}$ satisfies Assumption 1 and let $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$. Then the following statements hold.*

(a) The VI$(\mathscr{Q},\mathbf{F})$ *(the NEP $\mathscr{G}$) has a (possibly empty) closed solution set. If all strategy sets $\mathscr{Q}_i$ are bounded, the solution set is nonempty and thus compact;*

(b) If $\mathbf{F}(\mathbf{x})$ is monotone on $\mathscr{Q}$, then the VI$(\mathscr{Q},\mathbf{F})$ *(the NEP $\mathscr{G}$) has a convex solution set (possibly empty);*

(c) If $\mathbf{F}(\mathbf{x})$ is strictly monotone on $\mathscr{Q}$, then the VI$(\mathscr{Q},\mathbf{F})$ *(the NEP $\mathscr{G}$) has at most one solution;*

(d) If $\mathbf{F}(\mathbf{x})$ *is a uniformly-P function (or strongly monotone) on* $\mathcal{Q}$*, then the* $\mathrm{VI}(\mathcal{Q},\mathbf{F})$ *(the NEP* $\mathcal{G}$*) has a unique solution.*

Note that the uniqueness results stated in part (c)-(d) do not require that the set $\mathcal{Q}$ be bounded. Some sufficient conditions for $\mathbf{F}(\mathbf{x})$ being a (strictly, strongly) monotone or a uniformly-P function are given in the next section (see [30] for more details).

0.1.3 Monotonicity conditions for the vector function $\mathbf{F}$

Assuming that $\mathbf{F}$ is continuously differentiable on Ω, let $\mathrm{J}\mathbf{F}(\mathbf{x}) = \left(\mathrm{J}_{\mathbf{x}_j}\mathbf{F}_i(\mathbf{x})\right)_{i,j=1}^{Q}$ be the Jacobian of $\mathbf{F}$, where $\mathrm{J}_{\mathbf{x}_j}\mathbf{F}_i(\mathbf{x})$ is the partial Jacobian matrix of $\mathbf{F}_i$ with respect to the $\mathbf{x}_j$ vector. Note that when $\mathbf{F}(\mathbf{x}) = (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$, with each $f_i : \Omega_i \mapsto \mathbb{R}$ being a continuously differentiable function on Ω_i, we have $\mathrm{J}_{\mathbf{x}_j}\mathbf{F}_i(\mathbf{x}) = \nabla^2_{\mathbf{x}_i\mathbf{x}_j} f_i(\mathbf{x})$, for $i,j = 1,\ldots,Q$, where $\mathrm{J}_{\mathbf{x}_i}\mathbf{F}_i(\mathbf{x}) = \nabla^2_{\mathbf{x}_i\mathbf{x}_i} f_i(\mathbf{x})$ is the Hessian matrix of f_i.

It is well known and easy to show [23] that the following relations exist among the monotonicity properties of $\mathbf{F}$ and the definiteness properties of the Jacobian matrix $\mathrm{J}\mathbf{F}$.

$$\boxed{\begin{array}{lllll} \text{i)} & \mathbf{F}(\mathbf{x}) \text{ is monotone on } \mathcal{Q} & \Leftrightarrow & \mathrm{J}\mathbf{F}(\mathbf{x}) \succeq \mathbf{0},\ \forall \mathbf{x} \in \mathcal{Q}; \\ \text{ii)} & \mathbf{F}(\mathbf{x}) \text{ is strictly monotone on } \mathcal{Q} & \Leftarrow & \mathrm{J}\mathbf{F}(\mathbf{x}) \succ \mathbf{0},\ \forall \mathbf{x} \in \mathcal{Q}; \\ \text{iii)} & \mathbf{F}(\mathbf{x}) \text{ is strongly monotone on } \mathcal{Q} & \Leftrightarrow & \mathrm{J}\mathbf{F} - c_{\mathrm{sm}}\mathbf{I} \succeq \mathbf{0},\ \forall \mathbf{x} \in \mathcal{Q}; \end{array}} \tag{0.8}$$

where $\mathbf{A} \succeq \mathbf{B}$ ($\mathbf{A} \succ \mathbf{B}$) means that $\mathbf{A} - \mathbf{B}$ is a positive semidefinite (definite) matrix. In some applications we consider later on (cf. Section 0.3), it is useful to have at hand some further sufficient conditions that guarantee monotonicity properties of $\mathbf{F}$. Below we give two such conditions. Let define the matrix $\mathrm{J}\mathbf{F}_{\mathrm{low}}$ having the same dimension of $\mathrm{J}\mathbf{F}(\mathbf{x})$ as

$$[\mathrm{J}\mathbf{F}_{\mathrm{low}}]_{rs} \triangleq \begin{cases} \inf\limits_{\mathbf{x}\in\mathcal{Q}} [\mathrm{J}\mathbf{F}(\mathbf{x})]_{rr}, & \text{if } r = s, \\ -\sup\limits_{\mathbf{x}\in\mathcal{Q}} |[\mathrm{J}\mathbf{F}(\mathbf{x})]_{rs}|, & \text{otherwise.} \end{cases} \tag{0.9}$$

and let introduce the "condensed" $Q \times Q$ real matrices $\Upsilon_{\mathbf{F}}$ and Γ_F, given by

$$[\Upsilon_{\mathbf{F}}]_{ij} \triangleq \begin{cases} \alpha_i^{\min}, & \text{if } i = j, \\ -\beta_{ij}^{\max}, & \text{otherwise,} \end{cases} \tag{0.10}$$

and

$$[\Gamma_{\mathbf{F}}]_{ij} \triangleq \begin{cases} \dfrac{1}{1+\alpha_i^{\min}}, & \text{if } i = j, \\ \dfrac{\beta_{ij}^{\max}}{1+\alpha_i^{\min}}, & \text{otherwise,} \end{cases} \tag{0.11}$$

where

$$\alpha_i^{\min} \triangleq \inf_{\mathbf{z}\in\mathscr{Q}} \lambda_{\text{least}}\left(\mathrm{J}_{\mathbf{x}_i}\mathbf{F}_i(\mathbf{z})\right) \quad \text{and} \quad \beta_{ij}^{\max} \triangleq \sup_{\mathbf{z}\in\mathscr{Q}} \left\|\mathrm{J}_{\mathbf{x}_i}\mathbf{F}_j(\mathbf{z})\right\|, \tag{0.12}$$

with $\lambda_{\text{least}}(\mathbf{A})$ denoting the least eigenvalue of $\mathbf{A}$.[2] In order to explore the relationship between the two matrices $\Upsilon_{\mathbf{F}}$ and Γ_F, we need the following definition.

Definition 4. A matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ is called P-matrix if every principal minor of $\mathbf{M}$ is positive.

Many equivalent characterizations for a P-matrix can be given. The interested reader is referred to [7, 4] for more details. Here we note that any positive definite matrix is a P-matrix, but the reverse does not hold (unless the matrix is symmetric).

Building on the properties of the P-matrices *[7, Lemma 13.14],* one can show that $\Upsilon_{\mathbf{F}}$ is a P-matrix if and only if $\rho(\Gamma_{\mathbf{F}}) < 1$, where $\rho(\mathbf{A})$ denotes the spectral radius of $\mathbf{A}$. The P-property of matrix $\Upsilon_{\mathbf{F}}$ will be used to prove the uniqueness of the equilibrium of Nash games as well as the convergence of some of the proposed distributed algorithms. The positive definiteness property of $\Upsilon_{\mathbf{F}}$ and the strong monotonicity constant c_{sm} of $\mathbf{F}$ will be exploited to study the GNEPs in Section 0.2. Matrices $\mathrm{J}\mathbf{F}_{\text{low}}$ and $\Upsilon_{\mathbf{F}}$ are also instrumental to obtain sufficient conditions for the monotonicity of the mapping $\mathbf{F}$ (more results and milder conditions can be found in [30]), as given next.

Proposition 2. *Let* $\mathbf{F} : \Omega \supset \mathscr{Q} \to \mathbb{R}^n$ *be continuously differentiable with bounded derivatives on* $\mathscr{Q}$*. The following statements hold:*

(a) If either $J\mathbf{F}_{low}$ *or* $\Upsilon_{\mathbf{F}}$ *are positive semidefinite, then* $\mathbf{F}$ *is* monotone *on* $\mathscr{Q}$*;*

(b) If either $J\mathbf{F}_{low}$ *or* $\Upsilon_{\mathbf{F}}$ *are positive definite, then* $\mathbf{F}$ *is* strongly monotone *on* $\mathscr{Q}$*, with strong monotonicity constant given by either* $c_{\text{sm}} = \lambda_{least}(J\mathbf{F}_{low})$ or $c_{\text{sm}} = \lambda_{least}(\Upsilon_F)$;

(c) Let $\mathbf{F} = (\mathbf{F}_i(\mathbf{x}))_{i=1}^Q : \Omega = \prod_i \Omega_i \supset \mathscr{Q} = \prod_i \mathscr{Q}_i \ni \mathbf{x} \to \mathbb{R}^n$*. If* $\Upsilon_{\mathbf{F}}$ *is a P-matrix, then* $\mathbf{F}$ *is a* uniformly P-function *on* $\mathscr{Q}$*, for some positive* c_{uP}.

A lower bound of c_{uP} is given in [30].

Remark 2 (On the uniqueness of the NE - continued). Invoking Theorem 1 and Proposition 2, one can readily obtain sufficient conditions for the uniqueness of the NE of the game $\mathscr{G}$. According to Theorem 1(d) and Proposition 2(c) indeed, the NE of $\mathscr{G}$ is unique if the matrix $\Upsilon_{\mathbf{F}}$ in (0.10) with $\mathbf{F}(\mathbf{x}) = (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^Q$ is a *P*-matrix. It turns out that this condition is sufficient also for global convergence of best-response asynchronous distributed algorithms described in Section 0.1.4.1. To give additional insight into the uniqueness of the NE of $\mathscr{G}$ we provide the following diagonal dominance type conditions for the matrix $\Upsilon_{\mathbf{F}}$ to be a P-matrix (positive definite).

[2] The least eigenvalue of a real (not necessarily symmetric) matrix $\mathbf{A}$ is the smallest eigenvalue of the symmetric part of $\mathbf{A}$.

Proposition 3. *The matrix $\Upsilon_{\mathbf{F}}$ in* (0.10) *is a P-matrix* (*positive definite*) *if one of* (*both*) *the following two sets of conditions are satisfied: for some* $\mathbf{w} = (w_i)_{i=1}^{Q} > \mathbf{0}$*,*

$$\frac{1}{w_i}\sum_{j\neq i} w_j \frac{\beta_{ij}^{\max}}{\alpha_i^{\min}} < 1,\ \forall i = 1,\cdots,Q,\ \ \frac{1}{w_j}\sum_{i\neq j} w_i \frac{\beta_{ij}^{\max}}{\alpha_i^{\min}} < 1,\ \forall j = 1,\cdots,Q. \quad (0.13)$$

Note that if $\Upsilon_{\mathbf{F}}$ is a P matrix, it must be $\alpha_i^{\min} = \inf_{\mathbf{z}\in\mathscr{Q}}\left[\lambda_{\min}(\nabla^2_{\mathbf{x}_i} f_i(\mathbf{z}))\right] > 0$ for all *i,* where $\lambda_{\min}(\nabla^2_{\mathbf{x}_i} f_i(\mathbf{z}))$ denotes the minimum eigenvalue of $\nabla^2_{\mathbf{x}_i} f_i(\mathbf{z})$. Thus an implicit consequence of the P assumption of the matrix $\Upsilon_{\mathbf{F}}$ is the uniform positive definiteness of the matrices $\nabla^2_{\mathbf{x}_i} f_i$ on $\mathscr{Q}$, which implies the uniformly strong convexity of $f_i(\cdot, \mathbf{x}_{-i})$ for all $\mathbf{x}_{-i} \in \mathscr{Q}_{-i}$.

0.1.4 Distributed algorithms for Nash equilibria

In this section, we discuss some iterative algorithms for computing a NE of NEP (a solution of the VI). For the purposes of this chapter we restrict our attention to distributed algorithms, with special emphasis to best-response iterative algorithms.

0.1.4.1 Best response decomposition algorithms

We focus on asynchronous-iterative algorithms, since they are particularly suitable for CR applications. More specifically, we consider *totally asynchronous* schemes (in the sense specified in [5]), where some players may update their strategies more frequently than others and they may even use an outdated information about the strategy profile used by the others. To provide a formal description of the algorithm, we need to introduce some preliminary definitions. Let $\mathscr{T}_i \subseteq \mathscr{T} \subseteq \{0,1,2,\ldots\}$ be the set of times at which player i updates his own strategy $\mathbf{x}_i$, denoted by $\mathbf{x}_i^{(n)}$ (thus, implying that, at time $n \notin \mathscr{T}_i$, $\mathbf{x}_i^{(n)}$ is left unchanged). Let $t_j^i(n)$ denote the most recent time at which the strategy profile of player j is perceived by player i at the n-*th* iteration (observe that $t_j^i(n)$ satisfies $0 \leq t_j^i(n) \leq n$). Hence, if player i updates its strategy at the n-th iteration, then he minimizes his cost function using the following (possibly) outdated strategy profile of the other players:

$$\mathbf{x}_{-i}^{(\mathbf{t}^i(n))} \triangleq \left(\mathbf{x}_1^{(t_1^i(n))},\ldots,\mathbf{x}_{i-1}^{(t_{i-1}^i(n))},\mathbf{x}_{i+1}^{(t_{i+1}^i(n))},\ldots,\mathbf{x}_Q^{(t_Q^i(n))}\right). \quad (0.14)$$

Some standard conditions in asynchronous convergence theory, which are fulfilled in any practical implementation, need to be satisfied by the schedule $\mathscr{T}_i$'s and $t_j^i(n)$'s; we refer to [5, 32] for the details. Through the whole chapter we assume that these conditions are satisfied and call feasible such an updating schedule. Using the above definitions, the totally asynchronous algorithm based on the best-responses of the players is described in Algorithm 1. The convergence properties of the algorithm are given in Theorem 2.

Theorem 2 ([30]). *Let $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ satisfy Assumption 1 and let $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$. If $\Upsilon_{\mathbf{F}}$ defined in (0.10) is a P-matrix, any sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ generated by the asynchronous best-response algorithm described in Algorithm 1 converges to the unique NE of $\mathscr{G}$, for any given updating feasible schedule of the players.*

Algorithm 1: Asynchronous Best-Response Algorithm

Data : Choose any feasible starting point $\mathbf{x}^{(0)} = (\mathbf{x}_i^{(0)})_{i=1}^{Q}$; set $n = 0$.

`(S.1)` : If $\mathbf{x}^{(n)}$ satisfies a suitable termination criterion: `STOP`

`(S.2)` : For$i = 1, \ldots, Q$, compute $\mathbf{x}_i^{(n+1)}$ as

$$\mathbf{x}_i^{(n+1)} = \begin{cases} \mathbf{x}_i^{\star} \in \underset{\mathbf{x}_i \in \mathscr{Q}_i}{\operatorname{argmin}} f_i\left(\mathbf{x}_i, \mathbf{x}_{-i}^{(\mathbf{t}^i(n))}\right), & \text{if } n \in \mathscr{T}_i \\ \mathbf{x}_i^{(n)}, & \text{otherwise} \end{cases} \tag{0.15}$$

`end`

`(S.3)` : $n \leftarrow n+1$; go to `(S.1)`.

Remark 3 (Flexibility of the algorithm). Algorithm 1 contains as special cases a plethora of algorithms, each one obtained by a possible choice of the scheduling of the users in the updating procedure (i.e., the parameters $\{t_r^q(n)\}$ and $\{\mathscr{T}_q\}$). Examples are the *sequential* (Gauss-Seidel scheme) and the *simultaneous* (Jacobi scheme) updates, where the players update their own strategies *sequentially* and *simultaneously*, respectively. Interestingly, Theorem 2 states that all these algorithms are robust against missing or outdated updates of the players and are guaranteed to converge to the unique NE of the game under the same set of convergence conditions, since the matrix $\Upsilon_{\mathbf{F}}$ (or $\Gamma_{\mathbf{F}}$) does not depend on the particular choice of $\{t_r^q(n)\}$ and $\{\mathscr{T}_q\}$. This feature strongly relaxes the constraints on the synchronization of the players' updates; which makes this class of algorithms appealing in many practical distributed systems.

Remark 4 (On the convergence conditions of best-response algorithms). We have pointed out that the P property of $\Upsilon_{\mathbf{F}}$ (or equivalently $\rho(\Gamma_{\mathbf{F}}) < 1$) cannot be satisfied even if there is just one point where one player has a payoff function with singular Hessian. In fact, if this is the case, we have, $\alpha_i^{\min} = 0$ for some i, let us say $i = 1$ without loss of generality, which implies that the matrix $\Gamma_{\mathbf{F}}$ has a 1 in the left-upper corner. Since the matrix $\Gamma_{\mathbf{F}}$ is nonnegative, we have that this implies $\rho(\Gamma_{\mathbf{F}}) \geq 1$ [3, Th. 1.7.4]. Assuming that the element 1 is contained in an irreducible principal matrix, we will actually have $\rho(\Gamma_{\mathbf{F}}) > 1$. Note that the irreducibility assumption is extremely weak and trivially satisfied if the matrix $\Gamma_{\mathbf{F}}$ is positive, which is true in most of our applications. We thus focus in the next subsection on alternative distributed algorithms that are guaranteed to converge under milder assumptions that do not require the strict or strong convexity of the payoff functions. These milder conditions required on $\mathbf{F}$ to have convergence are traded for a slightly increasing computational/signaling complexity.

0.1.4.2 Proximal decomposition algorithms for monotone VIs

We focus now on distributed algorithms whose convergence is guaranteed under the monotonicity assumption on the mapping $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$. Let us introduce first the following assumption.

Assumption 2. The mapping $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$ is monotone on $\mathscr{Q} = \mathscr{Q}_1 \times \ldots \times \mathscr{Q}_Q$.

According to Proposition 1, solving the NEP $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ is equivalent to solving the VI$(\mathscr{Q}, \mathbf{F})$ that, under Assumption 2, is monotone. This can be done in a host of ways (see, e.g., [11, Vol. II]) but not directly by decomposition methods. Here our interest is on devising distributed solution methods. To pursue this goal, we propose the following approach. We consider a regularization of the VI$(\mathscr{Q}, \mathbf{F})$, given by VI$(\mathscr{Q}, \mathbf{F} + \tau(\mathbf{I} - \mathbf{y}))$, where $\mathbf{I}$ is the identity map (i.e., $\mathbf{I} : \mathbf{x} \to \mathbf{x}$), $\mathbf{y}$ is a fixed vector in $\mathbb{R}^n$, and τ is a positive constant. Under Assumption 2, this regularized problem is strongly monotone and thus has a unique solution (cf. Theorem 1); we denote by $\mathbf{S}_\tau(\mathbf{y}) \triangleq \text{SOL}(\mathscr{Q}, \mathbf{F} + \tau(\mathbf{I} - \mathbf{y}))$ such a unique solution. The relationship between the original game $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ and the regularized VI is given in the following.

Lemma 1. *Given the game $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$, suppose that Assumptions 1 and 2 hold. A tuple $\mathbf{x}^\star \in \mathscr{Q}$ is a NE of the game if and only if it is a fixed point of the vector function $\mathbf{S}_\tau(\mathbf{y})$, i.e., $\mathbf{x}^\star = \mathbf{S}_\tau(\mathbf{x}^\star)$.*

Under the monotonicity of $\mathbf{F}$, the mapping $\mathbf{S}_\tau(\mathbf{y})$ can be shown to be nonexpansive, meaning that, starting at a given iterate $\mathbf{y}^{(0)} \in \mathscr{Q}$, the sequence generated by a proper averaging of $\mathbf{S}_\tau(\mathbf{y}^{(n)})$ and $\mathbf{y}^{(n)}$ converges to a solution of the VI$(\mathscr{Q}, \mathbf{F})$. This idea is formalized in Algorithm 2 below, whose convergence properties are given in Theorem 3. Note that the convergence of the algorithm only requires the monotonicity of $\mathbf{F}$. Moreover, one can also replace the exact computation of the solution $\mathbf{S}_\tau(\mathbf{x}^{(n)})$ (see step 2) of the regularized VI$(\mathscr{Q}, \mathbf{F} + \tau(\mathbf{I} - \mathbf{x}^{(n)}))$ with an inexact solution, without affecting the convergence of Algorithm 2 (provided that the error bound goes to zero as $n \to \infty$).

Algorithm 2: Proximal Decomposition Algorithm (PDA)

Data : Let $\{\varepsilon_n\}_{n=0}^{\infty}$, $\{\rho_n\}_{n=0}^{\infty}$, and $\tau > 0$ be given, and choose any feasible starting point $\mathbf{x}^{(0)}$; set $n = 0$.
`(S.1)` : If $\mathbf{x}^{(n)}$ satisfies a suitable termination criterion: STOP.
`(S.2)` : Find a point $\mathbf{z}^{(n)}$ such that $\|\mathbf{z}^{(n)} - \mathbf{S}_\tau(\mathbf{x}^{(n)})\| \leq \varepsilon_n$.
`(S.3)` : Set $\mathbf{x}^{(n+1)} \triangleq (1 - \rho_n)\mathbf{x}^{(n)} + \rho_n \mathbf{z}^{(n)}$.
`(S.4)` : $n \leftarrow n + 1$; go to `(S.1)`.

Theorem 3 ([30]). *Let $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ satisfy Assumptions 1 and 2 and let $\mathbf{F} \triangleq (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$. Let $\{\varepsilon_n\} \subset [0, \infty)$ be a sequence such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and let ρ_n be such that $\{\rho_n\} \subset [R_m, R_M]$ with $0 < R_m \leq R_M < 2$, Then, the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ generated by the Proximal Decomposition Algorithm described in Algorithm 2 converges to a solution of the game $\mathscr{G}$.*

A key point now becomes how to compute, for any given $\mathbf{x}^{(n)}$, a(n approximated) solution $\mathbf{S}_\tau(\mathbf{x}^{(n)})$ of the regularized VI$(\mathscr{Q}, \mathbf{F}+\tau(\mathbf{I}-\mathbf{x}^{(n)}))$ in a distributed way. The interesting point is that, going backwards by applying Proposition 1 to the VI$(\mathscr{Q}, \mathbf{F}+\tau(\mathbf{I}-\mathbf{x}^{(n)}))$, one can see that $\mathbf{S}_\tau(\mathbf{x}^{(n)})$ coincides with the unique (under Assumption 2) NE of the following regularized game:

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) + \frac{\tau}{2}\|\mathbf{x}_i - \mathbf{x}_i^{(n)}\|^2 \\ \text{subject to} & \mathbf{x}_i \in \mathscr{Q}_i, \end{array} \qquad \forall i = 1, \ldots, Q. \tag{0.16}$$

It turns out that $\mathbf{S}_\tau(\mathbf{x}^{(n)})$ can be computed in a distributed way using any iterative algorithm falling in class of asynchronous algorithms described in Algorithm 1 and applied to the regularized game in (0.16); theorem 2 states that such a class of algorithms globally converges if the matrix

$$\Upsilon_{\mathbf{F},\tau} \triangleq \Upsilon_{\mathbf{F}} + \tau\mathbf{I}, \tag{0.17}$$

with $\Upsilon_{\mathbf{F}}$ defined in (0.10), is a P-matrix, which is guaranteed for any τ sufficiently large. Stated in mathematical terms, we have the following.

Corollary 1 ([30]). *In the setting of Theorem 3, if τ is chosen sufficiently large so that $\Upsilon_{\mathbf{F},\tau}$ is a P-matrix, then any Asynchronous Best-response Algorithm (see Algorithm 1) applied to the game in (0.16) converges to $\mathbf{S}_\tau(\mathbf{x}^{(n)})$.*

The only thing left to discuss at this point is how to check whether the condition $\|\mathbf{z}^{(n)} - \mathbf{S}_\tau(\mathbf{x}^{(n)})\| \leq \varepsilon_n$ in Step 2 is satisfied. This can be certainly accomplished but it is a rather technical issue and we refer to [30] for practical implementations of this check, under different level of signaling and supervision.

Remark 5 (On the structure of Algorithm 2). Algorithm 2 is *only conceptually* a double loop method. It is indeed very close to the iterative best-response algorithm applied to game (0.16) (see Algorithm 1); the only difference being that "from time to time" (more precisely when the inner termination test $\|\mathbf{z}^{(n)} - \mathbf{S}_\tau(\mathbf{x}^{(n)})\| \leq \varepsilon_n$ is satisfied), the objective functions of the players are changed by changing the regularizing term from $\frac{\tau}{2}\|\mathbf{x}_i - \mathbf{x}_i^{(n)}\|^2$ to $\frac{\tau}{2}\|\mathbf{x}_i - \mathbf{x}_i^{(n+1)}\|^2$.

Note that Algorithm 2 does not suffer of the main drawback of best-response based schemes (cf. Remark 4), since the convergence conditions as given in Theorem 3 do not require the strong convexity of the payoff functions $f_i(\cdot, \mathbf{x}_{-i})$. The only (sufficient) condition we need is the monotonicity of $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^Q$ on $\mathscr{Q}$ (cf. Proposition 2).

0.2 Generalized Nash Equilibrium Problems (GNEP)

In all previous developments we have assumed that the feasible set of each player is independent of the rival players' choices, but this is not always the case. There are many applications of interest where the feasible sets naturally depend on the variables of the player's rivals (see, e.g., Section 0.3). The GNEP extends the classical NEP setting described so far precisely by assuming that each player's strategy

set can depend on the rival players' strategies $\mathbf{x}_{-i}$. In order to describe a GNEP we denote by $\mathscr{Q}_i(\mathbf{x}_{-i}) \subseteq \mathbb{R}^{n_i}$ the feasible set of player i when the other players choose $\mathbf{x}_{-i}$. Analogously to the NEP case, the aim of each player i, given $\mathbf{x}_{-i}$, is to choose a strategy $\mathbf{x}_i \in \mathscr{Q}_i(\mathbf{x}_{-i})$ that solves the problem

$$\begin{array}{l} \underset{\mathbf{x}_i}{\text{minimize}}\ f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to}\ \mathbf{x}_i \in \mathscr{Q}_i(\mathbf{x}_{-i}). \end{array} \tag{0.18}$$

Definition 5. A Generalized Nash Equilibrium (GNE), or simply a solution of the GNEP, is a feasible point $\mathbf{x}^\star$ such that

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{x}_i \in \mathscr{Q}_i(\mathbf{x}_{-i}^\star) \tag{0.19}$$

holds for each player $i = 1, \ldots, Q$.

Due to the variability of the feasible sets, the GNEP is a much harder problem than an ordinary NEP. Indeed, in its full generality, the GNEP problem is almost intractable and also the VI approach is of no great help. We then restrict our attention to particular classes of (more tractable) equilibrium problems: The so-called GNEPs with *jointly convex shared constraints* (see [10] for a survey on GNEPs).

Definition 6. A GNEP is termed as *GNEP with jointly convex shared constraints* if the feasible sets are defined as:

$$\mathscr{Q}_i(\mathbf{x}_{-i}) \triangleq \left\{ \mathbf{x}_i \in \overline{\mathscr{Q}}_i : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0} \right\}, \tag{0.20}$$

where $\overline{\mathscr{Q}}_i \subseteq \mathbb{R}^{n_i}$ is the closed and convex set of individual constraints of player i and $\mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0}$ represents the set of shared coupling constraints (equal for all the players), with $\mathbf{g} \triangleq (g_j)_{j=1}^m : \mathbb{R}^n \to \mathbb{R}^m$ (jointly) convex in $\mathbf{x}$.

Note that if there are no shared constraints the problem reduces to a standard NEP. We can give a geometric interpretation to (0.20), as shown next. For a GNEP with shared constraints, let us define a set $\mathscr{Q}$ in the product space of all players:

$$\mathscr{Q} \triangleq \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \text{ and } \mathbf{x}_i \in \overline{\mathscr{Q}}_i,\ \forall i = 1, \ldots, Q \right\}. \tag{0.21}$$

It is easy to check that the set $\mathscr{Q}$ is closed and convex and that each feasible set is just a "slice" of the "big" set $\mathscr{Q}$:

$$\mathscr{Q}_i(\mathbf{x}_{-i}) = \left\{ \mathbf{x}_i \in \overline{\mathscr{Q}}_i : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0} \right\} = \{ \mathbf{x}_i \in \mathbb{R}^n : (\mathbf{x}_i, \mathbf{x}_{-i}) \in \mathscr{Q} \}. \tag{0.22}$$

Similarly to the NEP, throughout the chapter we make the following convexity/smoothness assumption for the GNEP with shared constraints.

Assumption 3. For each $i = 1, \ldots, Q$, the set $\overline{\mathscr{Q}}_i$ is a nonempty, closed and convex subset of $\mathbb{R}^{n_i}$ and the function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is twice continuously differentiable in $\mathbf{x}$ and convex in $\mathbf{x}_i$ for every fixed $\mathbf{x}_{-i}$; the functions $\mathbf{g}(\mathbf{x}) = (g_j(\mathbf{x}))_{j=1}^m$ are continuously differentiable and jointly convex in $\mathbf{x}$.

0.2.1 Connection to VIs: The variational solutions

GNEPs with shared constraints are still very difficult, but some types of solutions can be studied and calculated relatively easily by using a VI approach. More specifically, invoking the minimum principle, one can readily obtain the following connection between GNEPs and VIs (see, e.g., [9, 2]).

Lemma 2. *Let $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ be a GNEP with shared constraints. Suppose that $\mathscr{G}$ satisfies Assumption 3 and let VI($\mathscr{Q},\mathbf{F}$) be the VI with $\mathscr{Q}$ defined in (0.21) and $\mathbf{F} \triangleq (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$. Then, every solution of the VI($\mathscr{Q},\mathbf{F}$) is a solution of the GNEP $\mathscr{G}$.*

Note that passing from the GNEP to the associated VI not all the GNEP solutions are preserved: Lemma 2 in fact does not state that any solution of the GNEP is also a solution of the VI (see [9, 2, 13] for further details and examples). Solutions of the GNEP that are also solutions of the VI($\mathscr{Q},\mathbf{F}$) are termed as "variational solutions" [10] or "normalized solutions" [28]. Variational solutions enjoy some remarkable properties that make them particularly appealing in many applications. To discuss this issue further, assume that we have a GNEP with jointly convex shared constraints satisfying Assumption 3 and that some constraint qualification (CQ) (see, e.g., [11, Sec. 3.2]) holds at all elements in every set $\mathscr{Q}_i(\mathbf{x}_{-i})$ defined in (0.22). Under these assumptions the GNEP is equivalent to its KKT system that is obtained concatenating the KKT conditions of the convex optimization problem in (0.18) of each player: $\mathbf{x}^\star$ is a solution of the GNEP if and only if there exist multipliers $\lambda^\star = (\lambda^{(i)\star})_{i=1}^{Q} \in \mathbb{R}_+^{Qm}$ such that

$$\begin{array}{l} \mathbf{0} \in \nabla_{\mathbf{x}_i} \mathscr{L}_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star, \lambda^{(i)\star}) + \mathscr{N}(\mathbf{x}_i^\star, \overline{\mathscr{Q}}_i) \\ \mathbf{0} \leq \lambda^{\star(i)} \perp \mathbf{g}(\mathbf{x}^\star) \leq \mathbf{0} \end{array} \quad \forall i = 1,\ldots,Q, \tag{0.23}$$

where

$$\mathscr{L}_i(\mathbf{x}_i, \mathbf{x}_{-i}, \lambda^{(i)}) \triangleq f_i(\mathbf{x}) + \lambda^{(i)T}\, \mathbf{g}(\mathbf{x}) \tag{0.24}$$

is the Lagrangian function of player i's optimization problem (0.18), with $\lambda^{(i)} = (\lambda_k^{(i)})_{k=1}^{m}$ denoting the multipliers of the shared constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$, and $\mathscr{N}(\mathbf{x}_i^\star, \overline{\mathscr{Q}}_i) \triangleq \left\{ \mathbf{d} \in \mathbb{R}^{n_i} : \mathbf{d}^T(\mathbf{y} - \mathbf{x}_i^\star) \leq \mathbf{0}, \quad \forall \mathbf{y} \in \overline{\mathscr{Q}}_i \right\}$ is the normal cone to $\overline{\mathscr{Q}}_i$ at $\mathbf{x}_i^\star$. Similarly, assume some suitable CQ at all elements in the set $\mathscr{Q}$ defined in (0.21). Then, the VI($\mathscr{Q},\mathbf{F}$), with $\mathbf{F} \triangleq (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$, is equivalent to its KKT system [11]: $\bar{\mathbf{x}}$ is a solution of the VI($\mathscr{Q},\mathbf{F}$) if and only if there exist multipliers $\bar{\lambda} \in \mathbb{R}_+^m$ such that

$$\begin{array}{l} \mathbf{0} \in \nabla_{\mathbf{x}_i} \mathscr{L}_i\left(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}_{-i}, \bar{\lambda}\right) + \mathscr{N}(\bar{\mathbf{x}}_i, \overline{\mathscr{Q}}_i), \\ \mathbf{0} \leq \bar{\lambda} \perp \mathbf{g}(\bar{\mathbf{x}}) \leq \mathbf{0} \end{array} \qquad \forall i = 1,\ldots,Q. \tag{0.25}$$

Comparing (0.23) with (0.25) it is not difficult to see that the KKT system (0.25) is a special case of (0.23) with $\lambda^{\star(1)} = \ldots = \lambda^{\star(Q)} = \overline{\lambda}$, meaning that a solution $\bar{\mathbf{x}}$ of the GNEP is a variational solution if and only if the shared constraints have the same multipliers for all the players. More formally, we have the following.

Lemma 3. *Let $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ be a GNEP with shared constraints satisfying Assumption 3, and let VI$(\mathscr{Q},\mathbf{F})$ be the VI with $\mathscr{Q}$ defined in (0.21) and $\mathbf{F} \triangleq (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$. Then, the following hold:*
(i) Suppose that $\bar{\mathbf{x}}$ is a solution of the VI$(\mathscr{Q},\mathbf{F})$ at which the KKT (0.25) holds with multipliers $\overline{\lambda}$. Then $\bar{\mathbf{x}}$ is a solution of the GNEP at which the KKT (0.23) holds with $\lambda^{\star(1)} = \ldots = \lambda^{\star(Q)} = \overline{\lambda}$;
(ii) Conversely, suppose that $\mathbf{x}^\star$ is a solution of the GNEP at which the KKT (0.23) holds with $\lambda^{\star(1)} = \ldots = \lambda^{\star(Q)}$. Then $\mathbf{x}^\star$ os a solution of the VI$(\mathscr{Q},\mathbf{F})$ and the pair $(\mathbf{x}^\star, \lambda^{\star(1)})$ satisfies the KKT (0.25).

The importance of Lemma 2 and 3 is twofold. First, we can use VI results as given in Section 0.1 (see also [11, Sec. 2]) to obtain practical conditions ensuring the existence and the uniqueness of variational solutions for a GNEP with shared constraints; we leave to the reader the easy task to duplicate these results. Second, Lemma 3 gives rise to an interesting game theoretical pricing-based interpretation of the variational solutions, useful to design distributed algorithms, as detailed in the next section.

0.2.2 Distributed algorithms for variational solutions

In this section, we develop distributed algorithms to compute the variational solutions of a GNEP with jointly convex shared constraints. The presence of the coupling constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ prevents a direct application of the decomposition methods presented in Section 0.1.4 to the VI$(\mathscr{Q},\mathbf{F})$, because $\mathscr{Q}$ does not have a Cartesian structure. To overcome this issue, we rewrite the VI$(\mathscr{Q},\mathbf{F})$ in a more convenient form, as detailed next. Consider the problem of finding a couple $(\bar{\mathbf{x}},\bar{\lambda})$, with $\bar{\lambda} = (\bar{\lambda}_k)_{k=1}^{m}$ such that $\bar{\mathbf{x}}$ solves the NEP

$$\mathscr{G}_{\bar{\lambda}}: \quad \begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i,\mathbf{x}_{-i}) + \bar{\lambda}^T \mathbf{g}(\mathbf{x}) \\ \text{subject to} & \mathbf{x}_i \in \overline{\mathscr{Q}}_i, \end{array} \qquad \forall i = 1,\ldots,Q, \tag{0.26}$$

and furthermore

$$\mathbf{0} \leq \bar{\lambda} \perp \mathbf{g}(\bar{\mathbf{x}}) \leq \mathbf{0}. \tag{0.27}$$

We can interpret the $\bar{\lambda}$ as prices paid by the players for using the common "resource" represented by the shared constraints. Condition (0.27) says that the players actually have to pay only when the resource becomes scarce. In the following we refer to the NEP in (0.26) with fixed vector λ as $\mathscr{G}_\lambda$. A direct comparison of the KKT conditions of NEP (0.26) augmented with condition (0.27) and the KKT conditions (0.25) leads to the following interesting result.

Lemma 4. *Suppose that the GNEP with shared constraints $\mathscr{G} =< \mathscr{Q}, \mathbf{f} >$ satisfies assumptions in Lemma 2 and some CQ holds at all the elements in the set $\mathscr{Q}$ defined in (0.21). Then, $(\bar{\mathbf{x}},\bar{\lambda})$ is a solution of the problem (0.26)-(0.27) if and only if $\bar{\mathbf{x}}$ is a variational solution of the GNEP−the VI$(\mathscr{Q},\mathbf{F})$, with $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$− and $\bar{\lambda}$ is the corresponding multiplier of the shared constraints.*

Based on Lemma 4 we are now able to compute the variational solutions of the GNEP as solutions of (0.26)-(0.27), to which we can apply in principle the theory developed in Sections 0.1. Similarly to the NEP case, we consider both cases of strongly-monotone (uniformly-P) and monotone games.

0.2.2.1 Algorithms for strongly monotone pricing games

To describe the proposed algorithms we need the following preliminary definitions and results. Under the convexity of the shared constraints $\mathbf{g}(\mathbf{x})$ and the positive definiteness of matrix $\Upsilon_{\mathbf{F}}$ defined in (0.10), which implies the strongly monotonicity of $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$ (see Proposition 2), the game $\mathscr{G}_\lambda$ in (0.26)−the strongly monotone VI($\overline{\mathscr{Q}}, \mathbf{F} + \nabla_{\mathbf{x}}\mathbf{g}\lambda$), with $\overline{\mathscr{Q}} \triangleq \prod_{i=1}^{Q} \overline{\mathscr{Q}}_i$ and $\nabla_{\mathbf{x}}\mathbf{g}$ denoting the matrix whose i-th column is equal to $\nabla_{\mathbf{x}} g_i$−has a unique NE $\mathbf{x}^\star(\lambda)$, for any $\lambda \geq \mathbf{0}$. Under this condition, let define the map

$$\Phi(\lambda) : \mathbb{R}^m_+ \ni \lambda \to -\mathbf{g}(\mathbf{x}^\star(\lambda)) \tag{0.28}$$

which measures the (negative) violation of the shared constraints at $\mathbf{x}^\star(\lambda)$.

Based on (0.26)-(0.27), the key result to devise distributed algorithms is given in Theorem 4 below, where $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$, $\mathbf{x}^\star(\lambda)$ denotes the unique NE of $\mathscr{G}_\lambda$ (under the positive definiteness of matrix $\Upsilon_{\mathbf{F}}$), and $\Upsilon_{\mathbf{F}}$, $\mathscr{Q}$, and Φ are defined in (0.10), (0.21), and (0.28), respectively.

Theorem 4 ([30]). *Given the problem (0.26)-(0.27), suppose that Assumption 3 and some CQ at all elements of the set $\mathscr{Q}$ hold true. If $\Upsilon_{\mathbf{F}} \succ \mathbf{0}$, then the following hold:*
(a) The problem (0.26)-(0.27) is equivalent to the nonlinear complementarity problem (NCP) in the price tuple λ

$$\text{NCP}(\Phi): \qquad 0 \leq \lambda \perp \Phi(\lambda) \geq 0. \tag{0.29}$$

The equivalence is in the following sense: the NCP(Φ) must have a solution and, for any such a solution λ^{NCP}, the pair $(\mathbf{x}^\star(\lambda^{\text{NCP}}), \lambda^{\text{NCP}})$ is a solution of (0.26)-(0.27), with $\mathbf{x}^\star(\lambda^{\text{NCP}}) = \mathbf{x}^{\text{VI}}$, where $\mathbf{x}^{\text{VI}}$ is the unique solution of the VI($\mathscr{Q}, \mathbf{F}$)*; conversely, if $(\mathbf{x}^{\text{NE}}, \lambda^{\text{NE}})$ is a solution of (0.26)-(0.27) [$\mathbf{x}^{\text{NE}} = SOL(\mathscr{Q}, \mathbf{F})$], then λ^{NE} is a solution of the* NCP(Φ) *with $\mathbf{x}^\star(\lambda^{\text{NE}}) = \mathbf{x}^{\text{NE}}$;*
(b) The problem (0.26)-(0.27) has a unique least-norm price tuple, denoted by $\lambda^{\text{NE,ln}}$, such that $\|\lambda^{\text{NE,ln}}\|_2 \leq \|\lambda^{\text{NE}}\|_2$, for any price solution λ^{NE} of (0.26)-(0.27).

Remark 6 (On the uniqueness of the solution). Observe that, under $\Upsilon_{\mathbf{F}} \succ \mathbf{0}$, Theorem 4(a) implies only the uniqueness of the variables $\mathbf{x}$ of the problem (0.26)-(0.27) [the primal variables of the VI($\mathscr{Q}, \mathbf{F}$)], but not of the price tuple λ. The interesting result is that, in such a case, all these prices λ^{NCP}−the solutions of the NCP(Φ)−yield solution pairs $(\mathbf{x}^\star(\lambda^{\text{NCP}}), \lambda^{\text{NCP}})$ of (0.26)-(0.27) having the same optimal $\mathbf{x}^{\text{NE}}$, i.e., $\mathbf{x}^\star(\lambda^{\text{NCP}}) = \mathbf{x}^{\text{NE}}$. Part (b) of the theorem identifies a unique special price tuple $\lambda^{\text{NE,lm}}$.

The NCP reformulation of the problem (0.26)-(0.27) as stated by Theorem 4 offers the possibility of devising iterative algorithms that can be implemented in a distributed fashion among all players (because the feasible set of the NCP has

a Cartesian structure) and whose convergence can be studied using known results from the theory of VIs (cf. [11, Chapter 12]). An example is the *Projection Algorithm with variable steps* [11, Alg. 12.1.4] applied to the NCP(Φ) in (0.29) and formally described in Algorithm 3 (under the assumption $\Upsilon_{\mathbf{F}} \succ \mathbf{0}$), where $\overline{\mathscr{Q}} = \prod_i \overline{\mathscr{Q}}_i$ and $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$. The convergence properties of the algorithm are given in Theorem 5. For other algorithms we refer to [27, 30, 38].

Algorithm 3: Projection Algorithm with Variable Steps (PAVS)

Data : Choose any $\boldsymbol{\lambda}^{(0)} \geq \mathbf{0}$; set $n = 0$.

`(S.1)` : If $\boldsymbol{\lambda}^{(n)}$ satisfies a suitable termination criterion: `STOP`.

`(S.2)` : Given $\boldsymbol{\lambda}^{(n)}$, compute $\mathbf{x}^{\star}\left(\boldsymbol{\lambda}^{(n)}\right)$ as the unique NE of $\mathscr{G}_{\boldsymbol{\lambda}^{(n)}}$:

$$\mathbf{x}^{\star}(\boldsymbol{\lambda}^{(n)}) = \mathrm{SOL}(\overline{\mathscr{Q}}, \mathbf{F} + \nabla_{\mathbf{x}}\mathbf{g}\boldsymbol{\lambda}^{(n)}). \tag{0.30}$$

`(S.3)` : Choose $\tau_n > 0$ and update the price vectors $\boldsymbol{\lambda}$ according to

$$\boldsymbol{\lambda}^{(n+1)} = \left[\boldsymbol{\lambda}^{(n)} - \tau_n \boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)\right]^{+}. \tag{0.31}$$

`(S.4)` : Set $n \leftarrow n+1$; go to `(S.1)`.

Theorem 5 ([30]). *Suppose $\Upsilon_{\mathbf{F}} \succ \mathbf{0}$. If the scalars τ_n are chosen so that $0 < \inf_n \tau_n \leq \sup_n \tau_n < 2c_{\mathrm{sm}}/c_{\mathrm{Lip}}^2$, with $c_{\mathrm{Lip}} \triangleq \max_{\mathbf{x}\in\overline{\mathscr{Q}}} \left\|\nabla_{\mathbf{x}}\mathbf{g}(\mathbf{x})^T\right\|_2$ and c_{sm} defined in Proposition 2(b), then the sequence $\{\boldsymbol{\lambda}^{(n)}\}_{n=0}^{\infty}$ generated by Algorithm 3 converges to a solution of the* NCP(Φ).

Remark 7 (On the convergence of the inner loop via distributed algorithms). The implementation of Algorithm 3 requires the computation of the solution of the $\mathscr{G}_{\boldsymbol{\lambda}^{(n)}}$ (0.30) for a given set of prices, possibly in a distributed way. Given $\boldsymbol{\lambda} \geq \mathbf{0}$, the game $\mathscr{G}_{\boldsymbol{\lambda}}$ is a NEP and thus one can solve it by using any of the algorithms proposed in Section 0.1.4 for NEPs. We refer to [30] for a detailed study of the convergence of asynchronous algorithms applied to the VI($\overline{\mathscr{Q}}, \mathbf{F} + \nabla_{\mathbf{x}}\mathbf{g}\boldsymbol{\lambda}^{(n)}$). The interesting result is that, when the vector function $\mathbf{g}(\mathbf{x})$ is separable, i.e. $\mathbf{g}(\mathbf{x}) = \sum_{i=1}^{Q} \mathbf{g}_i(\mathbf{x}_i)$, conditions in Theorem 5 are sufficient for the convergence of both loops in Algorithm 3.

0.2.2.2 Algorithms for monotone pricing games

We focus now on the case in which the problem (0.26)-(0.27) (the associated VI) is monotone. In such a case, Algorithm 3 is not guaranteed to converge anymore; instead, the outer loop has to be complicated. To avoid this complication, here we consider a different approach, based on an equivalent reformulation of (0.26)-(0.27). To this end, observe first that the price complementarity condition in (0.27) is equivalent to $\boldsymbol{\lambda} \in \mathrm{argmin}_{\boldsymbol{\lambda}' \geq \mathbf{0}} \left\{-\boldsymbol{\lambda}'^{T} g(\mathbf{x})\right\}$. Then, consider the NEP with $Q+1$ players in which the "new" $(Q+1)$-th player controls the price variables $\boldsymbol{\lambda}$:

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i,\mathbf{x}_{-i})+\lambda^T\mathbf{g}(\mathbf{x}) \\ \text{subject to} & \mathbf{x}_i\in\overline{\mathscr{Q}}_i, \end{array} \qquad \forall i=1,\ldots,Q, \tag{0.32}$$

$$\underset{\lambda\geq 0}{\text{minimize}} -\lambda^T\mathbf{g}(\mathbf{x}).$$

The difference between the problem (0.26)-(0.27) and the NEP (0.32) is that, in the latter game there are no side constraints, but the complementarity condition is treated as an additional player of the game (at the same level of the other Q players), who solves a nonnegatively constrained linear program in the variable λ parametrized by $\mathbf{x}$. It is not difficult to see that this new extended game has the same solution set of the problem (0.26)-(0.27) [note that $\mathbf{g}(\mathbf{x})\leq\mathbf{0}$ in (0.32) is implicitly satisfied at the equilibrium]. Therefore, one can compute a variational solution of a GNEP with jointly convex shared constraints by finding a solution of the NEP in (0.32). This game is equivalent to the *partitioned* VI$(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}})$, where $\widetilde{\mathscr{Q}}\triangleq\prod_{i=1}^{Q}\overline{\mathscr{Q}}_i\times\mathbb{R}^m_+$ and $\widetilde{\mathbf{F}}(\mathbf{x},\lambda)$ is defined as

$$\widetilde{\mathbf{F}}(\mathbf{x},\lambda)\triangleq\begin{pmatrix}(\nabla_{\mathbf{x}_i}f_i(\mathbf{x})+\nabla_{\mathbf{x}_i}\mathbf{g}(\mathbf{x})\lambda)_{i=1}^{Q}\\ -\mathbf{g}(\mathbf{x})\end{pmatrix}=\begin{pmatrix}\mathbf{F}(\mathbf{x})+\nabla_{\mathbf{x}}\mathbf{g}(\mathbf{x})\lambda\\ -\mathbf{g}(\mathbf{x})\end{pmatrix}. \tag{0.33}$$

Proposition 4. *If $\mathbf{F}(\mathbf{x})=(\nabla_{x_i}f_i(\mathbf{x}))_{i=1}^{Q}$ is monotone on $\prod_{i=1}^{Q}\overline{\mathscr{Q}}_i$, then also $\widetilde{\mathbf{F}}(\mathbf{x},\lambda)$ is monotone on $\widetilde{\mathscr{Q}}$.*

It follows from Proposition 4 that, under Assumptions 2 and 3, one can compute a variational equilibrium of the GNEP by applying Algorithm 2 described in Section 0.1.4.2 to the VI$(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}})$, where $\widetilde{\mathbf{S}}_\tau(\mathbf{y}^{(n)})\triangleq\text{SOL}\left(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}}+\tau(\mathbf{I}-\mathbf{y}^{(n)})\right)$ in the step 2 of the algorithm is the unique solution of the regularized VI $\left(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}}+\tau(\mathbf{I}-\mathbf{y}^{(n)})\right)$, given $\mathbf{y}^{(n)}\triangleq(\mathbf{x}^{(n)},\lambda^{(n)})$. This is formalized in the following.

Theorem 6 ([30]). *Suppose that the GNEP with shared constraints $\mathscr{G}=<\mathscr{Q},\mathbf{f}>$ satisfies Assumptions 2 and 3, and some CQ holds at all the elements in the set $\mathscr{Q}$ defined in (0.21). Let $\{\varepsilon_n\}\subset[0,\infty)$ be a sequence such that $\sum_{n=1}^{\infty}\varepsilon_n<\infty$, and let ρ_n be such that $\{\rho_n\}\subset[R_m,R_M]$, with $0<R_m\leq R_M<2$. Then, the sequence $\{(\mathbf{x}^{(n)},\lambda^{(n)})\}_{n=0}^{\infty}$ generated by the PDA described in Algorithm 2 and applied to the VI$(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}})$ converges to a variational solution of the GNEP.*

The last thing left to discuss is how to compute, for any given $\mathbf{y}^{(n)}\triangleq(\mathbf{x}^{(n)},\lambda^{(n)})$, a(n approximated) solution $\widetilde{\mathbf{S}}_\tau(\mathbf{y}^{(n)})$ of the regularized VI $\left(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}}+\tau(\mathbf{I}-\mathbf{y}^{(n)})\right)$. Under Assumptions 2 and 3, the VI $\left(\widetilde{\mathscr{Q}},\widetilde{\mathbf{F}}+\tau(\mathbf{I}-\mathbf{y}^{(n)})\right)$ is equivalent to the following regularized NEP

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i,\mathbf{x}_{-i})+\boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x})+\frac{\tau}{2}\left\|\mathbf{x}_i-\mathbf{x}_i^{(n)}\right\|^2 \qquad \forall i=1,\ldots,Q \\ \text{subject to} & \mathbf{x}_i\in\overline{\mathscr{Q}}_i, \end{array} \tag{0.34}$$

$$\underset{\boldsymbol{\lambda}\geq 0}{\text{minimize}} -\boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x})+\frac{\tau}{2}\left\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{(n)}\right\|^2$$

whose Nash equilibria can be distributively computed by using the Asynchronous Best-response Algorithms described in Algorithm 1. The global convergence of such a class of algorithms is guaranteed for sufficiently large $\tau>0$, as stated in the following corollary, restricted to the case of separable $\mathbf{g}(\mathbf{x})=\sum_{i=1}^{Q}\mathbf{g}_i(\mathbf{x}_i)$ (the more general case of non separable $\mathbf{g}$ is addressed in [30]), where $\mathbf{F}=(\nabla_{\mathbf{x}_i}f_i)_{i=1}^{Q}$, the matrix $\bar{\Upsilon}_{\mathbf{F},\tau}$ is defined as

$$\bar{\Upsilon}_{\mathbf{F},\tau}\triangleq\left[\begin{array}{c|c} \Upsilon_{\mathbf{F}}+\tau\mathbf{I} & -\gamma \\ \hline -\gamma^T & \tau \end{array}\right] \tag{0.35}$$

with $\Upsilon_{\mathbf{F}}$ defined in (0.10), $\gamma\triangleq(\gamma_i)_{i=1}^{Q}$, and $\gamma_i\triangleq\sup_{\mathbf{z}_i\in\overline{\mathscr{Q}}_i}\|\nabla_{\mathbf{x}_i}\mathbf{g}_i(\mathbf{z}_i)\|_2$.

Corollary 2 ([30]). *In the setting of Theorem 6, if* $\mathbf{g}(\mathbf{x})=\sum_{i=1}^{Q}\mathbf{g}_i(\mathbf{x}_i)$ *and* τ *is chosen sufficiently large so that* $\bar{\Upsilon}_{\mathbf{F},\tau}$ *is a P-matrix, then any Asynchronous Best-response Algorithm (see Algorithm 1) applied to the game in (0.34) converges to* $\widetilde{\mathbf{S}}_{\tau}(\mathbf{y}^{(n)})$.

Remark 8 (Partial regularization schemes). In some situations, the optimization problems of some players might be "convex enough" to require no regularization. The study of the convergence properties of distributed schemes based on "partial regularizations" is addressed in [30].

0.3 Design of CR Systems based on Game Theory

In the last decade, CR has received considerable attention as a way to improve the efficiency of radio networks [21, 15]. CR networks adopt a hierarchical access structure where the primary users (PUs) are the legacy spectrum holders while the secondary users (SUs) are the unlicensed users who sense the electromagnetic environment and adapt their transceivers parameters as well as the resource allocation decisions in order to dynamically access temporally unoccupied spectrum regions.

We consider a hierarchical CR network composed of P PUs and Q SUs, each formed by a transmitter-receiver pair, coexisting in the same area and sharing the same band. We focus on (block) transmissions over SISO frequency-selective channels; more general results valid for MIMO channels can be found in [31, 35]. Because of the lack of coordination among the CR users and the competitive nature of the system, the set of SUs can be naturally modeled as a frequency-selective N-parallel Gaussian interference channel, where N is the number of available subcarriers. The transmission strategy of each SU i is then the power allocation vector $\mathbf{p}_i=(p_i(k))_{k=1}^{N}$ over the N subcarriers, subject to the transmit power constraints

$\sum_{k=1}^{N} p_i(k) \leq P_i$. Under mild conditions (see, e.g., [27]), the maximum information rate on link i for a specific power allocation profile $\mathbf{p}_1, \ldots, \mathbf{p}_Q$ is

$$r_i(\mathbf{p}_q, \mathbf{p}_{-q}) = \sum_{k=1}^{N} \log\left(1 + \frac{|H_{ii}(k)|^2 p_i(k)}{\sigma_i^2(k) + \sum_{j\neq i} |H_{ij}(k)|^2 p_j(k)}\right), \tag{0.36}$$

where $\sigma_i^2(k)$ is the thermal noise power over carrier k, $H_{ij}(k)$ is the channel transfer function between the secondary transmitter j and the receiver i, and $\mathbf{p}_{-i} \triangleq (\mathbf{p}_j)_{j\neq i}$ is the set of all the users power allocation vectors, except the i-th one.

Temperature-interference constraints: Differently from traditional static or centralized spectrum assignment, opportunistic communications in CR systems enable SUs to transmit with overlapping spectrum with PUs, provided that the degradation induced on the PUs' performance is null or tolerable [15]. In this chapter, we envisage the use of two classes of interference constraints termed *individual conservative* and *global flexible constraints*: for each $i = 1, \ldots, Q$,

- *Individual per-carrier interference constraints,*

$$p_i(k) \leq p_i^{\max}(k) \triangleq \min_{p=1,\ldots,P} \frac{I_{p,i}^{\text{peak}}(k)}{|H_{pi}^{(P,S)}(k)|^2}, \quad \forall k = 1, \cdots, N; \tag{0.37}$$

- *Individual overall bandwidth interference constraints*,

$$\sum_{k=1}^{N} |H_{pi}^{(P,S)}(k)|^2 p_i(k) \leq I_{p,i}^{\text{tot}} \quad \forall p = 1, \ldots, P; \tag{0.38}$$

- *Global per-carrier interference constraints*,

$$\sum_{i=1}^{Q} |H_{pi}^{(P,S)}(k)|^2 p_i(k) \leq I_p^{\text{peak}}(k), \quad \forall k = 1, \cdots, N, \ \forall p = 1, \ldots, P; \tag{0.39}$$

- *Global overall bandwidth interference constraints*,

$$\sum_{i=1}^{Q} \sum_{k=1}^{N} |H_{pi}^{(P,S)}(k)|^2 p_i(k) \leq I_p^{\text{tot}} \quad p = 1, \ldots, P \tag{0.40}$$

where $H_{pi}^{(P,S)}(k)$ is the channel transfer function between the secondary transmitter i and the primary receiver p over carrier k; $I_{p,i}^{\text{peak}}(k)$ [$I_p^{\text{tot}}(k)$] and $I_{p,i}^{\text{tot}}$ [I_p^{tot}] are the maximum interference allowed to be generated by the SU i [all the SUs] that is tolerable at the primary receiver p over carrier k and over the whole spectrum (licensed to the PU p), respectively. The values of $I_{p,i}^{\text{peak}}(k)$'s and $I_{p,i}^{\text{tot}}$'s can be obtained at the SUs' transmitters if the type of PUs are known. Methods to obtain the interference limits when the SUs do not have this knowledge are discussed in [15]. To avoid a trivial solution, we assume w.l.o.g. that, for all $k = 1, \ldots, N$ and $i = 1, \ldots, Q$, and some $p = 1, \ldots, P$,

$$\begin{aligned} &\text{i)} \qquad p_i^{\max}(k) < \min\left\{ \frac{I_{p,i}^{\text{peak}}(k)}{|H_{pi}^{(P,S)}(k)|^2}, P_i \right\} \\ &\text{ii)} \qquad \textstyle\sum_{k=1}^N p_i^{\max}(k) > P_i, \quad \text{or} \quad \sum_{k=1}^N |H_{pi}^{(P,S)}(k)|^2 p_i^{\max}(k) > I_{p,i}^{\text{tot}}. \end{aligned} \tag{0.41}$$

Individual interference constraints are motivated by all CR scenarios where primary terminals are oblivious to the presence of SUs (also called commons model [14, 15]). These constraints lead to totally distributed algorithms with no signaling among the SUs, as we will show later on. However, sometimes, they may be too restrictive and thus marginalize the potential gains offered by the dynamic resource assignment mechanism. The global interference constraints limit instead the *aggregate* interference, which is indeed the interference experimented by the PUs. These constraints will be shown to lead to better performance of SUs than those achievable by imposing individual constraints. However, this gain comes at a price: The resulting algorithms require some signaling between the PUs and SUs. Thus, they can be employed in CR networks where an interaction between the PUs and the SUs is allowed, as, e.g., in the so-called property-right CR model (or spectrum leasing) [14].

Game Theoretical formulation under individual interference constraints. Let define the strategy set of each SU i as

$$\mathscr{P}_i \triangleq \left\{ \mathbf{p}_i \in \mathbb{R}_+^N : \mathbf{1}^T \mathbf{p}_i \le P_i, \sum_{k=1}^N |H_{pi}^{(P,S)}(k)|^2 p_i(k) \le I_{p,i}^{\text{tot}} \, \forall p, \quad \mathbf{p}_i \le \mathbf{p}_i^{\max} \right\} \tag{0.42}$$

and consider the NEP $\mathscr{G}_{\text{ind}} = < \prod_i \mathscr{P}_i, (r_i)_{i=1}^Q >$, where each SU i, given the strategy profile $\mathbf{p}_{-i}$ of the other users, aims at maximizing his information rate $r_i(\mathbf{p})$ in (0.36) under local power and interference constraints in $\mathscr{P}_i$: for all $i = 1, \ldots, Q$,

$$\begin{aligned} &\underset{\mathbf{p}_i}{\text{maximize}} \quad r_i(\mathbf{p}_i, \mathbf{p}_{-i}) \\ &\text{subject to} \quad \mathbf{p}_i \in \mathscr{P}_i. \end{aligned} \tag{0.43}$$

First of all observe that, for any fixed $\mathbf{p}_{-i} \ge \mathbf{0}$, the single-user optimization problem in (0.43) admits a unique solution [indeed, the feasible set is convex and compact and $r_i(\mathbf{p}_i, \mathbf{p}_{-i})$ is strictly concave in $\mathbf{p}_i \in \mathscr{P}_i$], given by the (multilevel) waterfilling expression [30]:

$$p_i^\star(k) = \left[\frac{1}{\mu_i + \sum_{p=1}^P \lambda_{ip} |H_{pi}^{(P,S)}(k)|^2} - \frac{\sigma_i^2(k) + \sum_{j \ne i} |H_{ij}(k)|^2 p_j(k)}{|H_{ii}(k)|^2} \right]_0^{p_i^{\max}(k)} \tag{0.44}$$

with $k = 1, \ldots, N$, where $[x]_a^b \triangleq \min(\max(a,x), b)$ for $a \le b$ and the water-levels μ_i and $\{\lambda_{ip}\}$ are chosen to satisfy the following complementarity constraints: $0 \le \mu_i \perp P_i - \sum_{k=1}^N p_i^\star(k) \ge 0$ and $0 \le \lambda_{ip} \perp I_{pi}^{\text{tot}} - \sum_{k=1}^N |H_{pi}^{(P,S)}(k)|^2 p_i^\star(k) \ge 0$, for all $p = 1, \ldots, P$; see [30] for efficient algorithms to compute these water-levels. The Nash equilibria $\mathbf{p}^\star$ of the NEP $\mathscr{G}_{\text{ind}}$ are thus the fixed-points of the waterfilling mapping (0.44). The study of such a game can be carried out using the VI framework

developed in Section 0.1. Before stating the main results, we introduce the following definitions. For the NEP $\mathscr{G}_{\rm ind}$, matrix $\Upsilon_{\mathbf{F}}$ in (0.10) becomes

$$[\Upsilon_{\rm ind}]_{ij} \triangleq \begin{cases} \min\limits_{k=1,\ldots,N} \left(\dfrac{|H_{ii}(k)|^2}{\sigma_i^2(k)+\sum_{j=1}^N |H_{ij}(k)|^2 p_j^{\max}(k)} \right)^2, & \text{if } i=j, \\ -\max\limits_{k=1,\ldots,N} \dfrac{|H_{ij}(k)|^2}{\sigma_i^2(k)} \dfrac{|H_{ii}(k)|^2}{\sigma_i^2(k)}, & \text{otherwise.} \end{cases} \tag{0.45}$$

We also introduce the per-tone $\Upsilon_{\rm ind}(k)$, defined for each $k=1,\ldots,N$ as

$$[\Upsilon_{\rm ind}(k)]_{ij} \triangleq \begin{cases} \left(\dfrac{|H_{ii}(k)|^2}{\sigma_i^2(k)+\sum_{j=1}^N |H_{ij}(k)|^2 p_j^{\max}(k)} \right)^2, & \text{if } i=j, \\ -\dfrac{|H_{ij}(k)|^2}{\sigma_i^2(k)} \dfrac{|H_{ii}(k)|^2}{\sigma_i^2(k)}, & \text{otherwise,} \end{cases} \tag{0.46}$$

Note that $\Upsilon_{\rm ind} \leq \Upsilon_{\rm ind}(k)$ for all k. Building on Theorem 1 and Theorem 2 we obtain the following results for the game $\mathscr{G}_{\rm ind}$.

Theorem 7. *Given the NEP $\mathscr{G}_{ind} =< \prod_i \mathscr{P}_i, (r_i)_{i=1}^Q >$, suppose w.l.o.g. that condition i) and (the first of) ii) in (0.41) are satisfied. Then the following hold.*

(a) The NEP has a nonempty and bounded solution set;
(b) Suppose that Υ_{ind} in (0.45) is a P-matrix. Then the NEP has a unique NE and any sequence $\{\mathbf{p}^{(n)}\}_{n=0}^{\infty}$ generated by the asynchronous algorithm described in Algorithm 1 and based on the waterfilling best-response (0.44) converges to this equilibrium, for any given feasible updating schedule of the players.

Remark 9 (On the uniqueness/convergence conditions). Theorem 7 provides a physical interpretation of the conditions guaranteeing the uniqueness of the NE (and convergence of the asynchronous IWFA): the uniqueness of the NE is ensured if the interference among the SUs is sufficiently small. Sufficient conditions for $\Upsilon_{\rm ind}$ being a P-matrix easier to be checked can be readily obtained applying Corollary 3 to $\Upsilon_{\rm ind}$; we leave this task to the reader.

If the channels and the power/interference constraints in the game $\mathscr{G}_{\rm ind}$ are such that matrix $\Upsilon_{\rm ind}$ is not a P-matrix, we can still guarantee convergence of distributed algorithms to a NE of the game at the cost of some additional computational complexity. In fact, instead of the waterfilling best-response algorithm described in Algorithm 1, we can use the proximal algorithm described in Algorithm 2. According to Theorem 3, the monotonicity of $\mathbf{F}(\mathbf{p}) = (-\nabla_{\mathbf{p}_i} r_i(\mathbf{p}_i, \mathbf{p}_{-i}))_{i=1}^Q$ on $\prod_i \mathscr{P}_i$ is enough to guarantee the global convergence of this algorithm.

Theorem 8. *Given the NEP $\mathscr{G}_{ind} =< \prod_i \mathscr{P}_i, (r_i)_{i=1}^Q >$, suppose that the mapping $\mathbf{F}(\mathbf{p}) = (-\nabla_{\mathbf{p}_i} r_i(\mathbf{p}_i, \mathbf{p}_{-i}))_{i=1}^Q$ is monotone on $\prod_i \mathscr{P}_i$. Let $\{\varepsilon_n\} \subset [0,\infty)$ be a sequence such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, let ρ_n be such that $\{\rho_n\} \subset [R_m, R_M]$ with $0 < R_m \leq R_M < 2$, let τ be sufficiently large so that $\Upsilon_{ind} + \tau \mathbf{I}$ is a P matrix, and let $\mathbf{S}_\tau(\mathbf{p}^{(n)})$ in Step 2 be computed using the Asynchronous Best-response Algorithm described in Algorithm*

1. Then, the sequence $\{\mathbf{p}^{(n)}\}_{n=0}^{\infty}$ *generated by Algorithm 2 applied to* $\mathscr{G}_{ind}$ *converges to a NE of* $\mathscr{G}_{ind}$.

A sufficient condition for $\mathbf{F}(\mathbf{p})$ *being a monotone mapping on* $\prod_i \mathscr{P}_i$ *is that* $\Upsilon_{ind}(k) \succ \mathbf{0}$ *for all* $k = 1, \ldots, N$ *(cf. Proposition 2).*

Numerical results. In Figure 0.1, we compare the convergence properties of the following algorithms: i) The simultaneous (multilevel) IWFA (best-response algorithm) described in Algorithm 2 and based on the mapping in (0.44); ii) the simultaneous proximal-response algorithm applied to game $\mathscr{G}_{\text{ind}}$, according to which the players solve their regularized optimization problems simultaneously, while changing the "center" of their regularization at each iteration; iii) the simultaneous version of the Proximal Decomposition Algorithm described in Algorithm 2 and applied to game $\mathscr{G}_{\text{ind}}$; and iv) the iterative Tikhonov algorithm applied to the VI$(\prod_i \mathscr{P}_i, (-\nabla_{\mathbf{p}_i} r_i)_{i=1}^{Q})$ associated to $\mathscr{G}_{\text{ind}}$ and proposed in [38] (see also [19, Ch. 15.2]). We refer to these algorithms as Simultaneous Iterative Waterfilling Algorithm (SIWFA), Jacobi Proximal-response Algorithm (JPRA), Jacobi Proximal Decomposition Algorithm (JPDA), and Iterative Distributed Tikhonov Algorithm (IDTA), respectively. Note that JPRA and JPDA differ only in the rule used to update the center of the regularization: in the former, the players change the center at each iteration, whereas in the latter the center is kept fixed for a certain number of iterations (until the condition in step 2 of the algorithm is satisfied).

We consider a hierarchical CR network where there are two PUs (the base stations of two cells) and ten SUs, randomly distributed in the cell. The (cross-)channels among the secondary links and between the primary and the secondary links are simulated as FIR filter of order $L = 10$, where each tap has variance equal to $1/L^2$; the available bandwidth is divided in $N = 64$ subchannels. We focus on two scenarios, namely low/medium interference and high interference scenario. Low/medium interference means that secondary links are far enough from each other so that matrix Υ_{ind} defined in (0.45) is a P-matrix; whereas in the high interference scenario, matrices $\Upsilon_{\text{ind}}(k)$ are positive definite, but Υ_{ind} is not a P-matrix. In Figure 0.1(a) we plotted the average rate evolution of the SUs as a function of the iteration index in the low/medium interference case, corresponding to $\text{snr}_i \triangleq P_i/(\sigma_i^2 d_{ii}^2) = 5\text{dB}$ and $\text{inr}_{ij} \triangleq P_j/(d_{ij}^2 \sigma_i^2) = 0\text{dB}$ for all i and $j \neq i$; whereas in the Figure 0.1(b) we compared the SIWFA with the JPDA in an high interference scenario, corresponding to $\text{snr}_i \triangleq P_i/(\sigma_i^2 d_{ii}^2) = 0\text{dB}$ and $\text{inr}_{ij} \triangleq P_j/(d_{ij}^2 \sigma_i^2) = 5\text{dB}$ for all i and $j \neq i$. To make the picture not excessively overcrowded, we plot only the curves of two out of the ten links. We examined the performance of the above algorithms under the following setup (guaranteeing the fastest convergence speed of each algorithm for the given scenario). In JPRA, we chose $\tau = 1$; in JPDA we set $\tau = 0.2$ and the error sequence defining the inner termination criterion has been chosen as $\varepsilon_n = {\varepsilon_0}^{-n}$ with $\varepsilon_0 = 1e-2$, where n is the outer iteration index; in IDTA we chose the variable step-size sequences $\gamma_n = n^{-0.4}$ and $\delta_n = n^{-0.49}$ so that (sufficient) conditions for the convergence of IDTA given in [19, Prop. 15.1] are satisfied (we use the same notation as in [38]; see therein for the details).

From Figure 0.1(a) we infer that in the low/medium interference regime, the SIWFA, the JPRA and the JPDA converge quite fast (less than 10 iteration) and exhibit

almost the same convergence speed. The IDTA instead requires much more iterations (about 1500 iterations) to converge, which makes it not applicable in practice. The same convergence properties as in Figure 0.1(a) has been experienced for all the channel realizations we simulated. Figure 0.1(b) shows an example where the SIWFA does not converge because of the high interference among the SUs; whereas the proposed JPDA still converges in a few iterations. Interestingly, we experienced convergence of our JPDA even when the matrices $\Upsilon_{\text{ind}}(k)$ are not positive semidefinite, provided that the error sequence $\{\varepsilon_n\}$ are properly chosen.

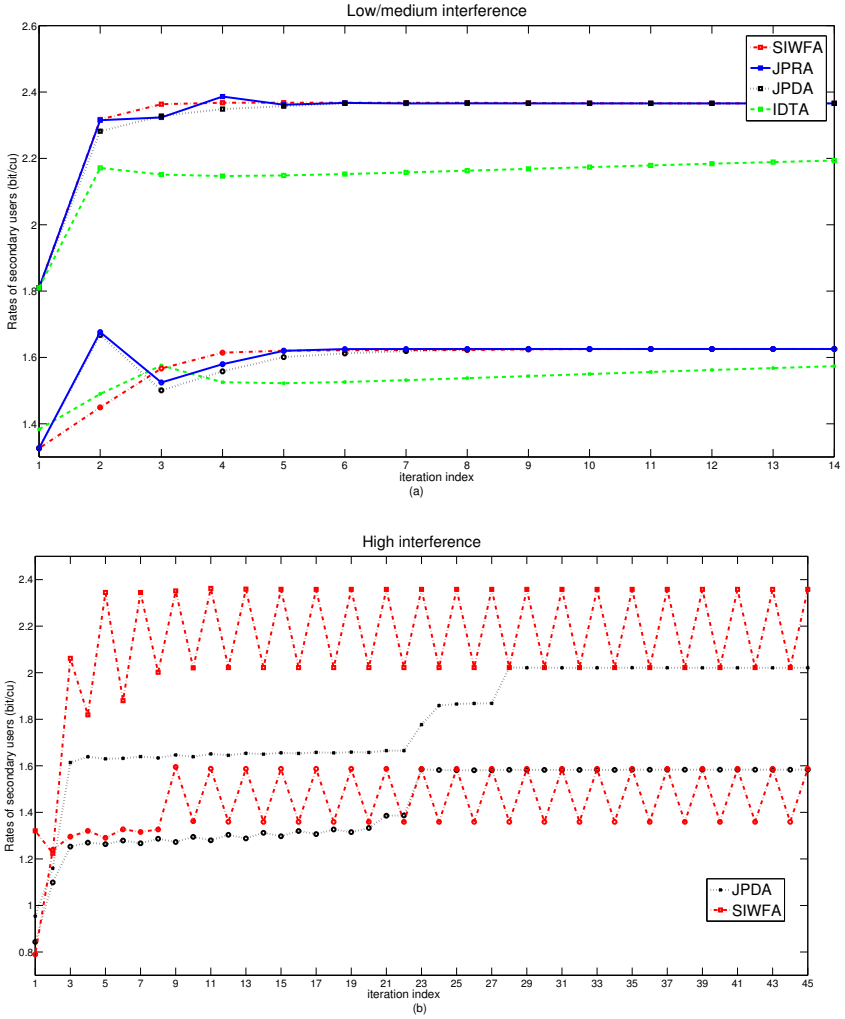

Fig. 0.1 Comparison of distributed algorithms solving the game $\mathscr{G}_{\text{ind}}$ in (0.43): Rates of the SUs' versus the iteration index of two out ten users, achieved by the Simultaneous Iterative Waterfilling Algorithm (SIWFA), the Jacobi Proximal-response Algorithm (JPRA), the Jacobi Proximal Decomposition Algorithm (JPDA), and the Iterative Distributed Tikhonov Algorithm (IDTA) in the low/medium interference regime [subplot (a)] and high interference regime [subplot (b)].

Game Theoretical formulation under individual/global interference constraints. We focus now on the power control problem among the SUs in the presence of both individual and global interference constraints. Because of the global constraints, the game theoretical formulation of this problem leads to a GNEP with shared constraints (cf. Section 0.2), whose shared constrained set is given by

$$\hat{\mathscr{P}} \triangleq \mathscr{P} \cap \left\{ \begin{array}{l} \mathbf{p}: \displaystyle\sum_{i=1}^{Q}\sum_{k=1}^{N} |H_{pi}^{(P,S)}(k)|^2\, p_i(k) \le I_p^{\text{tot}} \quad \forall p = 1,\cdots,P \\ \\ \displaystyle\sum_{i=1}^{Q} |H_{pi}^{(P,S)}(k)|^2\, p_i(k) \le I_p^{\text{peak}}(k),\ \forall p = 1,\cdots,P,\ k = 1,\cdots,N \end{array} \right\}, \tag{0.47}$$

where $\mathscr{P} \triangleq \prod_i \mathscr{P}_i$ and $\mathscr{P}_i$ is defined in (0.42).

Aiming at finding distributed algorithms, we focus our interest on the variational solutions of the GNEP, which, according to Lemma 4, correspond to the solutions of the following NEP with pricing:

$$\begin{array}{l} \underset{\mathbf{p}_i}{\text{maximize}} \quad r_i(\mathbf{p}_i,\mathbf{p}_{-i}) - \displaystyle\sum_{p=1}^{P}\sum_{k=1}^{N} \lambda_{p,k}^{\text{peak}} |H_{pi}^{(P,S)}(k)|^2\, p_i(k) - \sum_{p=1}^{P} \lambda_{p,\text{tot}} \sum_{k=1}^{N} |H_{pi}^{(P,S)}(k)|^2\, p_i(k) \\ \text{subject to} \quad \mathbf{p}_i \in \mathscr{P}_i \end{array} \tag{0.48}$$

for all $i = 1,\cdots,Q$, where $r_i(\mathbf{p}_i,\mathbf{p}_{-i})$ is defined in (0.36), and the prices $\lambda \triangleq \left(\lambda_{\text{tot}};\lambda^{\text{peak}}\right)$, with $\lambda_{\text{tot}} \triangleq (\lambda_{p,\text{tot}})_{p=1}^{P}$ and $\lambda^{\text{peak}} \triangleq ((\lambda_{p,k}^{\text{peak}})_{p=1}^{P})_{k=1}^{N}$, are chosen such that the following complementary conditions are satisfied:

$$\mathbf{0} \le \lambda \ \perp\ \Psi(\mathbf{p}) \ge \mathbf{0} \qquad \Leftrightarrow \qquad \underset{\lambda \ge 0}{\text{minimize}}\, \lambda^T \Psi(\mathbf{p}) \tag{0.49}$$

where

$$\Psi(\mathbf{p}) \triangleq \left(\begin{array}{c} \left(I_p^{\text{tot}} - \displaystyle\sum_{i=1}^{Q}\sum_{k=1}^{N} |H_{pi}^{(P,S)}(k)|^2\, p_i(k) \right)_{p=1}^{P} \\ \left(\left(I_p^{\text{peak}}(k) - \displaystyle\sum_{i=1}^{Q} |H_{pi}^{(P,S)}(k)|^2\, p_i(k) \right)_{k=1}^{N} \right)_{p=1}^{P} \end{array} \right). \tag{0.50}$$

With a slight abuse of terminology, we will refer in the following to the NEP (0.48) with the complementarity constraints (0.49) as game $\mathscr{G}_{\text{glob}}$.

To study the above game we need the following intermediate definitions, based on results in Section 0.2.2. Given the column vector $\mathbf{h} \triangleq (||\sum_{p=1}^{P} \mathbf{H}_{pi}^{(P,S)}||_2)_{i=1}^{Q}$ with $\mathbf{H}_{pi}^{(P,S)} \triangleq (|H_{pi}^{(P,S)}(k)|^2)_{k=1}^{N}$, the matrix Υ_τ in (0.35) associated to the problem (0.48)-(0.49) becomes

$$\Upsilon_{\text{ind},\tau} \triangleq \left[\begin{array}{c|c} \Upsilon_{\text{ind}} + \tau \mathbf{I} & -\mathbf{h} \\ \hline -\mathbf{h}^T & \tau \end{array} \right] \tag{0.51}$$

where $\Upsilon_{\rm ind}$ is defined in (0.45). Among the several distributed schemes proposed in the first part of the chapter to compute the solutions of $\mathscr{G}_{\rm glob}$, here we focus on the following, leaving to the reader the task of specializing the other proposed algorithms to game $\mathscr{G}_{\rm glob}$: i) Projection Algorithm with Variable Steps (PAVS), described in Algorithm 3; and ii) Proximal Decomposition Algorithm (PDA) described in Algorithm 2. The Step 2 of Algorithm 3−the computation of the NE of the game in (0.48) for $\lambda = \lambda^{(n)}$−and the Step 2 of Algorithm 2−the computation of the NE $\mathbf{S}_\tau((\mathbf{p}^{(n)}, \lambda^{(n)}))$ of the game obtained by the proximal regularization of (0.48)-(0.49) [see (0.34)]−can be efficiently computed using the asynchronous best-response algorithm described in Algorithm 1, whose best-response for both games has a closed form (multi-level waterfilling) expression [30]. Building on Theorems 4, 5, and 6, we obtain the following results for the problem $\mathscr{G}_{\rm glob}$.

Theorem 9. *Given the problem $\mathscr{G}_{glob}$, suppose w.l.o.g. that conditions (0.41) are satisfied; let $\mathbf{F}(\mathbf{p}) \triangleq (-\nabla_{\mathbf{p}_i} r_i(\mathbf{p}_i, \mathbf{p}_{-i}))_{i=1}^{Q}$. Then, the following statements hold.*

(a) $\mathscr{G}_{glob}$ has a nonempty and bounded solution set;

(b) Suppose that $\Upsilon_{ind} \succ \mathbf{0}$. Then: i) the power vector $\mathbf{p}^\star$ at every solution of $\mathscr{G}_{glob}$ is unique; and ii) any sequence $\{\lambda^{(n)}, \mathbf{p}^\star(\lambda^{(n)})\}_{n=0}^{\infty}$ generated by the PAVS converges to a solution of $\mathscr{G}_{glob}$, provided that the scalars τ_n are chosen so that $0 < \inf_n \tau_n \le \sup_n \tau_n < 2\lambda_{least}(\Upsilon_{ind})/\|\mathbf{h}\|_2^2$;

(c) Suppose that the mapping $\mathbf{F}(\mathbf{p})$ is monotone on $\hat{\mathscr{P}}$. Let $\{\varepsilon_n\} \subset [0,\infty)$ be a sequence such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, let ρ_n be such that $\{\rho_n\} \subset [R_m, R_M]$ with $0 < R_m \le R_M < 2$, and let τ be sufficiently large so that $\Upsilon_{{\rm ind},\tau}$ defined in (0.51) is a P-matrix. Then, the sequence $\{\mathbf{p}^{(n)}, \lambda^{(n)}\}_{n=0}^{\infty}$ generated by the PDA converges to a solution of $\mathscr{G}_{glob}$.

A sufficient condition for $\mathbf{F}(\mathbf{p})$ being a monotone mapping on $\hat{\mathscr{P}}$ is that $\Upsilon_{ind}(k) \succ \mathbf{0}$, for all $k = 1, \ldots, N$ (cf. Proposition 2).

Remark 10 (Implementation of the algorithms). In the PAVS and PDA, there are two levels of updates: 1) the computation of the optimal power allocations of the SUs, given the prices λ; and 2) the updates of the price vector, given the interference generated by the SUs over the N subcarriers. The former can be performed directly by the SUs via the asynchronous best-response algorithm described in Algorithm 1. Note that, once $\sum_{p=1}^{P}(\lambda_{p,k}^{\rm peak} + \lambda_{p,{\rm tot}})|H_{pi}^{(P,S)}(k)|^2$ are given, these algorithms are totally distributed, since the SUs only need to measure the received MUI over the N subcarriers [27]. The update of the price vector can be performed either by the PUs or by the SUs themselves, depending on the debate position assumed for the CR network (the property-right model or the common model [14]). More specifically, in a property-right model, an interaction between the PUs and the SUs is allowed. It is thus natural that the update of the prices is performed by the PUs. Note that the signaling from the SUs to the PUs is implicit, since the PUs to update the prices only need to locally measure the global received interference [the function $\Psi(\mathbf{p})$]. The signaling from the PUs to the SUs users, however, is explicit: the PUs have to broadcast the prices and the SUs receive and estimate their values. In a CR network based on the common model, the PUs are oblivious of the presence of the

SUs and thus the update of prices needs to be performed by the SUs. Building on consensus algorithms [22, 29], at the price of additional signaling among the SUs and computational complexity, the proposed algorithms still can be implemented in a distributed fashion by the SUs. Details can be found in [27].

Numerical results. In Figure 0.2 we compare the convergence properties of the following algorithms applied to the game $\mathscr{G}_{\text{glob}}$: i) PAVS; ii) PDA; and iii) the algorithm proposed in [16]; we refer to these algorithms as *Gradient Projection IWFA* (GP-IWFA), *Full Proximal IWFA* (FP-IWFA), and *Price based IWFA* (PB-IWFA) respectively. The setup is the same as the one considered in Figure 0.1 for the low/medium interference regime. In addition to the individual constraints, now the PUs impose global per-carrier interference constraints, assumed for the sake of simplicity equal over all the subcarriers. For the scenario considered in the picture, one can see that the convergence of both GP-IWFA and FP-IWFA is reasonable fast (less than 30 iterations), whereas the PB-IWFA requires much more iterations (more than 500).

Thanks to less stringent constraints on the transmit powers of the SUs, the performance of the SUs achievable under global interference constraints (game $\mathscr{G}_{\text{glob}}$) are expected to be much better than those achievable under the more conservative individual interference constraints (game $\mathscr{G}_{\text{ind}}$). Figure 0.3 confirms this intuition, where we plot the sum-rate of the SUs (averaged over 500 random i.i.d. Gaussian channel realizations) achievable in the two aforementioned cases as a function of the maximum tolerable interference at the PUs, within the same setup of Figure 0.1.

Concluding Remarks

The first part of this chapter was devoted to provide the (basic) theoretical tools and methods based on VIs to analyze some fundamental issues of NEPs and GNEPs with jointly shared convex constraints, such as the existence and uniqueness of a solution, and the design of iterative distributed algorithms along with their convergence properties. The second part of the chapter made these theoretical results practical by showing how the proposed framework can be successfully applied to solve some challenging equilibrium problems in CR networks.

Acknowledgments

The work of Scutari and Pang is based on research supported by the U.S.A. National Science Foundation grant CMMI 0969600 and by the Air Force Office of Sponsored Research award No. FA9555-09-10329. The work of Palomar was supported by the Hong Kong RGC 618709 research grant. The work of Facchinei was partially supported by MIUR, Research program PRIN 2007-9PLLN7 Nonlinear Optimization Variational Inequalities and Equilibrium Problems, Italy.

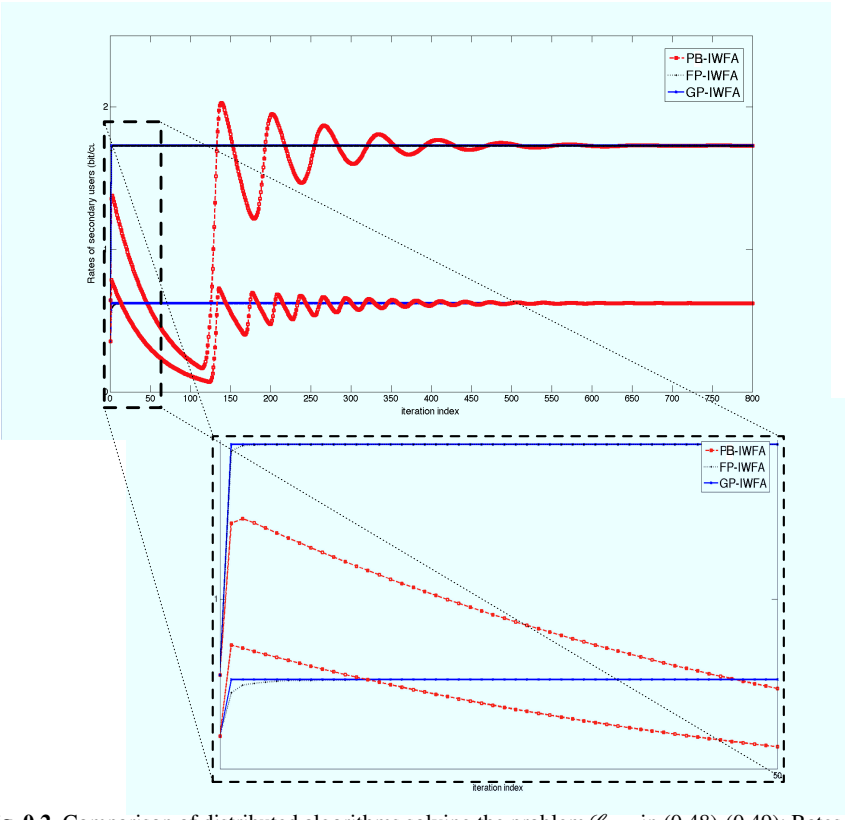

Fig. 0.2 Comparison of distributed algorithms solving the problem $\mathscr{G}_{\text{glob}}$ in (0.48)-(0.49): Rates of the SUs' versus the iteration index of two out ten users, achieved by the Price based IWFA (PB-IWFA) [16], the Gradient Projection IWFA (GP-IWFA) and the Full Proximal IWFA (FP-IWFA) in the low/medium interference regime.

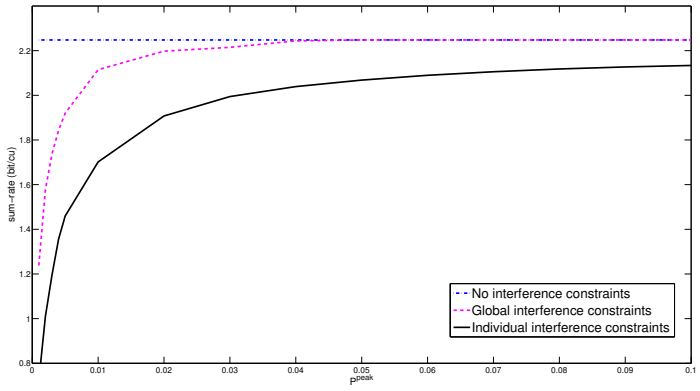

Fig. 0.3 Average sum-rate versus the (peak) interference constraint achievable under no interference, local, and global interference constraints.

References

1. E. Altman, T. Boulogne, R. El Azouzi, T. Jimenez, and L. Wynter. A survey on networking games. *Computers and Operations Research*, 33:286–311, 2006.
2. Jean-Pierre Aubin. *Mathematical Method for Game and Economic Theory*. Dover Publications, Nov. 2007.
3. R. B. Bapat and T. E. S. Raghavan. *Nonnegative Matrices and Applications*. Cambridge University Press, 1997.
4. Abraham Berman and Robert J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Society for Industrial Mathematics (SIAM), 1987.
5. D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific Press, 2nd edition, 1989.
6. R. Cendrillon, J. Huang, M. Chiang, and M. Moonen. Autonomous spectrum balancing for digital subscriber lines. *IEEE Trans. on Signal Processing*, 55(8):4241–4257, Aug. 2007.
7. R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Cambridge Academic Press, 1992.
8. R. Etkin, A. Parekh, and D. Tse. Spectrum sharing for unlicensed bands. *IEEE Jour. on Selected Areas in Communications*, 25(3):517–528, April 2007.
9. F. Facchinei, A. Fischer, and V. Piccialli. On generalized nash games and variational inequalities. *Operations Research Letters*, 35:159 –164, 2007.
10. F. Facchinei and C. Kanzow. Generalized Nash equilibrium problems. *A Quarterly Journal of Operations Research (4OR)*, 5(3):173–210, Sept. 2007.
11. F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problem*. Springer-Verlag, New York, 2003.
12. F. Facchinei and Jong-Shi Pang. Nash equilibria: The variational approach. In D. P. Palomar and Y. C. Eldar, editors, *Convex Optimization in Signal Processing and Communications*, chapter 12, pages 443–493. Cambridge University Press, London, 2009.
13. Francisco Facchinei and Simone Sagratella. On the computation of all solutions of jointly convex generalized nash equilibrium problems. *Optimization Letters*, (to appear).
14. A. Goldsmith, S. A. Jafar, I. Maric, and S. Srinivasa. Breaking spectrum gridlock with cognitive radios: An information theoretic perspective. *Proceedings of the IEEE*, 2009.
15. S. Haykin. Cognitive radio: Brain-empowered wireless communications. *IEEE Jour. on Selected Areas in Communications*, 23(2):201–220, Feb. 2005.
16. M. Hong and A. Garcia. Dynamic pricing of interference in cognitive radio networks. (submitted) 2010.
17. J. Huang, D. P. Palomar, N. Mandayam, J. Walrand, S. B. Wicker, and T. Basar. *IEEE Jour. on Selected Areas in Communications, special issue on Game Theory in Communication Systems*, 26(7), September 2008.
18. E. A. Jorswieck, E. G. Larsson, M. Luise, and H. V. Poor. *IEEE Signal Processing Magazine, special issue on Game Theory in Signal Proc. and Comm.*, 26(5), September 2009.

19. I. V. Konnov. *Equilibrium Models and Variational Inequalities*. Elsevier B.V., Amsterdam, 2007.
20. Z.-Q. Luo and J.-S. Pang. Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines. *EURASIP Jour. on Applied Signal Processing*, 2006:1–10, May 2006.
21. J. Mitola. Cognitive radio for flexible mobile multimedia communication. In *IEEE 1999 International Workshop on Mobile Multimedia Communications (MoMuC 1999)*, pages 3–10, Sandiego, California, Usa, Nov. 15-17 1999.
22. R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–223, Jan. 2007.
23. J. M. Ortega and W. C. Rheinboldt. *Iterative Solution of Nonlinear Equations in Several Variables*. Society for Industrial Mathematics (SIAM), Phila, USA, 1987.
24. J.-S. Pang. Asymmetric variational inequality problems over product sets: Applications and iterative methods. *Mathematical Programming*, 31(2):206–219, June 1985.
25. J.-S. Pang and D. Chan. Iterative methods for variational and complementarity problems. *Mathematical Programming*, 24(1):284–313, Dec. 1982.
26. J.-S. Pang, G. Scutari, F. Facchinei, and C. Wang. Distributed power allocation with rate constraints in Gaussian parallel interference channels. *IEEE Trans. on Information Theory*, 54(8):3471–3489, Aug. 2008.
27. J.-S. Pang, G. Scutari, D. P. Palomar, and F. Facchinei. Design of cognitive radio systems under temperature-interference constraints: A variational inequality approach. *IEEE Trans. on Signal Processing*, 58(6):3251–3271, June 2010.
28. J. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33(3):520–534, July 1965.
29. G. Scutari, S. Barbarossa, and L. Pescosolido. Distributed decision through self-synchronizing sensor networks in the presence of propagation delays and asymmetric channels. *IEEE Trans. on Signal Processing*, 56(4):1667–1684, April 2008.
30. G. Scutari, F. Facchinei, J.-S. Pang, and D. P. Palomar. Monotone communication games: Theory, algorithms, and models. *IEEE Trans. on Information Theory (submitted)*, 2010.
31. G. Scutari and D. P. Palomar. MIMO cognitive radio: A game theoretical approach. *IEEE Trans. on Signal Processing*, 58(2):761–780, February 2010.
32. G. Scutari, D. P. Palomar, and S. Barbarossa. Asynchronous iterative water-filling for Gaussian frequency-selective interference channels. *IEEE Trans. on Information Theory*, 54(7):2868–2878, July 2008.
33. G. Scutari, D. P. Palomar, and S. Barbarossa. Competitive design of multiuser MIMO systems based on game theory: A unified view. *IEEE Jour. on Selected Areas in Communications*, 26(7):1089–1103, Sept. 2008.
34. G. Scutari, D. P. Palomar, and S. Barbarossa. Optimal linear precoding strategies for wideband noncooperative systems based on game theory—part I & II: Nash equilibria & Algorithms. *IEEE Trans. on Signal Processing*, 56(3):1230–1249 & 1250–1267, March 2008.
35. G. Scutari, D. P. Palomar, and S. Barbarossa. Competitive optimization of cognitive radio MIMO systems via game theory. In D. P. Palomar and Y. C. Eldar, editors, *Convex Optimization in Signal Processing and Communications*, chapter 11, pages 387–442. Cambridge University Press, London, 2009.
36. G. Scutari, D. P. Palomar, and S. Barbarossa. The MIMO iterative waterfilling algorithm. *IEEE Trans. on Signal Processing*, 57(5), May 2009.
37. G. Scutari, D. P. Palomar, F. Facchinei, and J.-S. Pang. Convex optimization, game theory, and variational inequality theory in multiuser communication systems. *IEEE Signal Processing Magazine*, 27(4):35–49, May 2010.
38. H. Yin, U.V. Shanbhag, and P. G. Mehta. Nash equilibrium problems with congestion costs and shared constraints. *IEEE Transactions on Automatic Control*, (to appear). Available at https://netfiles.uiuc.edu/udaybag/www/Papers/FlowControl_tac_v12.pdf.
39. W. Yu, G. Ginis, and J. M. Cioffi. Distributed multiuser power control for digital subscriber lines. *IEEE Jour. on Selected Areas in Communications*, 20(5):1105–1115, June 2002.