Noncooperative and Cooperative Optimization of Distributed Energy Generation and Storage in the Demand-Side of the Smart Grid

Italo Atzeni, Luis G. Ordóñez, Gesualdo Scutari, Daniel P. Palomar, and Javier R. Fonollosa

***Abstract*—The electric energy distribution infrastructure is undergoing a startling technological evolution with the development of the smart grid concept, which allows more interaction between the supply- and the demand-side of the network and results in a great optimization potential. In this paper, we focus on a smart grid in which the demand-side comprises traditional users as well as users owning some kind of distributed energy source and/or energy storage device. By means of a day-ahead demand-side management mechanism regulated through an independent central unit, the latter users are interested in reducing their monetary expense by producing or storing energy rather than just purchasing their energy needs from the grid. Using a general energy pricing model, we tackle the grid optimization design from two different perspectives: a user-oriented optimization and an holistic-based design. In the former case, we optimize each user individually by formulating the grid optimization problem as a noncooperative game, whose solution analysis is addressed building on the theory of variational inequalities. In the latter case, we focus instead on the joint optimization of the whole system, allowing some cooperation among the users. For both formulations, we devise distributed and iterative algorithms providing the optimal production/storage strategies of the users, along with their convergence properties. Among all, the proposed algorithms preserve the users' privacy and require very limited signaling with the central unit.**

***Index Terms*—Demand-side management, distributed pricing algorithm, game theory, proximal decomposition algorithm, smart grid, variational inequality.**

I. Introduction

THE term *"smart grid"* refers to a manifold of concepts, solutions, and products. Still, no internationally unified definition for smart grids has been adopted [1]. Energy regulators describe the smart grid as an electricity network that can cost-efficiently integrate all users connected to it—generators, consumers, and those who do both—in order to ensure economically-efficient, sustainable power systems with low losses, high levels of quality and security of supply, and improved safety [2]. The smart grids task force set up by the European Commission goes one step beyond and includes smart metering and bidirectional communication capabilities as inherent parts of smart grids [3]. Indeed, smart metering and the related smart communication infrastructure provide information to the different grid users (distribution system operators, retailers, service-providers, and end users) and allow interactions among all of them. This opens up unprecedented possibilities for optimizing the energy grid and the energy usage at different network levels.

Not surprisingly, these premises are arousing the interest of the signal processing community. Indeed, the smart grid concept has been recognized as "*a major initiative related to the field of energy with significant signal processing content*" which requires expertise in the fields of communication, sensing, analysis, and actuation [4]. The first publications were mainly focused on the communication aspects of the smart grid. However, these technologies are only an enabler of the envisioned smart grid and, most importantly, they are not the sole aspects that can benefit from the contribution of the signal processing community.[1]

Recently, there has been a growing interest in adopting cooperative and noncooperative game theory to model the interaction among the smart grid users (see [6], [7] for an overview on this topic). In particular, real-time and day-ahead energy consumption scheduling (ECS) techniques, common demand-side management (DSM) procedures that intend to modify the demand profile by shifting energy consumption to off-peak hours, have been recently studied in literature using game theoretical approaches (see, e.g., [8]–[11]). However, since the users' inconvenience[2] must be taken into account, ECS presents limitations in terms of flexibility that can be overcome by incorporating distributed generation (DG) and distributed storage (DS) into the demand-side of the network.

In this paper, we propose a DSM method consisting in a day-ahead optimization process that corresponds to energy pro-

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[1]As evidence in support of this statement, the IEEE Signal Processing Magazine published a special issue entitled "Signal Processing Techniques for the Smart Grid" [5] during the reviewing process of the present paper.

[2]Note that ECS implies no cost for the residential customer, but this is not the case for the industrial customer, for whom the rescheduling of activities may result in monetary loss [12].

duction and energy storage scheduling rather than shifting energy consumption as in ECS techniques. We associate to each demand-side user, possibly owning a DG and/or a DS device, an energy consumption vector containing his energy requirements for each time-slot in which the time period of analysis is divided. Here, we assume that this vector is set a priori by each user according to his needs or as the result of an ECS algorithm. In doing so, we suppose that, by participating in the day-ahead optimization process, demand-side users commit to follow strictly the resulting consumption pattern.[3] The main objective of these end users is to reduce their monetary expense during the time period of analysis by producing or storing energy rather than just purchasing their energy needs from the grid.

DSM techniques have been traditionally formulated from the selfish point of view of the end users. However, it has been demonstrated that a collaborative approach can be more beneficial for all actors in the energy grid by minimizing, e.g., the peak-to-average ratio (PAR) of the energy demand or the total energy cost [10]. In this paper, we attack the grid optimization problem from two different perspectives, namely: a user-oriented optimization and an holistic-based design. More specifically, in the first approach, we formulate the DSM design as a noncooperative game where the end users act as players with objective functions and optimization variables given by their individual monetary expenses and production/storage strategies, respectively. Building on the variational inequality (VI) framework [14]–[16], we study the existence of a solution for the proposed game, the Nash equilibrium (NE); we obtain sufficient conditions on the energy cost functions guaranteeing the existence of Nash equilibria. Quite interestingly, we prove that all the solutions are equivalent, in the sense that the optimal value of the players' objective function is constant over the set of the Nash equilibria. We then focus on distributed algorithms solving the game; we propose a proximal-based best-response scheme and derive sufficient conditions guaranteeing its convergence to any of the (equivalent) Nash equilibria.

The second method we propose consists in formulating the DSM design as a standard nonlinear optimization problem, where one minimizes the overall expense incurred by the demand-side of the network. This approach is more suitable for "collaborative" contexts, where the users are willing to exchange some (limited) signaling in favor of better performance as, for example, when an energy retailer acts as intermediary between the supply-side and a group of subscribers. To solve the resulting nonconvex optimization problem, we build on the recent results in [17], [18] and introduce a distributed dynamic pricing-based algorithm (DDPA) that converges to a stationary solution of the problem.

The proposed algorithms have many desired (complementary) features, which make them applicable to alternative scenarios. For instance, the DDPA i) requires essentially the same signaling as the PDA (which is based on a noncooperative approach), ii) is proved to converge under very mild assumptions (always satisfied in practice), and iii) has fast convergence speed (considerably faster than the scheme presented in [10]). However, despite having the same communication cost as the PDA, the DDPA is not incentive compatible, implying that its best-response update must be imposed as a protocol to the demand-side users, in order to avoid selfish deviations from it. The PDA, instead, can be implemented by selfish users; moreover, quite surprisingly, numerical results show that it yields the same performance as the DDPA (at least for the scenarios simulated in this paper), but its convergence conditions are more stringent than those of the DDPA. Lastly, the PDA is based on a totally asynchronous update of the users' strategies, as opposed to the DDPA and the synchronous user-oriented DSM method presented in [19].

Notably, both approaches addressed in this paper are valid for a general energy pricing model, which includes the energy pricing used in [19] as a special case. Furthermore, they equivalently allow to achieve a generally flattened energy demand curve, from which both demand- and supply-side benefit in terms of reduced energy cost and CO_2 emissions, as well as overall power plants and capital cost requirements [1].

The rest of the paper is structured as follows. In Section II, we introduce the smart grid, the production, and the storage models, as well as the energy cost and pricing model. Section III formulates the grid optimization problem as a Nash game; we then derive sufficient conditions for the existence of a solution, propose a distributed algorithm solving the game, and study its convergence. In Section IV, we present an holistic-based optimization of the system and devise an efficient, distributed algorithm for computing its solutions. Section V shows some experiments, whereas Section VI draws the conclusions.

II. Smart Grid Model

The modern electric grid is a complex network comprising several subsystems, which, for our purposes, can be conveniently divided into [20]–[22] (see Fig. 1):

(i) *Supply-side*: it incorporates the utilities (energy producers) and the energy transmission network;
(ii) *Central unit*: it is the regulation authority that coordinates the grid optimization process. It serves both as independent system operator, by maintaining the reliability of a control area and optimally matching energy supply and demand, and as market operator, by fixing the energy price in the day-ahead market;
(iii) *Demand-side*: it includes the end users (energy consumers), possibly equipped with DG and/or DS, energy retailers, and the energy distribution network.

Since in this paper we are designing a DSM mechanism, we focus in particular on the end users, whereas the supply-side of the smart grid and the central unit are modeled as simply as possible.

A. Demand-Side Model

Demand-side users, whose associated set is denoted by $\mathcal{D}$, are characterized in the first instance by their individual *per-slot energy consumption profile* $e_n(h)$, defined as the energy needed by user $n \in \mathcal{D}$ to supply his appliances at time-slot h. Accordingly, we also introduce the *energy consumption vector* $\mathbf{e}_n$, which gathers the energy consumption profiles for the H

[3]We refer to [13] for an extended grid model that allows real-time deviations with respect to the negotiated demand, and where the day-ahead energy requirements follow from a bidding process based on the individual consumption statistics.

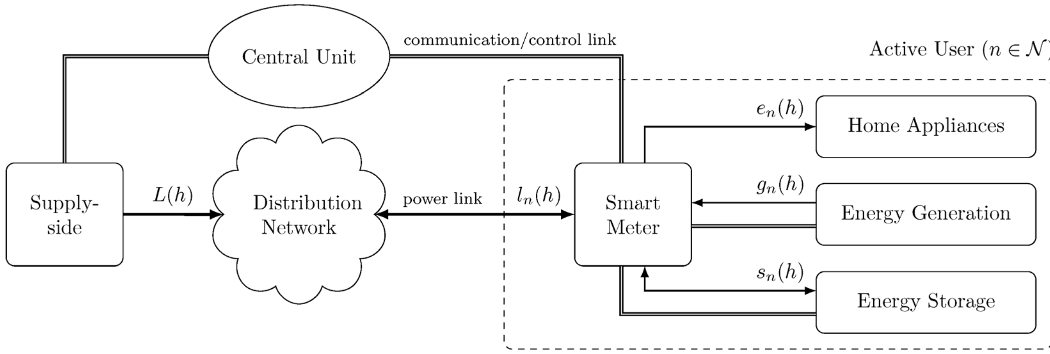

Fig. 1. Connection scheme between one end user and the smart grid.

TABLE I
CHARACTERISTICS OF THE DIFFERENT TYPES OF DEMAND-SIDE USERS

User subset		Energy load profile	Outg. capacity	Strategy set
$\mathcal{P}$	$\mathcal{P}\backslash\mathcal{G}_{\mathrm{R}}$	$l_n(h) = e_n(h)$	$l_n^{(\min)} = 0$	No strategy
	$\mathcal{G}_{\mathrm{R}}\backslash(\mathcal{N}\cap\mathcal{G}_{\mathrm{R}})$		$l_n^{(\min)} \geq 0$	
$\mathcal{N}$	$\mathcal{G}\backslash(\mathcal{G}\cap\mathcal{S}) = \mathcal{N}_{\mathcal{G}\backslash\mathcal{S}}$	$l_n(h) = e_n(h) - g_n(h)$	$l_n^{(\min)} \geq 0$	$\mathbf{g}_n \in \Omega_{\mathbf{g}_n}$
	$\mathcal{S}\backslash(\mathcal{G}\cap\mathcal{S}) = \mathcal{N}_{\mathcal{S}\backslash\mathcal{G}}$	$l_n(h) = e_n(h) + s_n(h)$		$\mathbf{s}_n \in \Omega_{\mathbf{s}_n}$
	$\mathcal{G}\cap\mathcal{S} = \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$	$l_n(h) = e_n(h) - g_n(h) + s_n(h)$		$(\mathbf{g}_n, \mathbf{s}_n) \in (\Omega_{\mathbf{g}_n} \times \Omega_{\mathbf{s}_n})$

time-slots in which the time period of analysis is divided, i.e., $\mathbf{e}_n \triangleq \left(e_n(h)\right)_{h=1}^{H}$. We assume that demand-side users know exactly their energy requirements at each time-slot in the time period of analysis in advance. A stochastic formulation that deals with the uncertainty induced by the end users' energy consumption and renewable generation is addressed in [13].

Our demand-side model distinguishes between *passive* and *active* users. Passive users are basically energy consumers and resemble traditional demand-side users, whereas active users denote those consumers participating in the optimization process, i.e., reacting to changes in the cost per unit of energy by modifying their demand. Hence, each active user is connected not only to the bidirectional power distribution grid, but also to a communication infrastructure that enables two-way communication between his smart meter and the central unit, as shown in Fig. 1. For convenience, we group the P passive users in the set $\mathcal{P} \subset \mathcal{D}$ and the N active users in the set $\mathcal{N} \triangleq \mathcal{D} \setminus \mathcal{P}$.

Furthermore, active users include two broad categories: dispatchable energy producers and energy storers. We use $\mathcal{G} \subseteq \mathcal{N}$ to denote the subset of users possessing some dispatchable energy generator. For users $n \in \mathcal{G}$, $g_n(h) \geq 0$ represents the *per-slot energy production profile* at time-slot h, to which corresponds the *energy production scheduling vector* $\mathbf{g}_n \triangleq \left(g_n(h)\right)_{h=1}^{H}$. Likewise, we introduce $\mathcal{S} \subseteq \mathcal{N}$ as the subset of users owning some energy storage device. Users $n \in \mathcal{S}$ are characterized by the *per-slot energy storage profile* $s_n(h)$ at each time-slot h: $s_n(h) > 0$ when the storage device is to be charged (implying an additional energy consumption), $s_n(h) < 0$ when the storage device is to be discharged (resulting in a reduction of the energy consumption), and $s_n(h) = 0$ when the storage device is inactive. The per-slot energy storage profiles are gathered in the *energy storage scheduling vector* $\mathbf{s}_n \triangleq \left(s_n(h)\right)_{h=1}^{H}$. It is worth remarking that $\mathcal{G} \cup \mathcal{S} = \mathcal{N}$, but we also contemplate the possibility of some active users being both dispatchable energy producers and storers, i.e., $\mathcal{G} \cap \mathcal{S} \neq \emptyset$, as shown in Fig. 2.

Finally, let us introduce the *per-slot energy load profile* as

$$l_n(h) \triangleq \begin{cases} e_n(h), & \text{if } n \in \mathcal{P} \\ e_n(h) - g_n(h) + s_n(h), & \text{if } n \in \mathcal{N} \end{cases} \quad (1)$$

which gives the energy flow between user n and the grid at time-slot h, as shown schematically in Fig. 1. Observe that $l_n(h) > 0$ if the energy flows from the grid to user n and $l_n(h) < 0$ otherwise. Due to physical constraints on the user's individual grid infrastructure, the per-slot energy load profile is bounded as

$$-l_n^{(\min)} \leq l_n(h) \leq l_n^{(\max)} \quad (2)$$

where $l_n^{(\min)} \geq 0$ and $l_n^{(\max)} > 0$ are the outgoing and the incoming capacities of user n's energy link, respectively. These capacities are negotiated between the users and the energy provider and are thus known to the central unit for each user $n \in \mathcal{D}$. The energy load profiles and capacities for the different demand-side users are provided in Table I.

B. Energy Production Model

Let us first characterize energy producers depending upon the type of DG they employ, as done in [19].

Non-dispatchable DG: $\mathcal{G}_{\mathrm{R}} \subset \mathcal{D}$, e.g, renewable resources of intermittent nature such as solar panels and wind turbines. These energy producers generate electricity at their maximum capacity whenever possible since they only have fixed costs and, therefore, they do not adopt any strategy regarding energy production. For convenience, we consider that the per-slot energy consumption profile $e_n(h)$ already takes into account the non-dispatchable energy production of each user $n \in \mathcal{G}_{\mathrm{R}}$.

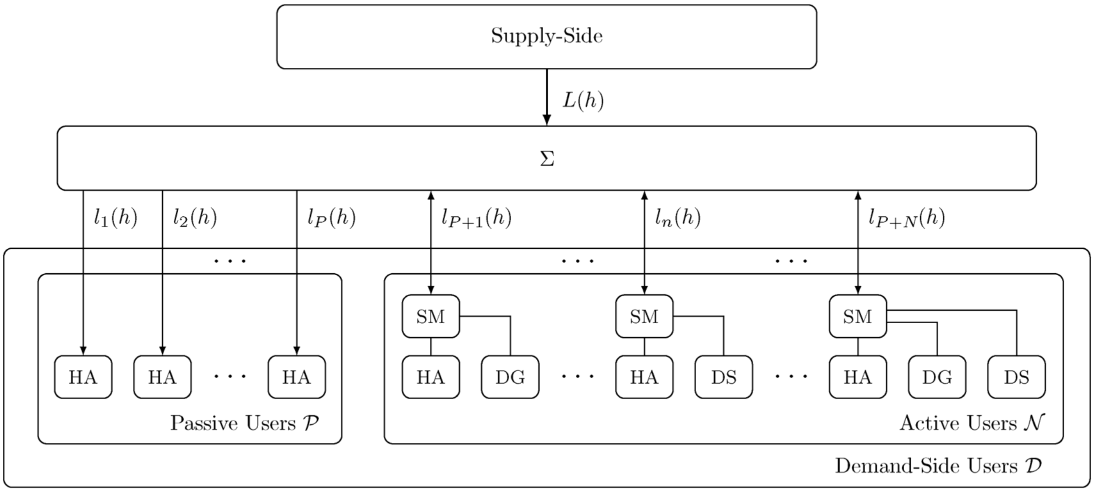

Fig. 2. Supply-side model and demand-side model including the sets of passive users $\mathcal{P}$ and active users $\mathcal{N}$.

Hence, for this type of users, we can have $e_n(h) \leq 0$ when the non-dispatchable energy production is greater than the energy consumption at a given time-slot h. Observe that any demand-side user can belong to $\mathcal{G}_{\mathrm{R}}$ regardless of his condition of passive or active participant in the day-ahead optimization process.

Dispatchable DG: $\mathcal{G} \subseteq \mathcal{N}$, e.g., internal combustion engines, gas turbines, or fuel cells, to be operated mostly during high demand hours in order to lower the peak in the load curve. These energy producers, beside fixed costs, have also variable production costs (due to, e.g., the fuel) and they are thus interested in optimizing their energy production strategies. We introduce accordingly the *production cost function* $W_n\big(g_n(h)\big)$, which gives the variable production costs for generating the amount of energy $g_n(h)$ at time-slot h, with $W_n(0) = 0$.

In the following, we provide, as an example, the dispatchable production model adopted in [19]. It is important to remark, however, that the optimization process analysis and algorithms provided in Sections III and IV hold for any production model resulting in a compact and convex strategy set. Dispatchable energy producers $n \in \mathcal{G}$ are characterized in [19] by their maximum energy production capability $g_n^{(\max)}$ and their capacity factor requirements, i.e., the minimum and maximum amount of energy generated during the time period of analysis, $\gamma_n^{(\min)}$ and $\gamma_n^{(\max)}$, so as to remain efficient. The strategy set $\Omega_{\mathbf{g}_n}$ for dispatchable energy producers $n \in \mathcal{G}$ is consequently defined as (see [19, Sec. II-B] for details)

$$\Omega_{\mathbf{g}_n} \triangleq \left\{\mathbf{g}_n \in \mathbb{R}_+^H : \mathbf{g}_n \preceq g_n^{(\max)} \mathbf{1}_H, \gamma_n^{(\min)} \leq \mathbf{1}_H^{\mathrm{T}} \mathbf{g}_n \leq \gamma_n^{(\max)}\right\} \tag{3}$$

where the operator $\preceq$ for vectors is defined componentwise, and $\mathbf{1}_H$ denotes the H-dimensional unit vector.

C. Energy Storage Model

Let us present, for illustration purposes, a simplified version of the energy storage model introduced in [19]. Nonetheless, as pointed out in the previous section for dispatchable energy producers, any storage model resulting in a compact and convex strategy set ensures the validity of the results in Sections III and IV.

We characterize storage devices by the following three attributes: leakage rate, capacity, and maximum charging rate.[4] The leakage rate $0 < \alpha_n \leq 1$ models the decrease in the energy level of the storage device with the passage of time: let $q_n(h)$ denote the charge level at time-slot h, indicating the amount of energy contained in the storage device of user $n \in \mathcal{S}$ at the end of time-slot h, then $q_n(h)$ reduces to $\alpha_n q_n(h)$ at the end of time-slot $h+1$. The capacity c_n denotes the maximum amount of energy that the storage device can accumulate. Lastly, the maximum charging rate $s_n^{(\max)}$ represents the maximum amount of energy that can be charged into the device during a time-slot. Observe that charging and discharging are mutually exclusive operations during the same time-slot, which results from the leakage of the storage device. Additionally, it is convenient to include a constraint on the desired charge level at the end of the time period of analysis $q_n(H)$. Following the discussion in [19, Sec. II-C], we impose that

$$\big|q_n(H) - q_n(0)\big| \leq \epsilon_n \tag{4}$$

where $q_n(0)$ denotes the initial charge level and ϵ_n is a sufficiently small constant. Finally, we can define the strategy set $\Omega_{\mathbf{s}_n}$ for energy storers $n \in \mathcal{S}$ as (see [19, Sec. II-C] for details)

$$\begin{aligned}\Omega_{\mathbf{s}_n} \triangleq \big\{&\mathbf{s}_n \in \mathbb{R}^H : \mathbf{s}_n \preceq s_n^{(\max)} \mathbf{1}_H, \\ &- q_n(0)\mathbf{b}_n \preceq \mathbf{A}_n \mathbf{s}_n \preceq c_n \mathbf{1}_H - q_n(0)\mathbf{b}_n, \\ &(1-\alpha_n^H) q_n(0) - \epsilon_n \leq \mathbf{a}_n^{\mathrm{T}} \mathbf{s}_n \leq (1-\alpha_n^H) q_n(0) + \epsilon_n\big\}\end{aligned} \tag{5}$$

where $\mathbf{A}_n$ is a $H \times H$ lower triangular matrix with elements $[\mathbf{A}_n]_{i,j} \triangleq \alpha_n^{(i-j)}$, and $\mathbf{a}_n$ and $\mathbf{b}_n$ are H-dimensional vectors with elements $[\mathbf{a}_n]_i \triangleq \alpha_n^{(H-i)}$ and $[\mathbf{b}_n]_i \triangleq \alpha_n^i$, respectively.

Now that we have gone through all possible types of users in the demand-side, we summarize their main characteristics in Table I.

[4]The storage model in [19] also takes into account charging and discharging efficiencies, which are not considered here for clarity of presentation.

TABLE II
LIST OF IMPORTANT SYMBOLS WITH CORRESPONDING DEFINITIONS AND DOMAIN

Symbol	Definition	Domain	
$l_n(h)$	Per-slot energy load profile	$0 \leq l_n(h) \leq l_n^{(\max)}$	$n \in \mathcal{P} \backslash \mathcal{G}_{\mathrm{R}}$
		$-l_n^{(\min)} \leq l_n(h) \leq l_n^{(\max)}$	$n \in \mathcal{N} \cup \mathcal{G}_{\mathrm{R}}$
$e_n(h)$	Per-slot energy consumption profile	$e_n(h) \geq 0$	$n \in \mathcal{D} \backslash \mathcal{G}_{\mathrm{R}}$
		possibly negative	$n \in \mathcal{G}_{\mathrm{R}}$
$g_n(h)$	Per-slot energy production profile	$g_n(h) \geq 0$	$n \in \mathcal{G}$
$s_n(h)$	Per-slot energy storage profile	$s_n(h) > 0$	$n \in \mathcal{S}$ (charging)
		$s_n(h) < 0$	$n \in \mathcal{S}$ (discharging)
$L(h)$	Aggregate per-slot energy load	$L^{(\min)} \leq L(h) \leq L^{(\max)}$	

D. Energy Cost and Pricing Model

This section describes the cost model on which depends the price of energy. Let us first define the *aggregate per-slot energy load* at time-slot h as

$$L(h) \triangleq L^{(\mathcal{P})}(h) + \sum_{n \in \mathcal{N}} l_n(h), \qquad h = 1, \ldots, H. \quad (6)$$

where $L^{(\mathcal{P})}(h) \triangleq \sum_{n \in \mathcal{P}} e_n(h)$ is the aggregate per-slot energy consumption associated with the passive users connected to the grid. Then, we can model the supply-side as a single utility that provides, at each time-slot h, a one-way energy flow $L(h)$ through the transmission grid to the demand-side (see Figs. 1 and 2). We work under the hypothesis that the aggregate energy demand is always guaranteed by the supply-side[5] and satisfies

$$L^{(\min)} \leq L(h) \leq L^{(\max)} \quad (7)$$

where $L^{(\min)} > 0$ is the *minimum aggregate energy load* throughout the grid, and $L^{(\max)} > 0$ is the *maximum aggregate energy load* that the grid can take before experiencing a blackout. Observe that both $L^{(\min)}$ and $L^{(\max)}$ are known to the central unit based on the actual grid infrastructure and on the available load statistics. A summary of the principal variables introduced throughout Section II, along with their main characteristics, is reported in Table II.

Given the aggregate per-slot energy load $L(h)$, let us now define the *cost per unit of energy* $C_h(L(h))$ as the price for a unit of energy at time-slot h resulting from the day-ahead market [20], [24], [25]. Then, $C_h(L(h)) l_n(h)$ represents the amount of money paid by user n to purchase the energy load $l_n(h)$ from the grid (if $l_n(h) > 0$) or received to sell the energy load $l_n(h)$ to the grid (if $l_n(h) < 0$) at time-slot h. Observe that $C_h(\cdot)$ can represent either the actual energy cost (as a result of energy generation, transmission, and distribution costs among other issues) or simply a pricing function designed to incentivize load-shifting by the end users [9]. In any case, $C_h(\cdot)$ is generally different at each time-slot h, since the energy production changes along the time period of analysis according to the energy demand and to the availability of intermittent sources. For instance, the energy price can be less during the night compared to the day time (as in the practical test case in Section V). Equivalent pricing models are given in [9], [10], [19].

We now have all the elements to introduce the cumulative expense of each group of users in the demand-side of the network. Let $p_n^{(\mathcal{N})}$ denote the *individual cumulative expense* over the time period of analysis of active user $n \in \mathcal{N}$, representing his cumulative monetary expense incurred for obtaining the desired amount of energy in the time period of analysis:

$$p_n^{(\mathcal{N})} \triangleq \sum_{h=1}^{H} \Big(C_h\big(L(h)\big)\big(e_n(h) - g_n(h) + s_n(h)\big) + W_n\big(g_n(h)\big) \Big) \quad (8)$$

where we have included the individual production costs $\big\{W_n\big(g_n(h)\big)\big\}_{h=1}^{H}$. Note that, in general, the amount of money paid/received by user n to purchase/sell the same amount of energy from/to the grid is different during distinct time-slots due to the fact that the grid cost function and the aggregate per-slot energy load vary along the day. Likewise, the aggregate cumulative expense incurred by the passive users is given by

$$p^{(\mathcal{P})} \triangleq \sum_{h=1}^{H} C_h\big(L(h)\big) L^{(\mathcal{P})}(h) \quad (9)$$

which indirectly depends on the strategies adopted by the active users through the cost per unit of energy at each time-slot $C_h(L(h))$. Lastly, we introduce the *aggregate cumulative expense* $p^{(\mathcal{D})}$, which expresses the overall grid expense over the time period of analysis, and which is related to the individual cumulative expenses of the active users in (8) and to the aggregate cumulative expense of the passive users in (9) as

$$\begin{aligned} p^{(\mathcal{D})} &\triangleq \sum_{h=1}^{H} \Big(C_h\big(L(h)\big) \Big(\sum_{n \in \mathcal{N}} l_n(h) + L^{(\mathcal{P})}(h) \Big) \\ &\quad + \sum_{n \in \mathcal{G}} W_n\big(g_n(h)\big) \Big) \\ &= \sum_{n \in \mathcal{N}} p_n^{(\mathcal{N})} + p^{(\mathcal{P})}. \end{aligned} \quad (10)$$

E. Introduction to the DSM Approaches

In the rest of the paper, we focus on the optimization problems posed by our DSM mechanisms, through which active users determine in advance their generation/storage strategies for the upcoming time period of analysis (corresponding usually to a

[5]The day-ahead optimization allows the supply-side to know in advance the amount of energy to be delivered to the demand-side over the upcoming time period of analysis, in order to plan its production accordingly [13], [23].

day [26]). Once the grid cost functions $\{C_h(\cdot)\}_{h=1}^H$ are fixed in the day-ahead market, active users react to the prices provided by the central unit by iteratively adjusting their generation and storage strategies $\mathbf{g}_n$ and $\mathbf{s}_n$ and, thus, their day-ahead energy demands $\{l_n(h)\}_{h=1}^H$, given the aggregate energy loads $\{L(h)\}_{h=1}^H$. The final objective of the active users is either i) to individually minimize their individual cumulative expense over the time period of analysis in (8) (see Section III), or ii) to jointly minimize the aggregate cumulative expense of all demand-side users in (10) (see Section IV). In the first method, active users act selfishly to reduce their cumulative monetary expenses without consulting or coordinating with each other. Despite the flexibility of this approach, the second solution may be more desirable from the point of view of both the individual users and the supply-side, since it takes into account the overall production costs and results in a more efficient demand-side energy consumption.

One could consider to solve the aforementioned optimization problems in a centralized fashion, with the central unit imposing every single user how much energy he must produce, charge, and discharge at each time-slot. Nonetheless, such solution requires every user to provide detailed information about his energy production and storage capabilities and this could lead to privacy issues. Besides, a centralized approach is not scalable and cannot account for an unpredictably increasing number of participants. In consequence, we adopt *distributed* solutions for both DSM techniques in Sections III and IV, respectively.

III. Noncooperative DSM Approach

In this section, we focus on the optimization problem posed by the noncooperative DSM mechanism through which active demand-side users aim at individually minimizing their individual cumulative expense over the time period of analysis introduced in (8).

For convenience, let us first distinguish three main groups among the users participating actively in the optimization (see Table I for details):

(i) *Dispatchable energy producers*: $\mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \triangleq \mathcal{G}\setminus(\mathcal{G}\cap\mathcal{S})$, for whom $\mathbf{g}_n \in \Omega_{\mathbf{g}_n}$ and $\mathbf{s}_n = \mathbf{0}$;

(ii) *Energy storers*: $\mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \triangleq \mathcal{S}\setminus(\mathcal{G}\cap\mathcal{S})$, for whom $\mathbf{s}_n \in \Omega_{\mathbf{s}_n}$ and $\mathbf{g}_n = \mathbf{0}$;

(iii) *Dispatchable energy producers-storers*: $\mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \triangleq \mathcal{G}\cap\mathcal{S}$, for whom $\mathbf{g}_n \in \Omega_{\mathbf{g}_n}$ and $\mathbf{s}_n \in \Omega_{\mathbf{s}_n}$.

Then, we can define the strategy vector and the per-slot strategy profile of a generic active user $n \in \mathcal{N}$ as

$$\mathbf{x}_n \triangleq \begin{cases} \mathbf{g}_n, & \text{if } n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \\ \mathbf{s}_n, & \text{if } n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \\ (\mathbf{g}_n, \mathbf{s}_n), & \text{if } n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \end{cases}$$

$$\mathbf{x}_n(h) \triangleq \begin{cases} g_n(h), & \text{if } n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \\ s_n(h), & \text{if } n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \\ (g_n(h), s_n(h))^{\mathrm{T}}, & \text{if } n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \end{cases}. \quad (11)$$

In addition, taking into account the limitations on the link capacity given in (2), we denote the corresponding strategy set by [see (12) at bottom of page] with dimension $\omega_{\mathbf{x}_n} \triangleq H\boldsymbol{\delta}_n^{\mathrm{T}}\boldsymbol{\delta}_n$, where we have introduced the auxiliary variables $\boldsymbol{\Delta}_n \triangleq \boldsymbol{\delta}_n^{\mathrm{T}} \otimes \mathbf{I}_H$, $\boldsymbol{\Delta}_{g,n} \triangleq \boldsymbol{\delta}_{g,n}^{\mathrm{T}} \otimes \mathbf{I}_H$, with

$$\boldsymbol{\delta}_n \triangleq \begin{cases} -1, & \text{if } n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \\ 1, & \text{if } n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \\ (-1,1)^{\mathrm{T}}, & \text{if } n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \end{cases}$$

$$\boldsymbol{\delta}_{g,n} \triangleq \begin{cases} 1, & \text{if } n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \\ 0, & \text{if } n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \\ (1,0)^{\mathrm{T}}, & \text{if } n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \end{cases} \quad (13)$$

and $\boldsymbol{\Delta}_{s,n} \triangleq (\mathbf{0}_H^{\mathrm{T}} \otimes \mathbf{0}_H \ \mathbf{I}_H)$, with $\mathbf{0}_H$ denoting the H-dimensional zero vector. Furthermore, let $\mathbf{x}_{-n} \triangleq (\mathbf{x}_m)_{n\neq m=1}^N$ be the vector including the strategies of the other users $m \in \mathcal{N}\setminus\{n\}$. Bearing in mind the individual cumulative expense given in (8), the objective function of user n is given by

$$f_n(\mathbf{x}_n, \mathbf{x}_{-n}) \triangleq (\mathbf{e}_n + \boldsymbol{\Delta}_n\mathbf{x}_n)^{\mathrm{T}}\mathbf{c}(\mathbf{x}) + \mathbf{1}_H^{\mathrm{T}}\mathbf{w}_n(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n) \quad (14)$$

with $\mathbf{x} \triangleq (\mathbf{x}_n)_{n=1}^N$ being the joint strategy vector and the vector functions $\mathbf{c}(\cdot)$ and $\mathbf{w}_n(\cdot)$ given by

$$\mathbf{c}(\mathbf{x}) \triangleq \left(C_h\left(L^{(\mathcal{P})}(h) + \sum_{m\in\mathcal{N}}\left(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}}\mathbf{x}_m(h)\right)\right)\right)_{h=1}^H \quad (15)$$

and

$$\mathbf{w}_n\left(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n\right) \triangleq \left(W_n\left(\boldsymbol{\delta}_{g,n}^{\mathrm{T}}\mathbf{x}_n(h)\right)\right)_{h=1}^H. \quad (16)$$

A. Game Theoretical and VI Formulation

Here, we model our DSM procedure as a (noncooperative) Nash game. Each active user $n \in \mathcal{N}$ is a player who competes against the others by choosing the production and storage strategies $\mathbf{g}_n$ and $\mathbf{s}_n$ that minimize his objective function $f_n(\mathbf{x}_n, \mathbf{x}_{-n})$ in (14), i.e., his cumulative expense over the time period of analysis. The formal definition of the game is the following: $\mathcal{G} = \langle\Omega_{\mathbf{x}}, \mathbf{f}\rangle$, where $\Omega_{\mathbf{x}} \triangleq \prod_{n=1}^N \Omega_{\mathbf{x}_n}$ is the $\omega_{\mathbf{x}}$-dimensional joint strategy set, $\omega_{\mathbf{x}} \triangleq \sum_{n\in\mathcal{N}}\omega_{\mathbf{x}_n}$, and $\mathbf{f} \triangleq (f_n(\mathbf{x}_n, \mathbf{x}_{-n}))_{n=1}^N$ is the vector of the objective functions. Each player $n \in \mathcal{N}$ aims at solving the following optimization problem, given $\mathbf{x}_{-n}$:

$$\begin{aligned} \min_{\mathbf{x}_n} \quad & f_n(\mathbf{x}_n, \mathbf{x}_{-n}) \\ \text{s.t.} \quad & \mathbf{x}_n \in \Omega_{\mathbf{x}_n} \end{aligned} \qquad \forall n \in \mathcal{N}. \quad (17)$$

Note that the dependence of the objective function in (14) on $\mathbf{x}_{-n}$ lies within the argument of the cost functions $C_h(\cdot)$ in (14), since $L(h) = \sum_{m\in\mathcal{N}\setminus\{n\}}\left(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}}\mathbf{x}_m(h)\right) + e_n(h) + \boldsymbol{\delta}_n^{\mathrm{T}}\mathbf{x}_n(h)$. The solution of the game $\mathcal{G} = \langle\Omega_{\mathbf{x}}, \mathbf{f}\rangle$ is given by the well-known concept of Nash equilibrium, which is a feasible strategy profile $\mathbf{x}^\star$ with the property that no single

$$\Omega_{\mathbf{x}_n} \triangleq \begin{cases} \left\{\mathbf{x}_n \in \Omega_{\mathbf{g}_n} : \boldsymbol{\Delta}_n\mathbf{x}_n \succeq -l_n^{(\min)}\mathbf{1}_H - \mathbf{e}_n\right\}, & \text{if } n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \\ \left\{\mathbf{x}_n \in \Omega_{\mathbf{s}_n} : -l_n^{(\min)}\mathbf{1}_H - \mathbf{e}_n \preceq \boldsymbol{\Delta}_n\mathbf{x}_n \preceq l_n^{(\max)}\mathbf{1}_H - \mathbf{e}_n\right\}, & \text{if } n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \\ \left\{\boldsymbol{\Delta}_{g,n}\mathbf{x}_n \in \Omega_{\mathbf{g}_n}, \boldsymbol{\Delta}_{s,n}\mathbf{x}_n \in \Omega_{\mathbf{s}_n} : -l_n^{(\min)}\mathbf{1}_H - \mathbf{e}_n \preceq \boldsymbol{\Delta}_n\mathbf{x}_n \preceq l_n^{(\max)}\mathbf{1}_H - \mathbf{e}_n\right\}, & \text{if } n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \end{cases} \quad (12)$$

player n can benefit by unilaterally deviating from his strategy $\mathbf{x}_n^\star$, if all other players act according to $\mathbf{x}_{-n}^\star$ [27], i.e.,:

$$f_n\left(\mathbf{x}_n^\star, \mathbf{x}_{-n}^\star\right) \leq f_n\left(\mathbf{x}_n, \mathbf{x}_{-n}^\star\right), \ \forall \mathbf{x}_n \in \Omega_{\mathbf{x}_n}, \ \forall n \in \mathcal{N}. \quad (18)$$

Variational inequality theory provides a general framework for investigating and solving various optimization problems and equilibrium models, even when classical game theory may fail. Throughout this and the next section, we refer extensively to [15]. For a detailed description of the subject, we refer the interested reader also to [14], [16], [28], [29], [30] for a comprehensive treatment of VIs.

In order to analyze the existence of the Nash equilibria as well as the convergence of distributed algorithms while keeping the pricing model general, it is very convenient to reformulate the game as a partitioned VI problem, which is formally defined next.

Definition 1 ([30] Def. 1.1.1): Let $\mathbf{F}(\mathbf{x}) : \Omega_{\mathbf{x}} \to \mathbb{R}^{\omega_{\mathbf{x}}}$ be a vector-valued function defined as $\mathbf{F}(\mathbf{x}) \triangleq \left(\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})\right)_{n=1}^N$, where $\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n}) : \Omega_{\mathbf{x}_n} \to \mathbb{R}^{\omega_{\mathbf{x}_n}}$ is the nth component block function of $\mathbf{F}(\mathbf{x})$, $\mathbf{x} \triangleq \left(\mathbf{x}_n\right)_{n=1}^N$, and $\Omega_{\mathbf{x}} = \prod_{n=1}^N \Omega_{\mathbf{x}_n}$. Then, the VI problem, denoted by $\mathrm{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, consists in finding $\mathbf{x}^\star \in \Omega_{\mathbf{x}}$ such that

$$(\mathbf{x} - \mathbf{x}^\star)^{\mathrm{T}} \mathbf{F}(\mathbf{x}^\star) \geq 0, \qquad \forall \mathbf{x} \in \Omega_{\mathbf{x}}. \quad (19)$$

The equivalence between the game theoretical and the VI formulation is established in the following lemma.

Lemma 1 ([15, Prop. 4.1], [30 Prop. 1.4.2]): The Game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ is equivalent to the VI problem $\mathrm{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, with $\mathbf{F}(\mathbf{x}) \triangleq \left(\nabla_{\mathbf{x}_n} f_n(\mathbf{x}_n, \mathbf{x}_{-n})\right)_{n=1}^N$, if:

(a) The strategy sets $\Omega_{\mathbf{x}_n}$ are closed and convex;
(b) For every fixed $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}} \triangleq \prod_{m \in \mathcal{N} \setminus \{n\}} \Omega_{\mathbf{x}_m}$, the objective function $f_n(\mathbf{x}_n, \mathbf{x}_{-n})$ is convex and twice continuously differentiable on $\Omega_{\mathbf{x}_n}$.

Since the individual strategy sets $\Omega_{\mathbf{x}_n}$ in (12) are nonempty polyhedra [31, Sec. 2.2.4], Lemma 1(a) is readily satisfied. On the other hand, Lemma 1(b) is satisfied if and only if the gradient of $f_n(\cdot, \mathbf{x}_{-n})$, $\mathbf{F}_n(\cdot, \mathbf{x}_{-n}) \triangleq \nabla_{\mathbf{x}_n} f_n(\cdot, \mathbf{x}_{-n})$, is monotone on $\Omega_{\mathbf{x}_n}$ for any given $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}$ [29],[6] where

$$\begin{aligned}\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n}) &\triangleq \nabla_{\mathbf{x}_n} f_n(\mathbf{x}_n, \mathbf{x}_{-n}) \\ &= \boldsymbol{\Delta}_n^{\mathrm{T}} \mathbf{c}(\mathbf{x}) + \boldsymbol{\Delta}_n^{\mathrm{T}} \mathbf{D}_{\mathbf{c}'}(\mathbf{x})(\mathbf{e}_n + \boldsymbol{\Delta}_n \mathbf{x}_n) \\ &\quad + \boldsymbol{\Delta}_{g,n}^{\mathrm{T}} \mathbf{w}_n'(\boldsymbol{\Delta}_{g,n} \mathbf{x}_n) \end{aligned} \quad (20)$$

with $\mathbf{D}_{\mathbf{c}'}(\mathbf{x}) \triangleq \mathrm{Diag}\big(\mathbf{c}'(\mathbf{x})\big)$. This requirement is accomplished under the conditions of Theorem 1 given in the next section.

Assuming that Lemma 1 holds, we can formulate the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ as the VI problem $\mathrm{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, where the vector function $\mathbf{F}(\mathbf{x})$ is

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \left(\nabla_{\mathbf{x}_n} f_n(\mathbf{x}_n, \mathbf{x}_{-n})\right)_{n=1}^N \\ &= \boldsymbol{\Delta}^{\mathrm{T}}\big(\mathbf{1}_N \otimes \mathbf{c}(\mathbf{x})\big) + \boldsymbol{\Delta}^{\mathrm{T}}\big(\mathbf{I}_N \otimes \mathbf{D}_{\mathbf{c}'}(\mathbf{x})\big)(\mathbf{e} + \boldsymbol{\Delta}\mathbf{x}) \\ &\quad + \boldsymbol{\Delta}_g^{\mathrm{T}} \mathbf{w}'(\mathbf{x}) \end{aligned} \quad (21)$$

[6]We say that $\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$ is monotone on $\Omega_{\mathbf{x}_n}$ when $(\mathbf{x}_n - \mathbf{y}_n)^{\mathrm{T}}(\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n}) - \mathbf{F}_n(\mathbf{y}_n, \mathbf{x}_{-n})) \geq 0$, $\forall \mathbf{x}_n, \mathbf{y}_n \in \Omega_{\mathbf{x}_n}$, for every fixed $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}$ [15, Def. 4.3(i)].

with $\boldsymbol{\Delta} \triangleq \mathrm{Diag}(\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_N)$, $\boldsymbol{\Delta}_g \triangleq \mathrm{Diag}(\boldsymbol{\Delta}_{g,1}, \ldots, \boldsymbol{\Delta}_{g,N})$, $\mathbf{e} \triangleq \left(\mathbf{e}_n\right)_{n=1}^N$, and $\mathbf{w}'(\mathbf{x}) \triangleq \left(\mathbf{w}_n'(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n)\right)_{n=1}^N$.

B. Nash Equilibria Analysis

Sufficient conditions on the grid cost functions per unit of energy and on the production cost functions that guarantee the existence of the Nash equilibria of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$, i.e., of the solutions of the VI problem $\mathrm{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, are derived in the next theorem.

Theorem 1: Given the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$, suppose that the following conditions hold:

(a) The grid cost functions per unit of energy $\left\{C_h(x)\right\}_{h=1}^H$ are increasing and convex on $[L^{(\min)}, L^{(\max)}]$, and satisfy

$$C_h'(x) \geq \frac{1}{2} \zeta^{(\min)} C_h''(x), \qquad \forall x \in \left[L^{(\min)}, L^{(\max)}\right] \quad (22)$$

where $\zeta^{(\min)} = \max_n l_n^{(\min)}$ denotes the maximum amount of energy that can be sold to the grid by any single user $n \in \mathcal{N}$ at any time-slot;

(b) The production cost function $W_n(x)$ is convex on $[0, g_n^{(\max)}]$, $\forall n \in \mathcal{G}$.

Then, the game has a nonempty and compact solution set.

Proof: See Appendix I-A.

Remark 1.1: Observe that any realistic grid cost function $C_h(\cdot)$ is increasing as required by Theorem 1(a) (see, e.g., the power price function in [32]). Actually, for non-strictly increasing $C_h(\cdot)$, a game-theoretical approach may not even be necessary since the individual optimization problems can possibly be decoupled (see details in Example 1.1(a)). The convexity of $C_h(\cdot)$ in Theorem 1(a) and the convexity of $W_n(\cdot)$ in Theorem 1(b) simply impose that the grid cost per unit of energy and the production cost function do not tend to saturate as the aggregate energy load and the per-slot energy production profile, respectively, increase, which is a very reasonable assumption. Still, the condition in (22) has to be verified case by case, although it is not difficult to be fulfilled (see Example 1.1(d)).

Example 1.1: Suppose, for instance, that the grid cost functions are given by $\left\{C_h(x) = K_h x^a\right\}_{h=1}^H$, with $\left\{K_h\right\}_{h=1}^H > 0$ and $a \geq 0$. Then:

(a) If $a = 0$, we have that $\left\{C_h'(x)\right\}_{h=1}^H = 0$: this means that the cost per unit of energy is constant at each time-slot h, and the resulting optimization problem for users in $\mathcal{N}$ does not depend on the aggregate energy load (and, in consequence, on the strategies of the other users $\mathbf{x}_{-n}$), but only on the energy cost at each time-slot h. In such trivial case, the game-theoretical approach proposed in this paper is not necessary.

(b) If $0 < a < 1$, the grid cost functions are not convex but concave. This is, however, unrealistic, since energy generation becomes less efficient as the aggregate demand increases (in fact, peaking power plants that allow to meet rapidly increasing demand are extremely expensive to operate [33, Sec. 3.9]).

(c) If $a = 1$, the grid cost functions are linear (hence strictly increasing and convex), and condition (22) is immediately satisfied since $\left\{C''_h(x)\right\}_{h=1}^{H} = 0$. This particular case is treated in detail in [19].

(d) If $a > 1$, the grid cost functions are strictly increasing and strictly convex and Theorem 1 guarantees the existence of the Nash equilibria of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ in (17) whenever

$$a \leq 1 + 2L(h)/\zeta^{(\min)}. \tag{23}$$

This is a very mild condition, since the ratio between the aggregate demand $L(h)$ and the maximum energy that can be individually injected into the grid $\zeta^{(\min)}$ can be very large in practice. Alternatively, this condition can be understood as a tradeoff between the minimum demand generated by the passive users and that coming from the active users, as explained in Remark 2.2.

Theorem 1 guarantees the existence of a solution of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ in (17), but not the uniqueness. Interestingly, all Nash equilibria for this problem happen to have the same quality in terms of optimal values of the players' objective functions, as stated in the following proposition.

Proposition 1.1: Given the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$, suppose that the conditions in Theorem 1 hold; let $\mathrm{NE}_{\mathcal{G}}$ be the set of the Nash equilibria of $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$. Then, the following holds: $f_n(\mathbf{x}^{(1)}) = f_n(\mathbf{x}^{(2)})$, $\forall \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathrm{NE}_{\mathcal{G}}$ and $\forall n \in \mathcal{N}$.

Proof: See Appendix I-B.

C. Proximal Decomposition Algorithm

We focus now on distributed algorithms to compute one of the (equivalent) Nash equilibria (see Proposition 1.1) of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$. We consider the class of *totally asynchronous* algorithms, where some users may update their strategies more frequently than others and they may even use outdated information about the strategy profiles adopted by the other users. This adds more flexibility and robustness with respect to the well-known Jacobi (simultaneous) and Gauss-Seidel (sequential) schemes, as the sequential ECS algorithm proposed in [10]. To provide a formal description of the algorithms, let $\mathcal{T}_n \subseteq \mathcal{T} \subseteq \{0, 1, 2, \ldots\}$ be the set of times at which user $n \in \mathcal{N}$ updates his own strategy $\mathbf{x}_n$, denoted by $\mathbf{x}_n^{(i)}$ at the ith iteration. We use $t_n(i)$ to denote the most recent time at which the strategy of user n is perceived by the central unit at the ith iteration. Each individual user n updates his strategy by minimizing his cumulative expense over the time period of analysis referring to the most recently available value of the per-slot aggregate energy load

$$L^{(\mathbf{t}(i))}(h) \triangleq L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} l_m^{(t_m(i))}(h), \qquad h = 1, \ldots, H \tag{24}$$

where $l_n^{(t_n(i))}(h)$ is the energy load of user $n \in \mathcal{N}$ as perceived by the central unit at time $t_n(i)$, which can possibly be outdated when computation occurs. Finally, to emphasize the dependence of the strategy of user n on the aggregate energy loads of the other users, we rewrite the objective function in (14) as

$$\begin{aligned} f_n&\left(\mathbf{x}_n, \left\{L^{(\mathbf{t}(i))}(h)\right\}_{h=1}^{H}\right) \\ &\triangleq \sum_{h=1}^{H} C_h\left(L^{(\mathbf{t}(i))}(h) + \boldsymbol{\delta}_n^{\mathrm{T}}\left(\mathbf{x}_n(h) - \mathbf{x}_n^{(t_n(i))}(h)\right)\right) \\ &\quad \times \left(e_n(h) + \boldsymbol{\delta}_n^{\mathrm{T}} \mathbf{x}_n(h)\right) + \sum_{h=1}^{H} W_n\left(\boldsymbol{\delta}_{g,n}^{\mathrm{T}} \mathbf{x}_n(h)\right). \end{aligned} \tag{25}$$

Some standard conditions in asynchronous convergence theory, which are fulfilled in any practical implementation, need to be satisfied by the schedule $\mathcal{T}_n$ and $t_n(i)$, $\forall n \in \mathcal{N}$ [14, Sec. 1.2.2] [34, Ch. 6], namely:

(A1) $0 \leq t_n(i) \leq i$: at any given iteration i, each user n can use only the aggregate energy loads $\left\{L^{(\mathbf{t}(i))}(h)\right\}_{h=1}^{H}$ resulting from the strategies adopted by the other players in the previous iterations;

(A2) $\lim_{k \to \infty} t_n(i_k) = +\infty$, where $\{i_k\}$ is a sequence of elements in $\mathcal{T}_n$ that tends to infinity: for any given iteration index i_k, the values of the components of $\left\{L^{(\mathbf{t}(i))}(h)\right\}_{h=1}^{H}$ generated prior to i_k are not used in the updates of the aggregate energy loads at the iteration i, when i becomes sufficiently larger than i_k;

(A3) $|\mathcal{T}_n| = \infty$: no player fails to update his own strategy as time i goes on.

Since all Nash equilibria are equivalent (in the sense of Proposition 1.1), we focus next on proximal-based best-response algorithms, whose convergence to some of the solutions is guaranteed even in the presence of multiple solutions. According to [15, Alg. 4.2], instead of solving the original game, i.e., the VI problem $\mathrm{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, one solves a sequence of regularized VI problems, each of them given by $\mathrm{VI}\left(\Omega_{\mathbf{x}}, \mathbf{F} + \tau(\mathbf{I} - \mathbf{x}^{(i)})\right)$, where $\mathbf{I}$ is the identity map (i.e., $\mathbf{I} : \mathbf{x} \to \mathbf{x}$), $\mathbf{x}^{(i)}$ is a fixed real vector, and τ is a positive constant. It can be shown that, under the monotonicity of $\mathbf{F}(\mathbf{x})$ on $\Omega_{\mathbf{x}}$, this regularized problem is strongly monotone and has thus a unique solution [15, Th. 4.1(d)] denoted by $\mathbf{S}_\tau(\mathbf{x}^{(i)})$; such a unique solution is a nonexpansive mapping, meaning that, starting at a given initial point $\mathbf{x}^{(0)} \in \Omega_{\mathbf{x}}$, the sequence generated by a proper averaging of $\mathbf{S}_\tau(\mathbf{x}^{(i)})$ and $\mathbf{x}^{(i)}$ converges to a solution of the $\mathrm{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, even when this is not unique.[7] Note also that, given $\mathbf{x}^{(i)} \triangleq \left(\mathbf{x}_n^{(i)}\right)_{n=1}^{N} \in \Omega_{\mathbf{x}}$, the solution $\mathbf{S}_\tau(\mathbf{x}^{(i)})$ of the regularized $\mathrm{VI}\left(\Omega_{\mathbf{x}}, \mathbf{F} + \tau(\mathbf{I} - \mathbf{x}^{(i)})\right)$ coincides with the unique Nash equilibrium of the regularized game, where each user n solves the following optimization problem:

$$\begin{array}{ll} \displaystyle\min_{\mathbf{x}_n} & f_n\left(\mathbf{x}_n, \left\{L^{(\mathbf{t}(i))}(h)\right\}_{h=1}^{H}\right) + \dfrac{\tau}{2}\|\mathbf{x}_n - \mathbf{x}_n^{(i)}\|^2 \\ \text{s.t.} & \mathbf{x}_n \in \Omega_{\mathbf{x}_n} \end{array} \qquad \forall n \in \mathcal{N}. \tag{26}$$

The solution $\mathbf{S}_\tau(\mathbf{x}^{(i)})$ can then be computed in a distributed way with convergence guarantee using any asynchronous best-response algorithm applied to the game (26) [15, Cor. 4.1] (see,

[7]Replacing the exact computation of the solution of the regularized VI with an inexact solution does not affect convergence of Algorithm 1, as long as the error bound goes to zero as $i \to \infty$ [14]–[16].

e.g., [15, Alg. 4.2]). The above scheme is formalized in Algorithm 1 below, whose convergence conditions are given in Theorem 2.

Algorithm 1 Asynchronous Proximal Decomposition Algorithm (PDA)

`Data` : Set $i=0$ and the initial centroid $\left(\bar{\mathbf{x}}_n\right)_{n=1}^{N}=\mathbf{0}$. Given $\left\{C_h(\cdot)\right\}_{h=1}^{H}$, $\left\{\rho^{(i)}\right\}_{i=0}^{\infty}$, $\tau>0$, and any feasible starting point $\mathbf{x}^{(0)}=\left(\mathbf{x}_n^{(0)}\right)_{n=1}^{N}$:

`(S.1)` : `If` a suitable termination criterion is satisfied: `STOP`.

`(S.2)` : `For` $n\in\mathcal{N}$, each user computes $\mathbf{x}_n^{(i+1)}$ as [see (27) at the bottom of the page].

`End`

`(S.3)` : `If` the NE is reached, then each user $n\in\mathcal{N}$ sets $\mathbf{x}_n^{(i+1)}\leftarrow(1-\rho^{(i)})\bar{\mathbf{x}}_n+\rho^{(i)}\mathbf{x}_n^{(i+1)}$ and updates his centroid: $\bar{\mathbf{x}}_n=\mathbf{x}_n^{(i+1)}$.

`(S.4)` : $i\leftarrow i+1$; `Go to (S.1)`.

Theorem 2: Given the game $\mathcal{G}=\langle\Omega_{\mathbf{x}},\mathbf{f}\rangle$, suppose that the conditions of Theorem 1 and the following hold:

(a) The grid cost functions per unit of energy $\{C_h(x)\}_{h=1}^{H}$ are strictly increasing and convex on $[L^{(\min)},L^{(\max)}]$, and additionally satisfy

$$C_h'(x)\geq N(\zeta^{(\min)}+\zeta^{(\max)})C_h''(x),\ \forall x\in[L^{(\min)},L^{(\max)}] \tag{28}$$

where N is the number of active users connected to the grid, $\zeta^{(\min)}\triangleq\max_{n\in\mathcal{N}}l_n^{(\min)}$ and $\zeta^{(\max)}\triangleq\max_{n\in\mathcal{N}}l_n^{(\max)}$ denote the maximum amount of energy that can be sold to or bought from the grid by any single user $n\in\mathcal{N}$ at any time-slot, respectively;

(b) The regularization parameter τ satisfies

$$\tau>2(N-1)\max_h C_h'(L^{(\max)})+2L^{(\max)}\max_h\left(\max_{L^{(\min)}\leq x\leq L^{(\max)}}C_h''(x)\right) \tag{29}$$

where $L^{(\max)}$ is the maximum aggregate energy load allowed by the grid infrastructure;

(c) $\rho^{(i)}$ is chosen such that $\{\rho^{(i)}\}\subset[R_m,R_M]$, with $0<R_m<R_M<2$ [15, Th. 4.3].

Then, any sequence $\left\{\mathbf{x}^{(i)}\right\}_{i=1}^{\infty}$ generated by Algorithm 1 converges to a Nash equilibrium of the game, for any given updating schedule of the users satisfying assumptions (A1)–(A3).

Proof: See Appendix I-C.

Remark 2.1 (on Algorithm 1): Algorithm 1 can be seen as an asynchronous algorithm with an occasional update of the individual centroids $\bar{\mathbf{x}}_n$, performed simultaneously $\forall n\in\mathcal{N}$. Nonetheless, it is a double-loop algorithm in nature: in the inner loop, the computation of $\mathbf{S}_\tau(\mathbf{x}^{(i)})$ requires the solution of the regularized game in (26) via asynchronous best-response algorithms (such as [15, Alg. 4.2]); in the outer loop, all users $n\in\mathcal{N}$ update their centroid $\bar{\mathbf{x}}_n$ and proceed to solve the inner game again, until an equilibrium is reached. Observe that the update of the centroids is performed locally by the users at the cost of no signaling exchange with the central unit. However, since this update must be simultaneous, some sort of synchronization must be provided by the central unit to the users (see [16] for a detailed discussion on synchronization methods for this class of distributed algorithms). The central unit also checks whether the termination criterion in step `(S.1)` is met, concluding thus the algorithm. Since the central unit only receives the individual energy loads from each user, a practical criterion can be to guarantee that the difference of the users' energy loads between two consecutive iterations is below the prescribed accuracy (c.f. Section V).

Summarizing, the proposed demand-side day-ahead optimization based on Algorithm 1 works as follows. At the beginning of the optimization process, τ is computed as in Theorem 2(b) and broadcast to each user $n\in\mathcal{N}$, together with the grid cost functions per unit of energy $\left\{C_h(\cdot)\right\}_{h=1}^{H}$. Then, at each iteration, any active user who wants to update his strategy solves his own optimization problem in (26) based on the most recent values of the aggregate energy loads $\left\{L^{(\mathtt{t}(i))}(h)\right\}_{h=1}^{H}$, which are calculated by the central unit referring to the (possibly outdated) individual demands, and communicates his new load to the central unit. When an equilibrium in the inner loop is reached, the central unit proceeds to the next iteration, and this process is repeated until convergence is reached.

Remark 2.2 (on Theorem 2(a)): The interpretation of the condition (28) given in Theorem 2(a) is twofold. First of all, it provides a guideline to choose the grid cost functions per unit of energy $\left\{C_h(\cdot)\right\}_{h=1}^{H}$. Second, it represents a tradeoff between the minimum demand generated by the passive users and that coming from the active users, as explained next. Suppose, for instance, that $C_h(x)=K_hx^a$ with $K_h>0$ and $a>1$; then (28) actually implies that

$$L^{(\min)}=L^{(\min,\mathcal{P})}+L^{(\min,\mathcal{N})}\geq N(a-1)\left(\zeta^{(\min)}+\zeta^{(\max)}\right) \tag{30}$$

where $L^{(\min,\mathcal{P})}$ and $L^{(\min,\mathcal{N})}$ denote the minimum aggregated demand of the passive and the active users, respectively. Observe that $L^{(\min,\mathcal{P})}$ is increasing with the number of passive users in the demand-side, whereas the right-hand side of (30) is

$$\mathbf{x}_n^{(i+1)}=\begin{cases}\mathbf{x}_n^{\star}\in\underset{\mathbf{x}_n\in\Omega_{\mathbf{x}_n}}{\arg\min}\left\{f_n\left(\mathbf{x}_n,\left\{L^{(\mathtt{t}(i))}(h)\right\}_{h=1}^{H}\right)+\dfrac{\tau}{2}\left\|\mathbf{x}_n-\bar{\mathbf{x}}_n\right\|^2\right\}, & \text{if } i\in\mathcal{T}_n\\ \mathbf{x}_n^{(i)}, & \text{otherwise}\end{cases} \tag{27}$$

not affected by it. On the other hand, when more active users are added, the previous condition becomes more restrictive, since the resulting increment of the right-hand side of (30) is always greater than the one of the left-hand side (as the individual demand of any active user satisfies $-\zeta^{(\min)} \leq l_n(h) \leq \zeta^{(\max)}$). It turns out that, for any given number of passive users, (28) provides an upper bound on the number of active users that can be tolerated in the demand-side of the network.

Remark 2.3 (on Theorem 2(b)): From the proof of Theorem 2(b), it follows that Algorithm 1 can converge under a milder bound on the regularization parameter than the one given in (29). However, the peculiarity of the provided expression of τ is that none of the terms in (29) depends on the particular energy generation or storage equipment owned by user n, but only on the transmission grid infrastructure. Thus, the regularization parameter can be calculated by the central unit a priori without interfering with the privacy of the users. These considerations apply also to the lower bound of τ provided in (36) for Algorithm 2.

IV. Cooperative DSM Approach

In contrast to the noncooperative approach discussed in Section III, we now consider an alternative DSM technique, in which demand-side users collaborate to minimize the aggregate cumulative expense over the time period of analysis introduced in (10).

Recalling the definitions of strategy vector and strategy set given in (11) and (12), respectively (see Section III), we formulate our DSM optimization problem as

$$\begin{aligned} \min_{\mathbf{x}} \quad & f^{(\mathcal{D})}(\mathbf{x}) \triangleq \sum_{n\in\mathcal{N}} f_n^{(\mathcal{D})}(\mathbf{x}_n, \mathbf{x}_{-n}) \\ \text{s.t.} \quad & \mathbf{x}_n \in \Omega_{\mathbf{x}_n}, \quad \forall n \in \mathcal{N} \end{aligned} \tag{31}$$

with

$$f_n^{(\mathcal{D})}(\mathbf{x}_n, \mathbf{x}_{-n}) \triangleq f_n(\mathbf{x}_n, \mathbf{x}_{-n}) + \frac{1}{N} p^{(\mathcal{P})}(\mathbf{x}) \tag{32}$$

where $f_n(\mathbf{x}_n, \mathbf{x}_{-n})$ represents the individual cumulative expense of user $n \in \mathcal{N}$ defined in (14) and where $p^{(\mathcal{P})}(\mathbf{x})$ denotes the aggregate cumulative expense of the passive users defined in (9), where we made explicit the dependence on the strategies of the active users. Note that in the objective function $f_n^{(\mathcal{D})}(\mathbf{x})$ there is a common term (equal for all users) $p^{(\mathcal{P})}(\mathbf{x})$, which is the cost associated with the aggregate load of the passive users. This cost is, in fact, a transferable utility and can be distributed among the active users in any arbitrary manner (e.g., as we did in (32)) without affecting the optimal value of the social function $f^{(\mathcal{D})}(\mathbf{x})$ in (31).

A. Distributed Dynamic Pricing Algorithm

Traditionally, optimization problems of the form of (31) have been tackled by using gradient-based algorithms, which solve a sequence of convex problems by convexifying the whole social function: because of that, they generally suffer from slow convergence. A faster algorithm can be obtained by following the approach recently proposed in [17] (see also [18] for more details): since each $f_n^{(\mathcal{D})}(\mathbf{x}_n, \mathbf{x}_{-n})$ is convex for any feasible $\mathbf{x}_{-n}$ (under the settings of Theorem 1), one can convexify only the nonconvex part, i.e., $\sum_{m\in\mathcal{N}\setminus\{n\}} f_m^{(\mathcal{D})}(\mathbf{x})$, and solve the sequence of resulting optimization problems. Since such a procedure preserves some structure of the original objective function, it is expected to be faster than classical gradient-based schemes. A formal description of the algorithm is given next.

Let us preliminary define $\mathbf{x}^{(i)} = (\mathbf{x}_n^{(i)})_{n=1}^N$ as the joint strategy vector at iteration i and the resulting aggregate load as

$$L^{(i)}(h) \triangleq L^{(\mathcal{P})}(h) + \sum_{m\in\mathcal{N}} l_m^{(i)}(h), \quad h = 1, \ldots, H \tag{33}$$

where $l_n^{(i)}(h)$ is the energy load of user $n \in \mathcal{N}$ at iteration i. We can then introduce the best-response mapping $\Omega_{\mathbf{x}} \ni \mathbf{x}^{(i)} \to \widehat{\mathbf{x}}_\tau(\mathbf{x}^{(i)}) \triangleq (\widehat{\mathbf{x}}_{\tau,n}(\mathbf{x}^{(i)}))_{n=1}^N$, where we have defined

$$\begin{aligned} \widehat{\mathbf{x}}_{\tau,n}(\mathbf{x}^{(i)}) \triangleq \underset{\mathbf{x}_n\in\Omega_{\mathbf{x}_n}}{\arg\min} \Big\{ & f_n^{(\mathcal{D})}\big(\mathbf{x}_n, \{L^{(i)}(h)\}_{h=1}^H\big) \\ & + \boldsymbol{\pi}_n\big(\{L^{(i)}(h)\}_{h=1}^H\big)^{\mathrm{T}} \mathbf{x}_n + \frac{\tau}{2}\|\mathbf{x}_n - \mathbf{x}_n^{(i)}\|^2 \Big\} \end{aligned} \tag{34}$$

and

$$\begin{aligned} & \boldsymbol{\pi}_n\big(\{L^{(i)}(h)\}_{h=1}^H\big) \\ & \triangleq \boldsymbol{\Delta}_n^{\mathrm{T}} \Big(C_h'\big(L^{(i)}(h)\big)\big(L^{(i)}(h) - l_n^{(i)}(h) - \frac{1}{N} L^{(\mathcal{P})}(h)\big)\Big)_{h=1}^H \end{aligned} \tag{35}$$

where $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n^{\mathrm{T}} \otimes \mathbf{I}_H$, with $\boldsymbol{\delta}_n$ defined as in (13). Note that each individual optimization in (34) is strongly convex under Theorem 1 and, therefore, has a unique solution (see Appendix II-A for details); (34) is thus well-defined. The proposed algorithm solving the social problem in (31) is formally described in Algorithm 2 below, whose convergence conditions are given in Theorem 3.

Algorithm 2 Distributed Dynamic Pricing Algorithm (DDPA)

`Data :` Set $i = 0$. Given $\{C_h(\cdot)\}_{h=1}^H$, $\{L^{(\mathcal{P})}(h)/N\}_{h=1}^H$, $\tau > 0$, and any feasible starting point $\mathbf{x}^{(0)} = (\mathbf{x}_n^{(0)})_{n=1}^N$:

`(S.1) :` `If` a suitable termination criterion is satisfied: `STOP`.

`(S.2) :` `For` $n \in \mathcal{N}$, each user computes $\mathbf{x}_n^{(i+1)}$ as

$$\begin{aligned} \mathbf{x}_n^{(i+1)} = \underset{\mathbf{x}_n\in\Omega_{\mathbf{x}_n}}{\arg\min} \Big\{ & f_n^{(\mathcal{D})}\big(\mathbf{x}_n, \{L^{(i)}(h)\}_{h=1}^H\big) \\ & + \boldsymbol{\pi}_n\big(\{L^{(i)}(h)\}_{h=1}^H\big)^{\mathrm{T}} \mathbf{x}_n + \frac{\tau}{2}\|\mathbf{x}_n - \mathbf{x}_n^{(i)}\|^2 \Big\} \end{aligned}$$

`End`

`(S.3) :` $i \leftarrow i + 1$; `Go to (S.1).`

Theorem 3: Given the social problem (31), suppose that the conditions of Theorem 1 hold and that the regularization parameter τ satisfies

$$\tau \geq \max_h \left((N+1) C'_h(L^{(\max)}) + \max_{L^{(\min)} \leq x \leq L^{(\max)}} \left(C''_h(x) x \right) \right) \tag{36}$$

where N is the number of active users connected to the grid and $L^{(\max)}$ is the maximum aggregate energy load allowed by the grid infrastructure. Then, either Algorithm 2 converges in a finite number of iterations to a stationary solution of (31) or every limit point of the sequence $\{\mathbf{x}^{(i)}\}_{i=1}^{\infty}$ is a stationary solution of (31).

Proof: See Appendix II-A.

Differently from Algorithm 1, Algorithm 2 is not incentive compatible, in the sense that demand-side users need to reach an agreement in following the best-response protocol (34). In addition, it differs from Algorithm 1 mainly in the synchronous update of the users' strategies. However, Algorithm 2 converges under consistently milder conditions on the grid cost functions than those of Algorithm 1 and, most importantly, it does not impose any limitation on the number of active users with respect to the total number of demand-side users, which means better scalability. Lastly, the signaling required by the two algorithms is essentially the same.

Let us summarize the proposed demand-side day-ahead optimization based on Algorithm 2. At the beginning of the optimization process, τ is computed as in (36) by the central unit and broadcast to each user $n \in \mathcal{N}$, together with the grid cost functions per unit of energy $\{C_h(\cdot)\}_{h=1}^{H}$ and the terms related to the transferable utility $\{L^{(\mathcal{P})}(h)/N\}_{h=1}^{H}$. Then, at each iteration, all users simultaneously update their strategies by solving their own optimization problems in (34) based on the aggregate energy loads $\{L^{(i)}(h)\}_{h=1}^{H}$, which are calculated by the central unit summing up the individual demands. Then, active users provide their new energy loads to the central unit, and this process is iterated until a suitable termination criterion imposed by the central unit is satisfied.

V. Evaluation of the DSM Approaches

A. Smart Grid Setup

Let us consider a smart grid consisting of 1000 demand-side users $n \in \mathcal{D}$, each one having a random energy consumption curve with average daily energy consumption $\sum_{h=1}^{24} e_n(h) = 12$ kWh [35], and ranging between 8 kWh and 16 kWh. We suppose that higher consumption occurs more likely during daytime hours, i.e., from 08:00 to 24:00, than during night-time hours, i.e., from 00:00 to 08:00, reaching peak demand generally between 17:00 and 23:00. The energy grid cost function per unit of energy is given by

$$C_h\big(L(h)\big) \triangleq K_h L^2(h) = \begin{cases} K_{\text{night}} L^2(h), & \text{for } h = 1, \ldots, 8 \\ K_{\text{day}} L^2(h), & \text{for } h = 9, \ldots, 24 \end{cases} \tag{37}$$

where $K_{\text{day}} = 1.5 K_{\text{night}}$ as in [10] and whose values are chosen so as to obtain an initial average price per kWh of 0.1412 £/kWh [36]. Additionally, we consider $\zeta^{(\min)} = 1$ kWh, $\zeta^{(\max)} = 1.5$ kWh, $L^{(\min)} = 300$ kWh, and $L^{(\max)} = 800$ kWh. With this setup, condition (22) is immediately satisfied, guaranteeing that the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ has a nonempty and compact set of Nash equilibria. Recalling Theorem 2, Algorithm 1 is ensured to converge to one of these Nash equilibria for any $L^{(\min)} \geq N(\zeta^{(\min)} + \zeta^{(\max)})$, which implies that the number of active users should satisfy $N \leq 120$, and for any $\tau > 4 \max_h K_h N L^{(\max)}$. Lastly, according to Theorem 3, Algorithm 2 converges to a stationary solution of the social problem in (30) for any $\tau \geq \max_h K_h (N+2) L^{(\max)}$.

In the following, we consider $N = 120$ active users, with $|\mathcal{N}_{\mathcal{G} \setminus \mathcal{S}}| = |\mathcal{N}_{\mathcal{S} \setminus \mathcal{G}}| = |\mathcal{N}_{\mathcal{G} \cap \mathcal{S}}| = 40$, and $P = 880$ passive users. This corresponds to having 12% of active users equally distributed among dispatchable energy producers, energy storers, and dispatchable energy producers-storers. For the sake of simplicity, we assume that all dispatchable energy producers and energy storers adopt generators and storage devices with the same features as in [19, Sec. IV]. In particular, all generators employed by users $n \in \mathcal{G}$ are characterized by a linear production cost function, resembling that of a combustion engine (e.g., a biomass generator [37]) working in the linear region:

$$W_n(x) = \eta_n x \tag{38}$$

with $\eta_n = 0.039$ £/kWh [38], $g_n^{(\max)} = 0.4$ kW, $\gamma_n^{(\min)} = 0$ kWh, and $\gamma_n^{(\max)} = 0.8 g_n^{(\max)} \times 24$ h. Likewise, we suppose that all energy storage devices adopted by users $n \in \mathcal{S}$ present the following parameters: leakage rate $\alpha_n = \sqrt[24]{0.9}$,[8] capacity $c_n = 4$ kWh (this value is also used in [24] and is equivalent to the capacity of the battery of a small PHEV), maximum charging rate $s_n^{(\max)} = 0.125 c_n/\text{h}$, $q_n(0) = 0.25 c_n$, and $\epsilon_n = 0$.

B. Simulation Results

In this section, we provide some numerical results that illustrate the performance of the proposed noncooperative and cooperative day-ahead DSM mechanisms formalized in Algorithms 1 and 2, respectively. In doing so, we delineate the overall results and examine the convergence of both schemes, comparing the benefits achieved by the different types of active users. In particular, we show that all active users substantially reduce their monetary expense by adopting distributed energy generation and/or storage.

Interestingly, the overall results produced by the noncooperative and the cooperative approaches happen to be equivalent in our case: beyond any doubt, this constitutes a major strength of Algorithm 1. Fig. 3 illustrates the global results obtained equivalently using Algorithms 1 and 2. In the specific, Fig. 3(a) shows, for each hour h, the aggregate per-slot energy consumption $\sum_{n \in \mathcal{D}} e_n(h)$ together with the aggregate per-slot energy load $L(h)$ resulting from both approaches. Likewise, Fig. 3(b) delineates the aggregate per-slot energy production $\sum_{n \in \mathcal{G}} g_n(h)$ and storage $\sum_{n \in \mathcal{S}} s_n(h)$ at each hour h. As expected, energy storers charge their battery at the valley of the energy cost, resulting in a substantially more flattened demand curve. Contrarily, they discharge it at peak hours, shaving off

[8]This value of α_n corresponds to having a leakage rate of 0.9 over the 24 hours.

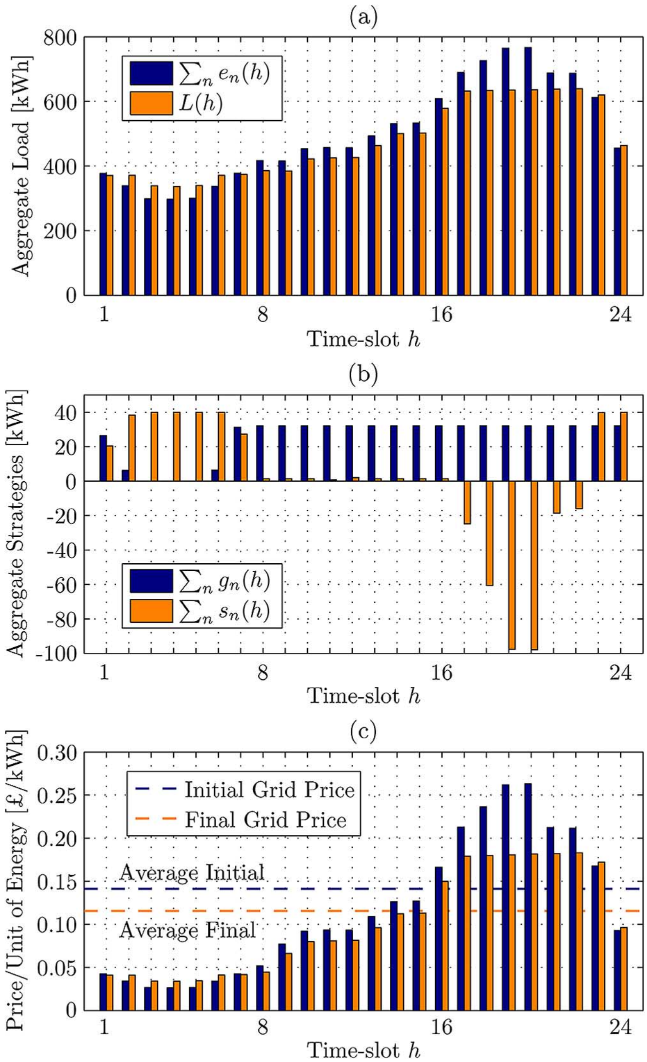

Fig. 3. (a) Initial aggregate per-slot energy consumption and aggregate per-slot energy load after both DSM optimizations at each h; (b) aggregate per-slot energy production and storage at each h; and (c) initial and final grid price per unit of energy at each h.

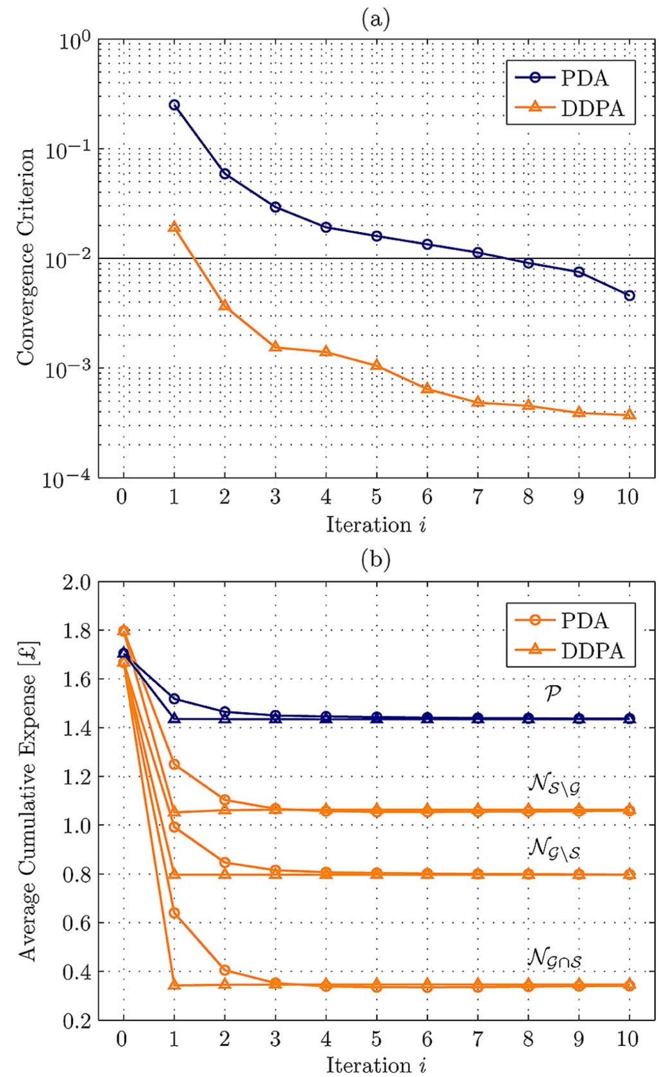

Fig. 4. (a) Convergence of Algorithm 1 (PDA) and Algorithm 2 (DDPA) with termination criterion $\|\mathbf{l}^{(i)} - \mathbf{l}^{(i-1)}\|_2/\|\mathbf{l}^{(i)}\|_2 \leq 10^{-2}$; and (b) average cumulative expense over the time period of analysis for each subset of active users, as a function of the iteration i.

the peak of the load. For the sake of comparison with ECS techniques [8]–[11], our day-ahead DSM optimization with just $|\mathcal{S}| = 80$ energy storers and the adopted storage capacities allows to shift 327 kWh from the peak hours to the valley of the demand curve: this is equivalent to having a shiftable load corresponding to 2.7% of the daily aggregate load among all 1000 demand-side users. On the other hand, dispatchable energy producers generate little energy during night-time hours, when they rather buy it from the grid. The average grid price per kWh reduces to 0.1156 £/kWh (i.e., 20.8% less) and, considering the individual energy production costs for users $n \in \mathcal{G}$, the overall price further decreases to 0.1116 £/kWh. The comparison between the initial and the final grid price at each hour h is illustrated in Fig. 3(c). Moreover, the aggregate cumulative expense $p^{(\mathcal{D})}$ reduces from £1705 to £1351. Finally, the peak-to-average ratio (PAR), defined as

$$\mathrm{PAR} \triangleq \frac{H \max_h L(h)}{\sum_{h=1}^{H} L(h)} \tag{39}$$

which expresses the ratio between the peak demand and the average energy demand calculated along the day, decreases from 1.5254 to 1.3337 (i.e., 12.6% less) resulting in a generally flattened demand curve.

We employ $\left\{\rho^{(i)}\right\}_{i=0}^{\infty} = 0.8$ for Algorithm 1, whereas the termination criterion used to finalize both algorithms is $\|\mathbf{l}^{(i)} - \mathbf{l}^{(i-1)}\|_2/\|\mathbf{l}^{(i)}\|_2 \leq 10^{-2}$. Fig. 4(a) plots this measure over the first 10 iterations. With the above setup, Algorithm 1 converges after 8 iterations and Algorithm 2 after just 2 iterations. In this regard, Fig. 4(b) shows how the average cumulative expenses over the time period of analysis for each type of active users, as well as for the passive users, converge to their final value: this further highlights the faster convergence of Algorithm 2, since the final values of the objective functions are approximately reached after the first iteration, even though active users keep adjusting their production and storage strategies until the above termination criterion is satisfied. From this figure it is also straightforward to conclude that active users with more degrees of freedom (i.e, both generation and storage equipment)

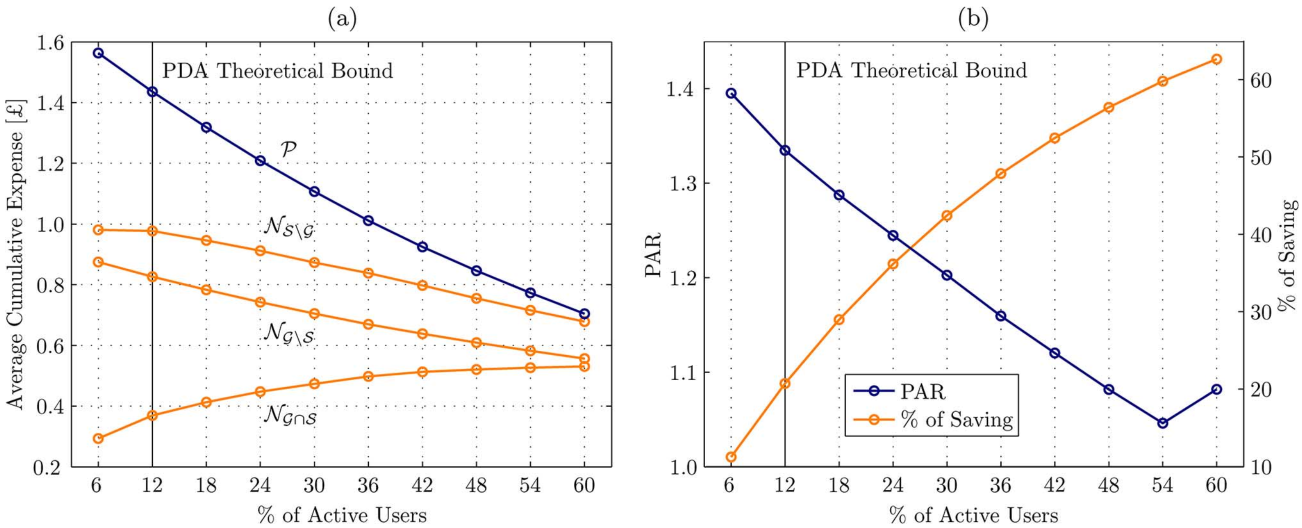

Fig. 5. (a) Average cumulative expense over the time period of analysis for each subset of users as a function of the percentage of active users; and (b) total percentage saving and PAR as functions of the percentage of active users. The active users are equally distributed among dispatchable energy producers, energy storers, and dispatchable energy producers-storers.

obtain better saving percentages, although the employment of distributed energy production and storage benefits all users in the smart grid. In particular, the average savings obtained for each subset of active users are: £1.3225 (i.e., 79.3% less) for users $n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$, £0.8717 (i.e., 52.3% less) for users $n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}}$, and £0.7348 (i.e., 40.9% less) for users $n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}}$. On the other hand, passive users $n \in \mathcal{P}$ save on average £0.2695 (i.e., 15.8% less) each. Evidently, the saving for users $n \in \mathcal{N}$ is greater than for users $n \in \mathcal{P}$, i.e., all demand-side users are incentivized to directly adopt distributed energy generation and/or storage. Moreover, users $n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$ save more than users $n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \cup \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}}$: this means that using both dispatchable energy sources and storage devices allows to further decrease the individual cumulative expense over the time period of analysis.

In Fig. 5(a) we plot the average cumulative expense over the time period of analysis for each subset of demand-side users versus different percentages of active users equally distributed among dispatchable energy producers, energy storers, and dispatchable energy producers-storers, with each demand-side user $n \in \mathcal{D}$ having the same consumption curve. Interestingly, Algorithm 1 keeps performing equivalently to Algorithm 2 even when the theoretical bound on the number of active users, $N > 120$, provided in Theorem 2 to ensure its convergence, is not fulfilled. Furthermore, we observe that the average cumulative expense of the active and passive users tend to the same value as the production and storage capacities increase. Besides, as illustrated in Fig. 5(b), the total saving of all (active and passive) users in the smart grid raises in inverse proportion with the decreasing PAR, which diminishes almost linearly as the percentage of active users increases. Note that, as the PAR approaches 1 with $N = 540$ (54%), its value raises unexpectedly when $N = 600$ (60%). This is due to the lower coefficients K_h adopted during $h = 1, \ldots, 8$ (c.f. (37)): in fact, once a perfectly flattened demand curve is achieved, active users naturally keep lowering the aggregate load during the last 16 hours when the price is higher in favor of the first 8 hours during which the price is lower.

Lastly, Fig. 6 depicts the number of iterations needed for the convergence of Algorithms 1 and 2 as a function of the percentage of active users, using the same termination criterion $\|\mathbf{l}^{(i)} - \mathbf{l}^{(i-1)}\|_2 / \|\mathbf{l}^{(i)}\|_2 \leq 10^{-2}$. In the first instance, the former always requires several more iterations than the latter, not to mention that each iteration in the proximal decomposition algorithm implies solving a (regularized) Nash game. Moreover, it is evident that the convergence speed of the proximal decomposition algorithm is substantially more related to the number of active participants than that of the distributed dynamic pricing algorithm, which emphasizes the better scalability properties of the latter.

VI. Conclusions

In this paper, we propose a general grid model that accommodates distributed energy production and storage, and a day-ahead DSM mechanism. In particular, we formulate the resulting grid optimization problem using a noncooperative method and a more classical nonlinear programming approach. In the first case, each active user on the demand-side selfishly minimizes his cumulative monetary expense for buying/producing his energy needs. We use noncooperative game theory and, building on the general framework of variational inequality, we derive (sufficient) conditions on the generalized energy cost functions that guarantee the existence of (multiple, yet equivalent) optimal strategies, as well as the convergence of the proposed asynchronous proximal decomposition algorithm. As for the second approach, we devise a distributed scheme based on the distributed dynamic pricing algorithm. Both methods allow to compute the optimal strategies of the users in a distributed fashion and with limited information exchange between the central unit and the demand-side of the network. Simulations on a realistic situation employing practical energy cost functions show that, despite their different (sufficient) convergence conditions, the two algorithms achieve equivalent overall results, sensibly flattening the demand curve and reducing the need for

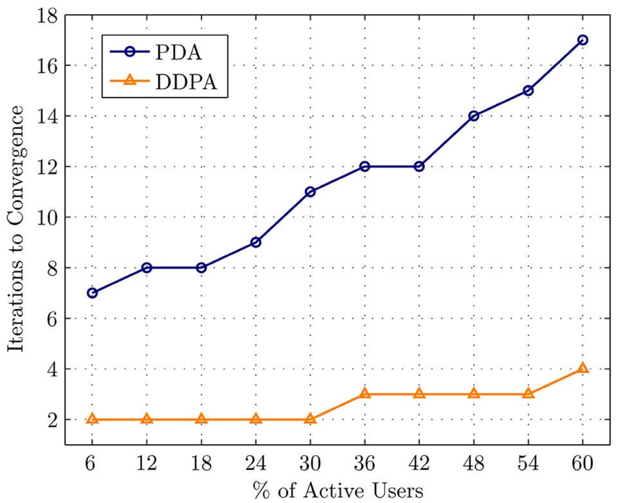

Fig. 6. Number of iterations required for the convergence of Algorithm 1 (PDA) and Algorithm 2 (DDPA), with termination criterion $\|\mathbf{l}^{(i)} - \mathbf{l}^{(i-1)}\|_2/\|\mathbf{l}^{(i)}\|_2 \leq 10^{-2}$, as a function of the percentage of active users.

carbon-intensive and expensive peaking power plants. Regardless, the two approaches present different characteristics in terms of strategy update and convergence speed that favor the employment of one over the other according to the situation. Finally, it is worth mentioning that the DSM techniques presented in this paper, being directly applicable to end users like households and small businesses, can also be extended to larger contexts, such as small communities or cities, by means of energy aggregators. In fact, flattening the energy demand along time is clearly beneficial at any layer of the energy grid.

Appendix I
Noncooperative DSM Approach

A. *Proof of Theorem* 1

In this appendix, we derive the conditions on the cost functions per unit of energy $\{C_h(\cdot)\}_{h=1}^{H}$ and on the production cost functions $\{W_n(\cdot)\}_{n\in\mathcal{G}}$ that guarantee the existence of the Nash equilibria of the game $\mathcal{G} = \langle\Omega_{\mathbf{x}}, \mathbf{f}\rangle$ in (17).

Recalling Lemma 1, the VI problem $\text{VI}(\Omega_{\mathbf{x}}, \mathbf{F})$, with $\mathbf{F}(\mathbf{x}) = \big(\nabla_{\mathbf{x}_n} f_n(\mathbf{x}_n, \mathbf{x}_{-n})\big)_{n=1}^{N}$, is equivalent to the game $\mathcal{G} = \langle\Omega_{\mathbf{x}}, \mathbf{f}\rangle$ if the objective function $f_n(\mathbf{x}_n, \mathbf{x}_{-n})$ in (14) is convex on $\Omega_{\mathbf{x}_n}$ for any $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}$, $\forall n \in \mathcal{N}$; note that the individual strategy sets $\Omega_{\mathbf{x}_n}$ in (12) are closed and convex. The convexity of each objective function is equivalent to the monotonicity of the associated mapping function $\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$ in (20) on $\Omega_{\mathbf{x}_n}$, for every given $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}$ [29], i.e.,

$$(\mathbf{x}_n-\mathbf{y}_n)^{\mathrm{T}}\big(\mathbf{F}_n(\mathbf{x}_n,\mathbf{x}_{-n})-\mathbf{F}_n(\mathbf{y}_n,\mathbf{x}_{-n})\big)\geq 0,\ \forall \mathbf{x}_n,\mathbf{y}_n\in\Omega_{\mathbf{x}_n}. \tag{40}$$

Next, we derive the conditions for $\mathbf{F}_n(\mathbf{x})$ to satisfy (40). We rewrite the left-hand side of (40) as

$$\begin{aligned}&(\mathbf{x}_n-\mathbf{y}_n)^{\mathrm{T}}\big(\mathbf{F}_n(\mathbf{x}_n,\mathbf{x}_{-n})-\mathbf{F}_n(\mathbf{y}_n,\mathbf{x}_{-n})\big)\\ &=\sum_{h=1}^{H}\Big(C_h\big(L_{-n}(h)+l_{\mathbf{x}_n}(h)\big)+C_h'\big(L_{-n}(h)+l_{\mathbf{x}_n}(h)\big)l_{\mathbf{x}_n}(h)\Big)\\ &\quad\times\big(l_{\mathbf{x}_n}(h)-l_{\mathbf{y}_n}(h)\big) \qquad (41)\\ &\quad-\sum_{h=1}^{H}\Big(C_h\big(L_{-n}(h)+l_{\mathbf{y}_n}(h)\big)+C_h'\big(L_{-n}(h)+l_{\mathbf{y}_n}(h)\big)l_{\mathbf{y}_n}(h)\Big)\\ &\quad\times\big(l_{\mathbf{x}_n}(h)-l_{\mathbf{y}_n}(h)\big) \qquad (42)\\ &\quad+\sum_{h=1}^{H}\big(W_n'\big(\boldsymbol{\delta}_{g,n}^{\mathrm{T}}\mathbf{x}_n(h)\big)-W_n'\big(\boldsymbol{\delta}_{g,n}^{\mathrm{T}}\mathbf{y}_n(h)\big)\big)\\ &\quad\times\boldsymbol{\delta}_{g,n}^{\mathrm{T}}\big(\mathbf{x}_n(h)-\mathbf{y}_n(h)\big) \qquad (43)\end{aligned}$$

where $L_{-n}(h) \triangleq L^{(\mathcal{P})}(h) + \sum_{m\in\mathcal{N}\setminus\{n\}} l_m(h)$, $l_{\mathbf{x}_n}(h) \triangleq e_n(h) + \boldsymbol{\delta}_n^{\mathrm{T}}\mathbf{x}_n(h)$, and $l_{\mathbf{y}_n}(h)$ is accordingly defined. Observe, then, that the term in (43) is nonnegative if $W_n(x)$ is convex, i.e., if $W_n'(x)$ is monotone:

$$W_n''(x) \geq 0, \qquad \forall x \in [0, g_n^{(\max)}] \tag{44}$$

since, under this condition, each element in the summation is nonnegative itself.

In addition, the sum of the terms in (41) and (42) is nonnegative if the function $C_h(x) + C_h'(x)\big(x - L_{-n}(h)\big)$ is increasing in $L^{(\min)} \leq x \leq L^{(\max)}$ for any time-slot h or, equivalently, if

$$2C_h'(x) + C_h''(x)\big(x - L_{-n}(h)\big) \geq 0, \quad \forall x \in [L^{(\min)}, L^{(\max)}]. \tag{45}$$

Assuming that for any time-slot h the grid cost function $C_h(x)$ is convex, i.e., $C_h''(x) \geq 0$, we can distinguish between two cases:

(i) When $C_h''(x) = 0$, the inequality in (45) is satisfied if $C_h'(x) \geq 0$, which forces $C_h(x)$ to be increasing;

(ii) When $C_h''(x) > 0$, it holds that

$$2C_h'(x) + C_h''(x)\big(x - L_{-n}(h)\big) \geq 2C_h'(x) - C_h''(x)l_n^{(\min)} \geq 0 \tag{46}$$

where $\max_n l_n^{(\min)}$ represents the maximum amount of energy that can be sold to the grid by any single user $n \in \mathcal{N}$ at any time-slot. Hence, (45) is verified if $C_h'(x) > 0$, i.e., if $C_h(x)$ is strictly increasing and, additionally, for any time-slot h it holds

$$C_h'(x) \geq \frac{1}{2}l_n^{(\min)}C_h''(x), \quad \forall x \in [L^{(\min)}, L^{(\max)}]. \tag{47}$$

So far, we have proved that $\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$ is monotone on $\mathbf{x}_n$, for any given $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}$, when the production cost function $W_n(x)$ is convex and the cost functions per unit of energy $\{C_h(x)\}_{h=1}^{H}$ are increasing and convex and satisfy (47). Nevertheless, this must be verified $\forall n \in \mathcal{N}$ and, therefore, constraint (47) becomes

$$C_h'(x) \geq \frac{1}{2}\big(\max_{n\in\mathcal{N}} l_n^{(\min)}\big)C_h''(x) = \frac{1}{2}\zeta^{(\min)}C_h''(x) \tag{48}$$

whereas the condition on the production cost function in (44) must be satisfied $\forall n \in \mathcal{G}$. Now that Lemma 1 holds, the solution set of $\mathcal{G} = \langle\Omega_{\mathbf{x}}, \mathbf{f}\rangle$ is nonempty and compact [15, Th. 4.1(a)], since the strategy sets $\Omega_{\mathbf{x}_n}$ are bounded $\forall n \in \mathcal{N}$. This concludes the proof of Theorem 1. $\square$

B. *Proof of Proposition* 1.1

Theorem 1 provides the conditions that guarantee the existence of the Nash equilibria of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ in (17). Although the solution is not unique, all Nash equilibria yield the same values of the objective functions in (14). In fact, consider a generic user $n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$: given two optimal strategy vectors $\mathbf{x}_{1,n}^\star \neq \mathbf{x}_{2,n}^\star$, with $\mathbf{x}_{1,n}^\star \triangleq (\mathbf{g}_{1,n}, \mathbf{s}_{1,n})$ and $\mathbf{x}_{2,n}^\star \triangleq (\mathbf{g}_{2,n}, \mathbf{s}_{2,n})$, we have that $f_n(\mathbf{x}_{1,n}^\star, \mathbf{x}_{-n}) = f_n(\mathbf{x}_{2,n}^\star, \mathbf{x}_{-n})$ if the following $H+2$ conditions hold (see the strategy sets (3) and (5) for details):

$$\sum_{h=1}^{H} W_n\big(g_{1,n}(h)\big) = \sum_{h=1}^{H} W_n\big(g_{2,n}(h)\big) \tag{49}$$

$$s_{1,n}(h) - g_{1,n}(h) = s_{2,n}(h) - g_{2,n}(h), \; h = 1, \ldots, H \tag{50}$$

$$\sum_{h=1}^{H} \alpha_n^{(H-h)} s_{1,n}(h) = \sum_{h=1}^{H} \alpha_n^{(H-h)} s_{2,n}(h). \tag{51}$$

Since in any realistic situation $H > 2$, and being $\mathbf{x}_n \in \mathbb{R}^{2H}$ for $n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$, it follows that $2H > H+2$, implying that user $n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$ can choose among infinitely many optimal strategy vectors $\mathbf{x}_n^\star$, each of them giving the same value of the objective function $f_n(\mathbf{x}_n^\star, \mathbf{x}_{-n})$. We can extend the previous considerations to all users: since all $\mathbf{x}_n^\star$ produce the same $\{l_n^\star(h)\}_{h=1}^H$, $\forall n \in \mathcal{N}$, the aggregate demands $\{L^\star(h)\}_{h=1}^H$, with $L^\star(h) \triangleq L^{\mathcal{P}} + \sum_{n\in\mathcal{N}}^{H} l_n^\star(h)$, are not affected by the multiplicity of the Nash equilibria. Hence, any $\mathbf{x}^\star \triangleq (\mathbf{x}_n^\star)_{n=1}^N$ leads to the same values of the objective functions $\{f_n(\mathbf{x}_n^\star, \mathbf{x}_{-n})\}_{n\in\mathcal{N}}$. $\square$

C. *Proof of Theorem* 2

It follows from [15, Th. 4.3] that the sequence generated by the proximal decomposition algorithm described in Algorithm 1 converges to a solution of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$ if the following conditions are satisfied: (a) the mapping function $\mathbf{F}(\mathbf{x})$ is monotone on $\Omega_{\mathbf{x}}$; and (b) the regularization parameter τ is such that the mapping $\mathbf{F}(\mathbf{x}) + \tau(\mathbf{I}_N - \mathbf{x}^{(i)})$ is strongly monotone on $\Omega_{\mathbf{x}}$, for any given $\mathbf{x}^{(i)} \in \Omega_{\mathbf{x}}$. Both conditions are proven next.

Proof of Theorem 2(a)*:* In this appendix, we derive additional conditions on the grid cost functions per unit of energy $\{C_h(\cdot)\}_{h=1}^H$ that guarantee the monotonicity of $\mathbf{F}(\mathbf{x}) = \big(\nabla_{\mathbf{x}_n} f_n(\mathbf{x}_n, \mathbf{x}_{-n})\big)_{n=1}^N$ on $\Omega_{\mathbf{x}}$, with $f_n(\mathbf{x}_n, \mathbf{x}_{-n})$ defined in (14). We assume next that the requirements given by Theorem 1 are satisfied.

The mapping $\mathbf{F}(\mathbf{x})$ is monotone on $\Omega_{\mathbf{x}}$ if and only if the Jacobian matrix $\mathrm{J}\mathbf{F}(\mathbf{x})$ satisfies [15, eq. (4.8(i))]

$$\frac{1}{2}\mathbf{z}^{\mathrm{T}}\big(\mathrm{J}\mathbf{F}(\mathbf{x}) + \mathrm{J}\mathbf{F}^{\mathrm{T}}(\mathbf{x})\big)\mathbf{z} \geq 0, \quad \forall \mathbf{x} \in \Omega_{\mathbf{x}}, \, \forall \mathbf{z} \in \mathbb{R}^{\omega_{\mathbf{x}}}. \tag{52}$$

Given $\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$ in (20), the partial Jacobian matrices of $\mathbf{F}(\mathbf{x})$ are

$$\begin{aligned}
&\mathrm{J}_{\mathbf{x}_n}\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n}) \\
&\quad \triangleq 2\boldsymbol{\Delta}_n^{\mathrm{T}}\mathbf{D}_{\mathbf{c}'}(\mathbf{x})\boldsymbol{\Delta}_n + \boldsymbol{\Delta}_n^{\mathrm{T}}\mathrm{Diag}\big(\mathbf{D}_{\mathbf{c}''}(\mathbf{x})(\mathbf{e}_n + \boldsymbol{\Delta}_n\mathbf{x}_n)\big)\boldsymbol{\Delta}_n \\
&\qquad + \boldsymbol{\Delta}_{g,n}^{\mathrm{T}}\mathbf{D}_{\mathbf{w}_n''}(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n)\boldsymbol{\Delta}_{g,n}
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\mathrm{J}_{\mathbf{x}_m}\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n}) \\
&\quad \triangleq \boldsymbol{\Delta}_n^{\mathrm{T}}\mathbf{D}_{\mathbf{c}'}(\mathbf{x})\boldsymbol{\Delta}_m + \boldsymbol{\Delta}_n^{\mathrm{T}}\mathrm{Diag}\big(\mathbf{D}_{\mathbf{c}''}(\mathbf{x})(\mathbf{e}_n + \boldsymbol{\Delta}_n\mathbf{x}_n)\big)\boldsymbol{\Delta}_m, \\
&\quad n \neq m
\end{aligned} \tag{54}$$

where $\mathbf{D}_{\mathbf{c}'}(\mathbf{x}) \triangleq \mathrm{Diag}\big(\mathbf{c}'(\mathbf{x})\big)$, $\mathbf{D}_{\mathbf{c}''}(\mathbf{x}) \triangleq \mathrm{Diag}\big(\mathbf{c}''(\mathbf{x})\big)$, and $\mathbf{D}_{\mathbf{w}_n''}(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n) \triangleq \mathrm{Diag}\big(\mathbf{w}_n''(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n)\big)$ are $H \times H$ diagonal matrices. By defining $\mathbf{J}(\mathbf{x}) \triangleq \frac{1}{2}\big(\mathrm{J}\mathbf{F}(\mathbf{x}) + \mathrm{J}\mathbf{F}^{\mathrm{T}}(\mathbf{x})\big)$ and decomposing vector $\mathbf{z}$ as $\mathbf{z} \triangleq (\mathbf{z}_n)_{n=1}^N$, where $\mathbf{z}_n \triangleq \big(z_n(1), \ldots, z_n(\boldsymbol{\delta}_n^{\mathrm{T}}\boldsymbol{\delta}_n H)\big)$, we can rewrite the left-hand side of (52) as in (55)–(56) at the bottom of the page. Observing the first term in (55), we are already in the position to state that, as long as $\mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \neq \emptyset$, $\mathbf{J}(\mathbf{x})$ cannot even be positive definite: in fact, we can have that $\boldsymbol{\Delta}_n\mathbf{z}_n = \mathbf{0}$ with $\mathbf{z}_n \neq \mathbf{0}$ for $n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}}$, whereas we cannot guarantee $W_n(x)$ to be strictly convex (i.e., $W_n''(x) > 0$) for these users.[9] Hence, let us now introduce

$$\tilde{z}_n(h) \triangleq [\boldsymbol{\Delta}_n\mathbf{z}_n]_h = \begin{cases} -z_n(h), & \text{if } n \in \mathcal{N}_{\mathcal{G}\setminus\mathcal{S}} \\ z_n(h), & \text{if } n \in \mathcal{N}_{\mathcal{S}\setminus\mathcal{G}} \\ \big(-z_n(h) + z_n(h+H)\big), & \text{if } n \in \mathcal{N}_{\mathcal{G}\cap\mathcal{S}} \end{cases} \tag{57}$$

so that we can express the left-hand side of (52) as

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}}\mathbf{J}(\mathbf{x})\mathbf{z} &= \sum_{h=1}^{H}\sum_{n\in\mathcal{N}} C_h'\big(L(h)\big)\tilde{z}_n^2(h) \\
&\quad + \sum_{h=1}^{H}\sum_{n\in\mathcal{G}} W_n''\big(g_n(h)\big)\tilde{z}_n^2(h) \qquad (58)\\
&\quad + \sum_{h=1}^{H}\sum_{n\in\mathcal{N}} \Big(C_h'\big(L(h)\big) + C_h''\big(L(h)\big)l_n(h)\Big) \\
&\quad \times \Big(\tilde{z}_n(h)\sum_{m\in\mathcal{N}}\tilde{z}_m(h)\Big). \qquad (59)
\end{aligned}$$

Let us now concentrate on the term in (56). Note that, under condition (22) in Theorem 1(a), $C_h'\big(L(h)\big) + C_h''\big(L(h)\big)l_n(h) \geq 0$ at any time-slot h, $\forall n \in \mathcal{N}$. Then, it follows that [see (60)

[9]Recall that best-response algorithms such as [15, Alg. 5.1] converge under sufficient conditions that imply the strict monotonicity of $\mathbf{F}(\mathbf{x})$ on $\Omega_{\mathbf{x}}$. It is not difficult to show that such requirement forces $\mathcal{N}_{\mathcal{G}\cap\mathcal{N}} = \emptyset$, which is too restrictive and cannot be guaranteed.

$$\mathbf{z}^{\mathrm{T}}\mathbf{J}(\mathbf{x})\mathbf{z} = \sum_{n\in\mathcal{N}}(\boldsymbol{\Delta}_n\mathbf{z}_n)^{\mathrm{T}}\mathbf{D}_{\mathbf{c}'}(\mathbf{x})(\boldsymbol{\Delta}_n\mathbf{z}_n) + \sum_{n\in\mathcal{G}}(\boldsymbol{\Delta}_{g,n}\mathbf{z}_n)^{\mathrm{T}}\mathbf{D}_{\mathbf{w}_n''}(\boldsymbol{\Delta}_{g,n}\mathbf{x}_n)(\boldsymbol{\Delta}_{g,n}\mathbf{z}_n) \tag{55}$$

$$+ \sum_{n,m\in\mathcal{N}}(\boldsymbol{\Delta}_n\mathbf{z}_n)^{\mathrm{T}}\Big(\mathbf{D}_{\mathbf{c}'}(\mathbf{x}) + \frac{1}{2}\mathrm{Diag}\big(\mathbf{D}_{\mathbf{c}''}(\mathbf{x})(\mathbf{e}_n + \mathbf{e}_m + \boldsymbol{\Delta}_n\mathbf{x}_n + \boldsymbol{\Delta}_m\mathbf{x}_m)\big)\Big)(\boldsymbol{\Delta}_m\mathbf{z}_m). \tag{56}$$

at the bottom of the page] where we have defined $\zeta^{(\min)} = \max_{n\in\mathcal{N}} l_n^{(\min)}$, $\zeta^{(\max)} = \max_{n\in\mathcal{N}} l_n^{(\max)}$, and the sets

$$\mathcal{N}^+ \triangleq \Big\{n \in \mathcal{N} : \tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h) \geq 0\Big\}$$
$$\mathcal{N}^- \triangleq \Big\{n \in \mathcal{N} : \tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h) < 0\Big\}. \tag{61}$$

Then, assuming for instance that $\sum_{m\in\mathcal{N}} z_m(h) \geq 0$ and recalling the inequality in (60), we have that

$$\begin{aligned}
&\sum_{n\in\mathcal{N}} \Big(C_h'\big(L(h)\big) + C_h''\big(L(h)\big)l_n(h)\Big)\Big(\tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big)\\
&\geq \Big(C_h'\big(L(h)\big) - \zeta^{(\min)} C_h''\big(L(h)\big)\Big)\\
&\quad\times \sum_{n\in\mathcal{N}^+} \tilde{z}_n(h)\Big(\sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big)\\
&\quad- \Big(C_h'\big(L(h)\big) + \zeta^{(\max)} C_h''\big(L(h)\big)\Big)\\
&\quad\times \sum_{n\in\mathcal{N}^-} |\tilde{z}_n(h)|\Big(\sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big) \qquad (62)\\
&\geq -\big(\zeta^{(\min)} + \zeta^{(\max)}\big) C_h''\big(L(h)\big)\\
&\quad\times \Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big) \qquad (63)
\end{aligned}$$

where in (63) we have used

$$\sum_{n\in\mathcal{N}^+} \big|\tilde{z}_n(h)\big| = \sum_{n\in\mathcal{N}^+} \tilde{z}_n(h) \geq \sum_{n\in\mathcal{N}^-} \big|\tilde{z}_n(h)\big| = -\sum_{n\in\mathcal{N}^-} \tilde{z}_n(h). \tag{64}$$

On the other hand, when $\sum_{m\in\mathcal{N}} \tilde{z}_m(h) < 0$, we know that

$$\sum_{n\in\mathcal{N}^+} \big|\tilde{z}_n(h)\big| = -\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h) < \sum_{n\in\mathcal{N}^-} \big|\tilde{z}_n(h)\big| = \sum_{n\in\mathcal{N}^-} \tilde{z}_n(h) \tag{65}$$

and, following similar steps, we obtain

$$\begin{aligned}
&\sum_{n\in\mathcal{N}} \Big(C_h'\big(L(h)\big) + C_h''\big(L(h)\big)l_n(h)\Big)\Big(\tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big)\\
&\geq -\big(\zeta^{(\min)} + \zeta^{(\max)}\big) C_h''\big(L(h)\big)\Big(\sum_{n\in\mathcal{N}^-} \tilde{z}_n(h)\Big|\sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big|\Big).
\end{aligned} \tag{66}$$

Let us consider the lower bound in (63): the term in (56) satisfies

$$\begin{aligned}
&\sum_{h=1}^{H} \sum_{n\in\mathcal{N}} \Big(C_h'\big(L(h)\big) + C_h''\big(L(h)\big)l_n(h)\Big)\Big(\tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big)\\
&\geq -\big(\zeta^{(\min)} + \zeta^{(\max)}\big)\\
&\quad\times \sum_{h=1}^{H} C_h''\big(L(h)\big)\Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big)
\end{aligned} \tag{67}$$

with

$$(67) \geq -\big(\zeta^{(\min)} + \zeta^{(\max)}\big) \sum_{h=1}^{H} C_h''\big(L(h)\big)\Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h)\Big)^2 \tag{68}$$

and, by substituting back in (56), it holds that

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}} \mathbf{J}(\mathbf{x})\mathbf{z} \geq &\sum_{h=1}^{H} C_h'\big(L(h)\big)\Big(\sum_{n\in\mathcal{N}} \tilde{z}_n^2(h)\Big)\\
&+ \sum_{h=1}^{H} \sum_{n\in\mathcal{G}} W_n''\big(g_n(h)\big)\tilde{z}_n(h)^2\\
&- \big(\zeta^{(\min)} + \zeta^{(\max)}\big) \sum_{h=1}^{H} C_h''\big(L(h)\big)\\
&\times \Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h)\Big)^2.
\end{aligned} \tag{69}$$

Then, invoking the Cauchy-Schwartz Inequality [39, eq. (3.2.9)]:

$$\sum_{n\in\mathcal{N}^+} \tilde{z}_n^2(h) \geq \frac{1}{|\mathcal{N}^+|}\Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h)\Big)^2 \geq \frac{1}{N}\Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n(h)\Big)^2 \tag{70}$$

it follows that

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}} \mathbf{J}(\mathbf{x})\mathbf{z} \geq &\sum_{h=1}^{H} \Big(C_h'\big(L(h)\big) - N\big(\zeta^{(\min)} + \zeta^{(\max)}\big) C_h''\big(L(h)\big)\Big)\\
&\times \Big(\sum_{n\in\mathcal{N}^+} \tilde{z}_n^2(h)\Big) + \sum_{h=1}^{H} \Big(C_h'\big(L(h)\big) \sum_{n\in\mathcal{N}^-} \tilde{z}_n^2(h)\Big)\\
&+ \sum_{h=1}^{H} \sum_{n\in\mathcal{G}} W_n''\big(g_n(h)\big)\tilde{z}_n^2(h) \qquad (71)\\
\geq &\sum_{h=1}^{H} \Big(C_h'\big(L(h)\big) - N\big(\zeta^{(\min)} + \zeta^{(\max)}\big) C_h''\big(L(h)\big)\Big)\\
&\times \Big(\sum_{n\in\mathcal{N}} \tilde{z}_n^2(h)\Big). \qquad (72)
\end{aligned}$$

The result in (72) can be equivalently obtained by considering the lower bound in (66), which simply corresponds to swapping $\mathcal{N}^+$ and $\mathcal{N}^-$ in (67)–(71). Finally, the inequality in (52) is satisfied as long as

$$C_h'(x) \geq N\big(\zeta^{(\min)} + \zeta^{(\max)}\big) C_h''(x), \; L^{(\min)} \leq x \leq L^{(\max)}. \tag{73}$$

Therefore, $\mathbf{F}(\mathbf{x})$ is monotone on $\Omega_{\mathbf{x}}$ if (73) is satisfied, and this completes the proof of Theorem 2(a). □

$$\Big(C_h'\big(L(h)\big) + C_h''\big(L(h)\big)l_n(h)\Big)\Big(\tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\Big) \geq \begin{cases} \Big(C_h'\big(L(h)\big) - \zeta^{(\min)} C_h''\big(L(h)\big)\Big)\big|\tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\big|, & \text{if } n \in \mathcal{N}^+\\ -\Big(C_h'\big(L(h)\big) + \zeta^{(\max)} C_h''\big(L(h)\big)\Big)\big|\tilde{z}_n(h) \sum_{m\in\mathcal{N}} \tilde{z}_m(h)\big|, & \text{if } n \in \mathcal{N}^- \end{cases} \tag{60}$$

Proof of Theorem 2(b): Here, we derive the condition on the regularization parameter τ for the convergence of Algorithm 1 to one of the Nash Equilibria of the game $\mathcal{G} = \langle \Omega_{\mathbf{x}}, \mathbf{f} \rangle$. By [15, Cor. 4.1], it is sufficient to choose τ large enough such that the matrix $\boldsymbol{\Upsilon}_{\mathrm{F},\tau} \triangleq \boldsymbol{\Upsilon}_{\mathrm{F}} + \tau \mathbf{I}_N$ is a P-matrix, where

$$[\boldsymbol{\Upsilon}_{\mathrm{F}}]_{nm} \triangleq \begin{cases} \upsilon_n^{(\min)}, & \text{if } n = m \\ -\upsilon_{nm}^{(\max)}, & \text{if } n \neq m \end{cases} \tag{74}$$

with

$$\begin{aligned} \upsilon_n^{(\min)} &\triangleq \min_{\mathbf{x} \in \Omega_{\mathbf{x}}} \lambda_{\min}\big(\mathsf{J}_{\mathbf{x}_n} \mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})\big) \\ \upsilon_{nm}^{(\max)} &\triangleq \max_{\mathbf{x} \in \Omega_{\mathbf{x}}} \|\mathsf{J}_{\mathbf{x}_m} \mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})\| \end{aligned} \tag{75}$$

where $\mathsf{J}_{\mathbf{x}_m} \mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$ is the partial Jacobian matrix defined in (53)–(54), and $\lambda_{\min}\big(\mathsf{J}_{\mathbf{x}_n} \mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})\big)$ denotes the smallest eigenvalue of $\mathsf{J}_{\mathbf{x}_n} \mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$.

In the proof of Theorem 2(a), we have shown that, under the conditions of Theorem 1, $\mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n})$ is monotone on $\mathbf{x}_n$, for any given $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}$, implying that $\mathsf{J}_{\mathbf{x}_n} \mathbf{F}_n(\mathbf{x}_n, \mathbf{x}_{-n}) \succeq 0$ [15, eq. (4.8(i))], $\forall \mathbf{x}_n \in \Omega_{\mathbf{x}_n}$, $\forall n \in \mathcal{N}$. Hence, we have that $\upsilon_n^{(\min)} \geq 0$.

Now, let us examine $\upsilon_{nm}^{(\max)}$ for $n \in \mathcal{N}_{\mathcal{G} \setminus \mathcal{S}} \cup \mathcal{N}_{\mathcal{S} \setminus \mathcal{G}}$, for whom $\boldsymbol{\Delta}_n^{\mathrm{T}} \boldsymbol{\Delta}_m = \mathbf{I}_H$ if $n, m \in \mathcal{N}_{\mathcal{G} \setminus \mathcal{S}}$ or if $n, m \in \mathcal{N}_{\mathcal{S} \setminus \mathcal{G}}$ and $\boldsymbol{\Delta}_n^{\mathrm{T}} \boldsymbol{\Delta}_m = -\mathbf{I}_H$ otherwise. Considering the first and worst case, we have:

$$\upsilon_{nm}^{(\max)} = \max_{\mathbf{x} \in \Omega_{\mathbf{x}}} \big\|\mathbf{D}_{\mathbf{c}'}(\mathbf{x}) + \mathrm{Diag}\big(\mathbf{D}_{\mathbf{c}''}(\mathbf{x})(\mathbf{e}_n + \boldsymbol{\Delta}_n \mathbf{x}_n)\big)\big\| \tag{76}$$

$$\leq \max_{\mathbf{x} \in \Omega_{\mathbf{x}}} \lambda_{\max}\Big(\mathbf{D}_{\mathbf{c}'}(\mathbf{x}) + \mathrm{Diag}\big(\mathbf{D}_{\mathbf{c}''}(\mathbf{x})(\mathbf{e}_n + \boldsymbol{\Delta}_n \mathbf{x}_n)\big)\Big) \tag{77}$$

with

$$\begin{aligned} (77) \leq{}& \max_h \Big(\max_{\mathbf{x} \in \Omega_{\mathbf{x}}} C'_h\big(L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}} \mathbf{x}_m(h)\big)\big)\Big) \\ &+ \max_h \Big(\max_{\mathbf{x} \in \Omega_{\mathbf{x}}} C''_h\big(L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}} \mathbf{x}_m(h)\big)\big) \\ &\times \big(e_n(h) + \boldsymbol{\delta}_n^{\mathrm{T}} \mathbf{x}_n(h)\big)\Big) \end{aligned} \tag{78}$$

$$\begin{aligned} \leq{}& \max_h C'_h(L^{(\max)}) + l_n^{(\max)} \\ &\times \max_h \Big(\max_{\mathbf{x} \in \Omega_{\mathbf{x}}} C''_h\big(L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}} \mathbf{x}_m(h)\big)\big)\Big). \end{aligned} \tag{79}$$

On the other hand, for $n \in \mathcal{N}_{\mathcal{G} \cap \mathcal{S}}$, we have that, for any $H \times H$ matrix $\mathbf{Q}$, it holds $\boldsymbol{\Delta}_n^{\mathrm{T}} \mathbf{Q} \boldsymbol{\Delta}_n = (2\mathbf{I}_2 - \mathbf{J}_2) \otimes \mathbf{Q}$, where $\mathbf{J}_2$ denotes the 2-dimensional unit matrix, and hence

$$\begin{aligned} \lambda_{\min}(\boldsymbol{\Delta}_n^{\mathrm{T}} \mathbf{Q} \boldsymbol{\Delta}_n) &= \begin{cases} 2\lambda_{\min}(\mathbf{Q}), & \text{if } \lambda_{\min}(\mathbf{Q}) < 0 \\ 0, & \text{otherwise} \end{cases} \\ \lambda_{\max}(\boldsymbol{\Delta}_n^{\mathrm{T}} \mathbf{Q} \boldsymbol{\Delta}_n) &= \begin{cases} 2\lambda_{\max}(\mathbf{Q}), & \text{if } \lambda_{\max}(\mathbf{Q}) > 0 \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \tag{80}$$

Combining the previous results, we can state that

$$\upsilon_n^{(\min)} \geq 0 \tag{81}$$

$$\begin{aligned} \upsilon_{nm}^{(\max)} \leq{}& 2 \max_h C'_h(L^{(\max)}) + 2 l_n^{(\max)} \\ &\times \max_h \Big(\max_{\mathbf{x} \in \Omega_{\mathbf{x}}} C''_h\big(L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}} \mathbf{x}_m(h)\big)\big)\Big) \end{aligned} \tag{82}$$

where we have considered the worst case of $\mathcal{N} = \mathcal{N}_{\mathcal{G} \cap \mathcal{S}}$.

Then, $\boldsymbol{\Upsilon}_{\mathrm{F},\tau}$ is a P-matrix if the condition in (83)–(84) at the bottom of the page is fulfilled [15, Prop. 4.3]. Evidently, the

$$\sum_{m \in \mathcal{N} \setminus \{n\}} \left(\frac{\upsilon_{nm}^{(\max)}}{\upsilon_n^{(\min)} + \tau} \right)$$

$$\leq \frac{2}{\tau} \sum_{m \in \mathcal{N} \setminus \{n\}} \bigg(\max_h C'_h(L^{(\max)}) + l_n^{(\max)} \max_h \Big(\max_{\mathbf{x} \in \Omega_{\mathbf{x}}} C''_h\big(L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}} \mathbf{x}_m(h)\big)\big)\Big)\bigg) \tag{83}$$

$$\leq \frac{2(N-1)}{\tau} \max_h C'_h(L^{(\max)}) + \frac{2L^{(\max)}}{\tau} \max_h \Big(\max_{\mathbf{x} \in \Omega_{\mathbf{x}}} C''_h\big(L^{(\mathcal{P})}(h) + \sum_{m \in \mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}} \mathbf{x}_m(h)\big)\big)\Big) < 1 \tag{84}$$

$$\begin{aligned} L_f \triangleq \big\|\mathsf{H}(\mathbf{x})\big\|_\infty = \max_i \sum_j \big|\mathsf{H}_{ij}(\mathbf{x})\big| \leq{}& \max_h \max_{\mathbf{x} \in \Omega_{\mathbf{x}}} \bigg(\boldsymbol{\delta}_n^{\mathrm{T}} \boldsymbol{\delta}_n \Big(2C'_h\big(L(h)\big) + C''_h\big(L(h)\big)\Big(l_n(h) + \frac{L^{(\mathcal{P})}(h)}{N}\Big)\Big) \\ &+ \sum_{m \in \mathcal{N} \setminus \{n\}} \boldsymbol{\delta}_m^{\mathrm{T}} \boldsymbol{\delta}_m \Big(C'_h\big(L(h)\big) + C''_h\big(L(h)\big)\Big(l_m(h) + \frac{L^{(\mathcal{P})}(h)}{N}\Big)\Big)\bigg) \end{aligned} \tag{87}$$

$$\leq 2 \max_h \max_{\mathbf{x} \in \Omega_{\mathbf{x}}} \Big((N+1) C'_h\big(L(h)\big) + C''_h\big(L(h)\big) L(h) \Big) \tag{88}$$

$$\leq 2 \max_h \Big((N+1) C'_h(L^{(\max)}) + \max_{L^{(\min)} \leq x \leq L^{(\max)}} \big(C''_h(x) x\big) \Big) \tag{89}$$

previous inequality is verified for any regularization parameter τ satisfying

$$\tau > 2(N-1)\max_h C_h'(L^{(\max)}) + 2L^{(\max)} \times \max_h \Big(\max_{\mathbf{x}\in\Omega_{\mathbf{x}}} C_h''\big(L^{(\mathcal{P})}(h) + \sum_{m\in\mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}}\mathbf{x}_m(h)\big)\big)\Big). \tag{85}$$

Finally, note that

$$\max_h \Big(\max_{\mathbf{x}\in\Omega_{\mathbf{x}}} C_h''\big(L^{(\mathcal{P})}(h) + \sum_{m\in\mathcal{N}} \big(e_m(h) + \boldsymbol{\delta}_m^{\mathrm{T}}\mathbf{x}_m(h)\big)\big)\Big) \leq \max_h \Big(\max_{L^{(\min)}\leq x \leq L^{(\max)}} C_h''(x)\Big). \tag{86}$$

In consequence, we can substitute the term on the right-hand side of (86) into (85), and this s completes the proof of Theorem 2(b). ☐

APPENDIX II
COOPERATIVE DSM APPROACH

A. *Proof of Theorem* 3

By [17, Th. 2], Algorithm 2 converges to a stationary solution of the social problem in (31) if the following conditions are satisfied: (a) the objective function $f_n^{(\mathcal{D})}(\mathbf{x}_n, \mathbf{x}_{-n})$ in (32) is convex on $\Omega_{\mathbf{x}_n}$ for any $\mathbf{x}_{-n} \in \Omega_{\mathbf{x}_{-n}}, \forall n \in \mathcal{N}$; (b) the regularization parameter τ satisfies $\tau \geq L_f/2 - \min_{n\in\mathcal{N}}(\upsilon_n^{(\min)})$, where L_f denotes the Lipschitz constant of $\nabla_{\mathbf{x}} f^{(\mathcal{D})}(\mathbf{x})$ on $\Omega_{\mathbf{x}}$ and $\upsilon_n^{(\min)}$ is defined in (75). Recall that the individual strategy sets $\Omega_{\mathbf{x}_n}$ in (12) are closed and convex and that the set $\Omega_{\mathbf{x}}$ is bounded.

Condition (a) is satisfied under the setting of Theorem 1. Therefore, we just need to prove that (36) implies condition (b) above. Recalling that $f^{(\mathcal{D})}(\mathbf{x}) = \sum_{n\in\mathcal{N}} f_n^{(\mathcal{D})}(\mathbf{x}_n, \mathbf{x}_{-n})$, and the definitions of the partial Jacobian matrices of $\mathbf{F}(\mathbf{x}) = \big(\nabla_{\mathbf{x}_n} f_n(\mathbf{x}_n, \mathbf{x}_{-n})\big)_{n=1}^N$ given in (53)–(54), the previous statement comes readily from (87)–(89) at the bottom of the previous page, with $\mathbf{H}(\mathbf{x})$ denoting the Hessian of $f^{(\mathcal{D})}(\mathbf{x})$ and $\boldsymbol{\delta}_n$ defined as in (13). This concludes the proof of Theorem 3.

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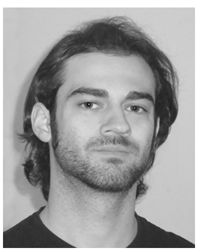

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