Lorentz-Positive Maps and Quadratic Matrix Inequalities With Applications to Robust MISO Transmit Beamforming

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***Abstract*—Consider a unicast downlink beamforming optimization problem with robust signal-to-interference-plus-noise ratio constraints to account for imperfect channel state information at the base station in a multiple-input single-output (MISO) communication system. The convexity of this robust beamforming problem remains unknown. A slightly conservative version of the robust beamforming problem is thus studied herein as a compromise. It is in the form of a semi-infinite second-order cone program (SOCP) and, more importantly, it possesses an equivalent and explicit convex reformulation, due to a linear matrix inequality (LMI) description of the cone of Lorentz-positive maps. Hence, the conservative robust beamforming problem can be efficiently solved by an optimization solver. Additional robust shaping constraints can also be easily handled to control the amount of interference generated on other co-existing users such as in cognitive radio systems.**

***Index Terms*—Lorentz-positive map, quadratic matrix inequality, robust MISO downlink beamforming, SDP, semi-infinite SOCP.**

I. Introduction

In multiuser communication systems, beamforming techniques provide a powerful approach to transmit signals that yield higher spectrum efficiency and larger downlink capacity for the system. The base station (BS) is equipped with multiple antennas, which allows the signals for different users to be spatially weighted with beamforming vectors (beamvectors) (see [1], [2]). In order to design the beamvectors, a basic beamforming optimization problem formulation is to minimize the transmission power while providing an acceptable quality-of-service (QoS) to each user, as well as keeping tolerable interference around some other directions.

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In a cellular unicast downlink (without considering limiting the interference caused along some directions), the beamforming design problem can be solved by convex optimization techniques, e.g., semidefinite programming (SDP) relaxation (see [1]–[5]), assuming that the perfect channel state information (CSI) is available at the BS when optimizing the beamvectors.

In practical situations, however, the available CSI is not perfect as it contains errors caused by estimation, limited channel state feedback quantization, or feedback delays. Thus, the design of beamforming robust to CSI errors is of great practical interest and has been recently considered in a large number of references (e.g., for multi-input single-output (MISO) systems in the cellular context, see [1], [6]–[8] and references therein, and for multi-input multi-output (MIMO) downlink systems, see [9], [10] and the references therein). However, most of the resulting robust downlink beamforming problems are inherently non-convex and, consequently, no global optimality of an efficient solution can be guaranteed theoretically. Nonetheless, in [8], sufficient conditions are presented to constrain some design parameters so that the robust beamforming problem becomes convex and, in [7], an ellipsoid method is proposed for a restricted (conservative) version of the robust problem. Note that the robust beamforming designs in [8] and [7] consider robust QoS constraints only (without external constraints), while minimizing the total transmission power.

In modern communication systems, several nearby systems employing a common frequency band may co-exist. This is to make the best use of the frequency shortage in communication systems. Examples include multiple access systems, peer-to-peer links, or cognitive wireless networks. When designing downlink beamforming vectors, one may reasonably want to take into consideration that the interference level caused to coexisting systems must be kept under tolerable limits. This motivates the introduction of the soft-shaping constraints on the beamvectors, under the assumption of either perfect or imperfect CSI (see, for example, [4], and cognitive radio (CR) [11]–[18] and references therein).

In this paper, we first revisit the robust beamforming problems of [8] and [7], and provide an equivalent and explicit convex reformulation for the robust optimization problem considered in [7]. The derived reformulation appears elegant and convenient due to a profound recent result of linear matrix inequality (LMI) description for the cone of Lorentz-positive maps [19]. Thanks to this novel explicit convex characterization, the beamforming optimization problem becomes very simple and one can make use of existing optimization solvers, e.g., `SeDuMi` [20]. Second, we consider the robust beamforming optimization problem involving not only the original QoS constraints, but also soft-shaping constraints to control the interference generated on co-existing systems, also under the assumption of imperfect CSI. The resulting robust beamforming optimization problem therefore comprises two types of robust second-order cone (SOC) constraints, and we reformulate it into a standard linear conic program, resorting to the LMI characterization for the cone of Lorentz-positive maps in [19] and a complex extension of the LMI representation for a class of robust quadratic matrix inequality (QMI) in [21]. The final reformulation of the problem results in a standard linear conic program that can be solved within polynomial-time computational complexity and implemented in a convenient fashion.

This paper is organized as follows. In Section II, we introduce the system model and formulate the optimal beamforming problems in the cellular multiuser downlink. In Section III, we study the LMI representation for the cone of Lorentz-positive maps and build the SDP reformulation of the conservative robust beamforming problem. In Section IV, we consider the robust optimal beamforming problem involving the additional robust soft-shaping constraints and reformulate it into a linear conic program via QMI. Section V contains some illustrative numerical results and Section VI draws some concluding remarks.

II. Problem Formulation

Consider a single-cell communication system with an N-antenna BS serving M decentralized single-antenna receivers (users). The signal transmitted by the BS is the vector $\boldsymbol{x}(t) = \sum_{m=1}^{M} \boldsymbol{w}_m s_m(t)$, where the information signal $s_m(t) \in \mathbb{C}$ intended for receiver m is temporally white with zero mean and unit variance, and $\boldsymbol{w}_m \in \mathbb{C}^N$ is the transmit beamforming vector for receiver m. The signal received by user m is given by

$$y_m(t) = \boldsymbol{h}_m^H \boldsymbol{x}(t) + n_m(t) \tag{1}$$

where $\boldsymbol{h}_m \in \mathbb{C}^N$ is the channel vector between the BS and receiver m, and $n_m(t)$ is the additive zero-mean noise with variance σ_m^2. The received signal-to-interference-plus-noise ratio (SINR) of user m is then

$$\mathrm{SINR}_m = \frac{\boldsymbol{w}_m^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_m}{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2}. \tag{2}$$

The downlink beamforming problem with perfect CSI is formulated as [6]:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{\boldsymbol{w}_m^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_m}{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2} \geq \gamma_m, \\ & m = 1, \ldots, M, \end{aligned} \tag{3}$$

where $\gamma_m > 0$ is the minimal acceptable SINR for user m. It is known that Problem (3) amounts to a second-order cone program (SOCP) as follows (which is convex and solvable):

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{1}{\sqrt{\gamma_m}} \Re\left(\boldsymbol{h}_m^H \boldsymbol{w}_m\right) \geq \sqrt{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2}, \\ & m = 1, \ldots, M. \end{aligned} \tag{4}$$

In the case that the CSI is not perfect, we model the m-th user's uncertain channel as $\boldsymbol{h}_m = \bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m$ where $\bar{\boldsymbol{h}}_m$ is the nominal channel vector (estimation) and $\boldsymbol{\delta}_m$ is the perturbation (channel estimation error) norm-bounded by ϵ_m, i.e., $\|\boldsymbol{\delta}_m\| \leq \epsilon_m$. Accordingly, the worst-case beamforming design problem is the following non-convex robust optimization problem (cf. [8])[1]:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{\boldsymbol{w}_m^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_m}{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_i + \sigma_m^2} \\ & \geq \gamma_m, \ \forall \boldsymbol{\delta}_m : \|\boldsymbol{\delta}_m\| \leq \epsilon_m, \ m = 1, \ldots, M. \end{aligned} \tag{5}$$

It is shown in [8] that when the perturbation bounds ϵ_m are small to some extent, or the number of transmit antennas is two (i.e., $N = 2$), the conventional SDP relaxation of (5) (cf. (P_ϵ) in [8]), obtained by applying the S-lemma (see e.g., ([22], p. 88), [24], [25], Lemma 4.1 in Section IV-A) and dropping the rank-one constraints $\boldsymbol{W}_m = \boldsymbol{w}_m \boldsymbol{w}_m^H$, is tight:

$$\underset{\{\boldsymbol{W}_m, \bar{\boldsymbol{W}}_m, \lambda_m\}}{\text{minimize}} \quad \sum_{m=1}^{M} \boldsymbol{I} \bullet \boldsymbol{W}_m \tag{6a}$$

$$\begin{aligned} \text{subject to} \quad & \begin{bmatrix} \bar{\boldsymbol{W}}_m & \boldsymbol{0} \\ \boldsymbol{0} & -\sigma_m^2 \end{bmatrix} + \lambda_m \begin{bmatrix} \boldsymbol{I} & -\bar{\boldsymbol{h}}_m \\ -\bar{\boldsymbol{h}}_m^H & \bar{\boldsymbol{h}}_m^H \bar{\boldsymbol{h}}_m - \epsilon_m^2 \end{bmatrix} \succeq \boldsymbol{0}, \\ & m = 1, \ldots, M, \end{aligned} \tag{6b}$$

$$\bar{\boldsymbol{W}}_m = \frac{1}{\gamma_m} \boldsymbol{W}_m - \sum_{i=1, i\neq m}^{M} \boldsymbol{W}_i, \tag{6c}$$

$$\lambda_m \geq 0, \ \boldsymbol{W}_m \succeq \boldsymbol{0}. \tag{6d}$$

Here, $\boldsymbol{W}_m \succeq \boldsymbol{0}$ means that the complex Hermitian matrix $\boldsymbol{W}_m$ is positive semidefinite, i.e., $\boldsymbol{W}_m \in \mathcal{H}_+^N$, and the notation bullet $\bullet$ denotes the inner product between two matrices. However, it remains to be understood whether (5) has an equivalent convex reformulation in general[2], notwithstanding existing numerical simulations (see, e.g., [8], [18], [26]) showing that the SDP relaxation always gives a rank-one optimal solution with some data sets.

Another interesting beamforming problem formulation is the robust extension of the optimal beamforming problem (4):

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{1}{\sqrt{\gamma_m}} \Re((\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_m) \\ & \geq \sqrt{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_i + \sigma_m^2}, \\ & \forall \boldsymbol{\delta}_m : \|\boldsymbol{\delta}_m\| \leq \epsilon_m, \ m = 1, \ldots, M. \end{aligned} \tag{7}$$

[1]Besides the worst-case formulation of SINR, there is a so-called chance constrained formulation of SINR to treat stochastic data uncertainty (see the robust optimization lecture notes [22], and the paper [23]). The feasible set of the chance constrained formulation very often is non-convex and a tractable safe (conservative) approximation is usually considered (see ([22], pp. 30–31))

[2]A similar difficulty also exists for the MIMO downlink beamforming problem, see e.g., [10] and [9].

Note that the feasible set of (5)[3] contains that of (7) (due to the simple observation $|(\bar{\boldsymbol{h}}_m+\boldsymbol{\delta}_m)^H\boldsymbol{w}_m| \geq \Re((\bar{\boldsymbol{h}}_m+\boldsymbol{\delta}_m)^H\boldsymbol{w}_m)$), and thus (7) is more conservative than the former. Observe that (7) is a semi-infinite SOCP and hence convex, but that does not mean that it can be solved efficiently.

Nevertheless, in [7], the authors present an iterative ellipsoidal method for (7) and show its convergence within polynomial complexity (see also [22]). In contrast, we herein show that Problem (7) possesses an equivalent convex LMI reformulation resorting to a recent profound result in [19], and thus can be solved efficiently, e.g., using a solver like `SeDuMi` [20].

III. Lorentz-Positive Maps and Robust Transmit Beamforming in a Cellular Multiuser Downlink

A. A Standard Form of Semi-Infinite SOCP for (7)

In this section, we will present an equivalent SDP reformulation of Problem (7), resorting to a result on LMI description of a robust SOC constraint recently obtained in [19]. To start with, let us rewrite (7) into a problem with real-valued design variables. We denote the real and imaginary parts of $\boldsymbol{w}_m = \Re\boldsymbol{w}_m + j\Im\boldsymbol{w}_m \in \mathbb{C}^N$ as follows:

$$\boldsymbol{w}_{m1} = \Re\boldsymbol{w}_m, \quad \boldsymbol{w}_{m2} = \Im\boldsymbol{w}_m, \quad m=1,\ldots,M, \tag{8}$$

and clearly $\boldsymbol{w}_{m1}\in\mathbb{R}^N$ and $\boldsymbol{w}_{m2}\in\mathbb{R}^N$. Likewise, $\bar{\boldsymbol{h}}_{m1}$, $\bar{\boldsymbol{h}}_{m2}$, $\boldsymbol{\delta}_{m1}$, $\boldsymbol{\delta}_{m2}$ are defined such that $\bar{\boldsymbol{h}}_m = \bar{\boldsymbol{h}}_{m1} + j\bar{\boldsymbol{h}}_{m2}$ and $\boldsymbol{\delta}_m = \boldsymbol{\delta}_{m1} + j\boldsymbol{\delta}_{m2}$, respectively. Denote

$$\boldsymbol{W}_{-m,1} = [\boldsymbol{w}_{11}\cdots\boldsymbol{w}_{m-1,1}\boldsymbol{w}_{m+1,1}\cdots\boldsymbol{w}_{M1}] \in \mathbb{R}^{N\times(M-1)}, \tag{9}$$

and $\boldsymbol{W}_{-m,2}\in\mathbb{R}^{N\times(M-1)}$ and $\boldsymbol{W}_{-m}\in\mathbb{C}^{N\times(M-1)}$ are defined analogously. Therefore, by letting $\boldsymbol{C}_m^T(\boldsymbol{w}_m,\boldsymbol{W}_{-m}) =$

$$\begin{bmatrix} \frac{1}{\sqrt{\gamma_m}}\boldsymbol{w}_{m1} & \boldsymbol{W}_{-m,1} & \boldsymbol{W}_{-m,2} & \boldsymbol{0} \\ \frac{1}{\sqrt{\gamma_m}}\boldsymbol{w}_{m2} & \boldsymbol{W}_{-m,2} & -\boldsymbol{W}_{-m,1} & \boldsymbol{0} \end{bmatrix} \in \mathbb{R}^{2N\times 2M} \tag{10}$$

and $\boldsymbol{c}_m^T(\boldsymbol{w}_m,\boldsymbol{W}_{-m}) =$

$$\left[\bar{\boldsymbol{h}}_{m1}^T\bar{\boldsymbol{h}}_{m2}^T\right]\boldsymbol{C}_m^T(\boldsymbol{w}_m,\boldsymbol{W}_{-m}) + [0,\ldots,0,\sigma_m] \in \mathbb{R}^{1\times 2M}, \tag{11}$$

we can express Problem (7) as the following real-valued optimization problem:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\},t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \left[t\ \boldsymbol{w}_{11}^T\ \boldsymbol{w}_{12}^T\ \cdots\ \boldsymbol{w}_{M1}^T\ \boldsymbol{w}_{M2}^T\right]^T \in \mathbb{L}^{2MN+1}, \\ & \boldsymbol{C}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})\begin{bmatrix}\boldsymbol{\delta}_{m1}\\ \boldsymbol{\delta}_{m2}\end{bmatrix} + \boldsymbol{c}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m}) \in \mathbb{L}^{2M} \\ & \forall \left\|\begin{bmatrix}\boldsymbol{\delta}_{m1}\\ \boldsymbol{\delta}_{m2}\end{bmatrix}\right\| \leq \epsilon_m,\ m=1,\ldots,M, \end{aligned} \tag{12}$$

where $\mathbb{L}^K$ represents the K-dimensional SOC (also termed Lorentz cone):

$$\mathbb{L}^K = \left\{\boldsymbol{x}\in\mathbb{R}^K \mid x_1 \geq \sqrt{x_2^2+\cdots+x_K^2}\right\}. \tag{13}$$

[3]For the prefect CSI case (cf. (3)), there is a full characterization of the feasible region in [27]; since there are infinitely many SINR constraints in robust problem (5), the feasible set could be empty especially when the perturbation bounds ϵ_m are large to some extent.

Note that $\boldsymbol{C}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})$ and $\boldsymbol{c}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})$ are affine with respect to (w.r.t.) $\{\boldsymbol{w}_m\}$, and that the optimal value of (12) is the square root of that of (7).

To simplify the notations, we drop the arguments $\{\boldsymbol{w}_m,\boldsymbol{W}_{-m}\}$ in $\boldsymbol{C}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})$ and $\boldsymbol{c}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})$, and rewrite the second set of constraints of (12) equivalently into:

$$[\boldsymbol{c}_m\ \boldsymbol{C}_m]\begin{bmatrix}1\\ \boldsymbol{\delta}_{m1}\\ \boldsymbol{\delta}_{m2}\end{bmatrix} \in \mathbb{L}^{2M}, \quad \forall \begin{bmatrix}\epsilon_m\\ \boldsymbol{\delta}_{m1}\\ \boldsymbol{\delta}_{m2}\end{bmatrix} \in \mathbb{L}^{2N+1}, \quad m=1,\ldots,M, \tag{14}$$

which is equivalent to

$$[\boldsymbol{c}_m\ \epsilon_m\ \boldsymbol{C}_m]\begin{bmatrix}1\\ \boldsymbol{\delta}_{m1}\\ \boldsymbol{\delta}_{m2}\end{bmatrix} \in \mathbb{L}^{2M}, \quad \forall \begin{bmatrix}1\\ \boldsymbol{\delta}_{m1}\\ \boldsymbol{\delta}_{m2}\end{bmatrix} \in \mathbb{L}^{2N+1}, \quad m=1,\ldots,M, \tag{15}$$

which in turn is recast into

$$[\boldsymbol{c}_m\ \epsilon_m\ \boldsymbol{C}_m]\tilde{\boldsymbol{\delta}}_m \in \mathbb{L}^{2M}, \quad \forall\tilde{\boldsymbol{\delta}}_m \in \mathbb{L}^{2N+1}, \quad m=1,\ldots,M, \tag{16}$$

where $\tilde{\boldsymbol{\delta}}_m = [\alpha_m, \boldsymbol{\delta}_{m1}^T, \boldsymbol{\delta}_{m2}^T]^T \in \mathbb{R}^{2N+1}$. To see the equivalence between (15) and (16), we assume that (15) holds and $\tilde{\boldsymbol{\delta}}_m \in \mathbb{R}^{2N+1}$. If $\alpha_m = 0$, then $\boldsymbol{\delta}_{m1} = \boldsymbol{\delta}_{m2} = \boldsymbol{0}$ and we have $[\boldsymbol{c}_m\epsilon_m\boldsymbol{C}_m]\tilde{\boldsymbol{\delta}}_m = \boldsymbol{0} \in \mathbb{L}^{2M}$. Now, suppose that $\alpha_m > 0$. From the observation $\tilde{\boldsymbol{\delta}}_m/\alpha_m \in \mathbb{L}^{2N+1}$ and (15), it follows that (16) is true. The implication from (16) to (15) is trivial.

Let us denote

$$\boldsymbol{B}_m = [\boldsymbol{c}_m\ \epsilon_m\boldsymbol{C}_m], \tag{17}$$

keeping in mind that $\boldsymbol{B}_m$ is indeed $\boldsymbol{B}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})$ affine w.r.t. the design variables. We can finally rewrite (12) (or (7)) using the Lorentz cone notation as:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\},t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \left[t\ \boldsymbol{w}_{11}^T\ \boldsymbol{w}_{12}^T\ \cdots\ \boldsymbol{w}_{M1}^T\ \boldsymbol{w}_{M2}^T\right]^T \in \mathbb{L}^{2MN+1}, \\ & \boldsymbol{B}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m})\tilde{\boldsymbol{\delta}}_m \in \mathbb{L}^{2M},\ \forall\tilde{\boldsymbol{\delta}}_m \in \mathbb{L}^{2N+1}, \\ & m=1,\ldots,M. \end{aligned} \tag{18}$$

In order to solve (18), let us deal with the second set of constraints, i.e., the robust SOC constraints. Define the set

$$\mathcal{B} = \left\{\boldsymbol{B}_m \in \mathbb{R}^{2M\times(2N+1)} \mid \boldsymbol{B}_m\boldsymbol{y}_m \in \mathbb{L}^{2M},\ \forall\boldsymbol{y}_m \in \mathbb{L}^{2N+1}\right\}. \tag{19}$$

The set (19) contains linear maps (or matrices) which take $\mathbb{L}^{2N+1}$ to $\mathbb{L}^{2M}$. Note that any Lorentz cone is self-dual. Thus, we have another equivalent expression for the set $\mathcal{B}$ as follows.

Lemma 3.1: The set $\mathcal{B}$ in (19) is equivalent to the matrix $\boldsymbol{B}_m$ being *Lorentz-positive*, i.e.,

$$\boldsymbol{x}_m^T\boldsymbol{B}_m\boldsymbol{y}_m \geq 0,\ \forall\boldsymbol{x}_m \in \mathbb{L}^{2M},\ \forall\boldsymbol{y}_m \in \mathbb{L}^{2N+1}.$$

The set $\mathcal{B}$ in (19) of all Lorentz-positive matrices forms a closed convex cone, and the cone has an LMI description, as shown in ([19], Theorem 5.6). Having such a convex LMI description of (19), we can claim that (18) has an equivalent SDP (or linear conic) reformulation. In order to present the theorem and reformulate it in an implementable way, we need some basic notions and facts to be introduced.

B. An LMI Characterization of the Cone of Lorentz-Positive Maps

Let $\mathcal{S}^N$ and $\mathcal{A}^N$ be the set of all $N \times N$ symmetric matrices and the set of all $N \times N$ skew-symmetric matrices, respectively, and let $\mathcal{L}_{L,K}$ stand for the $KL(K+1)(L+1)/4$-dimensional linear space of biquadratic forms (cf. ([21], p. 1148)):

$$\mathcal{L}_{L,K} = \left\{ \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \cdots & \boldsymbol{M}_{1L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{L1} & \cdots & \boldsymbol{M}_{LL} \end{bmatrix} \in \mathcal{S}^{KL} \;\middle|\; \boldsymbol{M}_{ln} \in \mathcal{S}^K \right\}. \tag{20}$$

The set $\mathcal{L}_{L,K}$ is a subspace of $\mathcal{S}^{KL}$, and the orthogonal complement of it within $\mathcal{S}^{KL}$ is clearly the $KL(K-1)(L-1)/4$-dimensional subspace:

$$\mathcal{L}_{L,K}^{\perp} = \left\{ \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \cdots & \boldsymbol{M}_{1L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{L1} & \cdots & \boldsymbol{M}_{LL} \end{bmatrix} \in \mathcal{S}^{KL} \;\middle|\; \boldsymbol{M}_{ln} \in \mathcal{A}^K \right\}. \tag{21}$$

That is, $\mathcal{S}^{KL} = \mathcal{L}_{L,K} \oplus \mathcal{L}_{L,K}^{\perp}$. By the notations, it is immediately seen that $\mathcal{S}_+^L \otimes \mathcal{S}_+^K \subseteq \mathcal{L}_{L,K}$ where $\mathcal{S}_+^K$ stands for the set of $K \times K$ positive semidefinite real symmetric matrices, and

$$\mathcal{A}^L \otimes \mathcal{A}^K \subseteq \mathcal{L}_{L,K}^{\perp}, \tag{22}$$

where $\otimes$ denotes the Kronecker product throughout the paper.

Given the vector $\boldsymbol{a} = [a_1, \ldots, a_K]^T$ (with $K \geq 3$), it is evident that $\boldsymbol{a} \in \mathbb{L}^K$ amounts to the arrow matrix (generated by $\boldsymbol{a}$)

$$\boldsymbol{A}(\boldsymbol{a}) = \begin{bmatrix} a_1 + a_2 & a_3 & a_4 & \cdots & a_K \\ a_3 & a_1 - a_2 & 0 & \cdots & 0 \\ a_4 & 0 & a_1 - a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_K & 0 & 0 & \cdots & a_1 - a_2 \end{bmatrix} \in \mathcal{S}^{K-1} \tag{23}$$

being positive semidefinite.

Let $\boldsymbol{G} \in \mathbb{R}^{L \times K}$ with $\boldsymbol{g}_l^T \in \mathbb{R}^K$ denoting the l-th row of $\boldsymbol{G}$ (i.e., $\boldsymbol{G}^T = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_L]$), and denote by $\hat{\boldsymbol{A}}(\boldsymbol{G})$ the arrow matrix generated by the L arrow matrices $[\boldsymbol{A}(\boldsymbol{g}_1), \ldots, \boldsymbol{A}(\boldsymbol{g}_L)]$ (with $L \geq 3$):

$$\hat{\boldsymbol{A}}(\boldsymbol{G}) = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{g}_0) & \boldsymbol{A}(\boldsymbol{g}_3) & \boldsymbol{A}(\boldsymbol{g}_4) & \cdots & \boldsymbol{A}(\boldsymbol{g}_L) \\ \boldsymbol{A}(\boldsymbol{g}_3) & \boldsymbol{A}(\boldsymbol{g}_{-1}) & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{A}(\boldsymbol{g}_4) & \boldsymbol{0} & \boldsymbol{A}(\boldsymbol{g}_{-1}) & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}(\boldsymbol{g}_L) & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}(\boldsymbol{g}_{-1}) \end{bmatrix} \in \mathcal{L}_{L-1,K-1}, \tag{24}$$

where $\boldsymbol{g}_0 = \boldsymbol{g}_1 + \boldsymbol{g}_2$ and $\boldsymbol{g}_{-1} = \boldsymbol{g}_1 - \boldsymbol{g}_2$.

It is easily seen that $\boldsymbol{A}(\boldsymbol{g}_0) = \boldsymbol{A}(\boldsymbol{g}_1) + \boldsymbol{A}(\boldsymbol{g}_2)$ and $\boldsymbol{A}(\boldsymbol{g}_{-1}) = \boldsymbol{A}(\boldsymbol{g}_1) - \boldsymbol{A}(\boldsymbol{g}_2)$, and that $\hat{\boldsymbol{A}}(\boldsymbol{b}\boldsymbol{a}^T) = \boldsymbol{A}(\boldsymbol{b}) \otimes \boldsymbol{A}(\boldsymbol{a})$, where $\boldsymbol{b} \in \mathbb{R}^L$ is given. It is clear also that $\hat{\boldsymbol{A}}(\boldsymbol{G})$ is matrix $\hat{\boldsymbol{A}}(\boldsymbol{b}\boldsymbol{a}^T)$ with element $b_l a_k$ substituted by G_{lk} (cf. ([22], p. 95)).

With the above notations in hand, we are ready to cite the following result ([19], Theorem 5.6) as a lemma, albeit written in a different way.

Lemma 3.2: Suppose that $\min\{L, K\} \geq 3$. Then a matrix $\boldsymbol{G} \in \mathbb{R}^{L \times K}$ is Lorentz-positive, if and only if there is $\boldsymbol{X} \in \mathcal{A}^{L-1} \otimes \mathcal{A}^{K-1}$ such that

$$\hat{\boldsymbol{A}}(\boldsymbol{G}) + \boldsymbol{X} \succeq \boldsymbol{0} \; \left(\in \mathcal{S}_+^{(K-1)(L-1)} \right), \tag{25}$$

where $\hat{\boldsymbol{A}}(\cdot)$ is defined in (24).

By exploiting the notation in (21), the relationship in (22), and Theorem 3.1 in [19], one can show the following which appears easily implementable.

Proposition 3.3: Suppose that $\min\{L, K\} \geq 3$. Then a matrix $\boldsymbol{G} \in \mathbb{R}^{L \times K}$ is Lorentz-positive, if and only if

$$\hat{\boldsymbol{A}}(\boldsymbol{G}) \in \mathcal{S}_+^{(K-1)(L-1)} + \mathcal{L}_{L-1,K-1}^{\perp}. \tag{26}$$

Proof: See Appendix A. ■

Capitalizing on Proposition 3.3, we can derive an equivalent condition for $\boldsymbol{B}_m$ complying with (19). In other words, $\boldsymbol{B}_m$ belonging to set $\mathcal{B}$ in (19) is tantamount to the condition that there is $\boldsymbol{X}_m \in \mathcal{L}_{2M-1,2N}^{\perp}$ such that

$$\hat{\boldsymbol{A}}(\boldsymbol{B}_m) + \boldsymbol{X}_m \in \mathcal{S}_+^{2(2M-1)N}, \tag{27}$$

which is an implementable LMI description for $\boldsymbol{B}_m \in \mathcal{B}$ (considering that $\hat{\boldsymbol{A}}(\boldsymbol{B}_m)$ is affine in $\boldsymbol{B}_m$).

C. Equivalent SDP Reformulation of Problem (7)

Considering that the robust beamforming problem (7) amounts to (18), we obtain an identical form of linear conic program (cf. [22]) for (7) and summarize it as follows.

Proposition 3.4: The robust MISO downlink beamforming problem (7) is equivalent to the following linear conic program:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m, \boldsymbol{X}_m\}, t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \left[t \; \boldsymbol{w}_{11}^T \; \boldsymbol{w}_{12}^T \; \cdots \; \boldsymbol{w}_{M1}^T \; \boldsymbol{w}_{M2}^T \right]^T \in \mathbb{L}^{2MN+1}, \\ & \hat{\boldsymbol{A}}(\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m})) + \boldsymbol{X}_m \succeq \boldsymbol{0}, m = 1, \ldots, M, \\ & \boldsymbol{X}_m \in \mathcal{L}_{2M-1,2N}^{\perp}. \end{aligned} \tag{28}$$

We remark that $\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) = [\boldsymbol{c}_m \epsilon_m \boldsymbol{C}_m] \in \mathbb{R}^{2M \times (2N+1)}$ is affine w.r.t. the design variables $\{\boldsymbol{w}_m\}$, and hence so is $\hat{\boldsymbol{A}}(\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}))$, and that the set $\mathcal{L}_{2M-1,2N}^{\perp}$ defined in (21) is easily characterized as symmetric matrices with skew-symmetric blocks.

IV. Quadratic Matrix Inequalities and Robust Optimal Beamforming

In a modern communication system[4], one may be interested in Problem (7) with some additional soft-shaping constraints to limit the interference generated around certain co-existing external users (e.g., the primary users who are the license owners of operating spectrum in the context of a spectrum sharing-based CR network). In such a practical scenario, the following

[4]For example, the secondary transmissions in a spectrum sharing-based CR network, see [16], [18]

optimal beamforming problem is of interest (assuming that perfect CSI is available at the transmitter):

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{\boldsymbol{w}_m^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_m}{\sum_{i=1,i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2} \geq \gamma_m, \\ & \qquad m = 1,\dots,M, \\ & \boldsymbol{g}_k^H \left(\boldsymbol{w}_1\boldsymbol{w}_1^H + \cdots + \boldsymbol{w}_M\boldsymbol{w}_M^H\right)\boldsymbol{g}_k \leq \eta_k, \\ & \qquad k = 1,\dots,K, \end{aligned} \tag{29}$$

where $\eta_k > 0$ is the tolerable interference generated to external user k. The soft-shaping (the second set of) constraints in (29) is also termed interference temperature (IT) constraints or CR constraints in the literature of CR network (cf. [12], [11]). Evidently, Problem (29) is equivalent to the following SOCP:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{1}{\sqrt{\gamma_m}} \Re\left(\boldsymbol{h}_m^H \boldsymbol{w}_m\right) \geq \sqrt{\sum_{i=1,i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2}, \\ & \qquad m = 1,\dots,M, \\ & \sqrt{\boldsymbol{g}_k^H \left(\boldsymbol{w}_1\boldsymbol{w}_1^H + \cdots + \boldsymbol{w}_M\boldsymbol{w}_M^H\right)\boldsymbol{g}_k} \leq \sqrt{\eta_k}, \\ & \qquad k = 1,\dots,K, \end{aligned} \tag{30}$$

thus can be solved with polynomial-time complexity.

When the CSI is not perfectly known at the transmitter, the following robust optimization to account for the uncertainty of CSI is considered:

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{\boldsymbol{w}_m^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_m}{\sum_{i=1,i\neq m}^{M} \boldsymbol{w}_i^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_i + \sigma_m^2} \\ & \quad \geq \gamma_m,\ \forall \boldsymbol{\delta}_m : \|\boldsymbol{\delta}_m\| \leq \epsilon_m, m = 1,\dots,M, \\ & (\bar{\boldsymbol{g}}_k + \boldsymbol{\delta}'_k)^H \left(\boldsymbol{w}_1\boldsymbol{w}_1^H + \cdots + \boldsymbol{w}_M\boldsymbol{w}_M^H\right)(\bar{\boldsymbol{g}}_k + \boldsymbol{\delta}'_k) \\ & \quad \leq \eta_k,\ \forall \boldsymbol{\delta}'_k : \|\boldsymbol{\delta}'_k\| \leq \epsilon'_k,\ k = 1,\dots,K, \end{aligned} \tag{31}$$

where $\bar{\boldsymbol{h}}_m$ and $\bar{\boldsymbol{g}}_k$ are the nominal estimations (they are known to the transmitter). In what follows, we will remove the superscript prime in the robust soft-shaping constraints (as long as the meaning of the notation is clear in the context) for the sake of notational simplicity. Several related robust designs have been studied in ([18], Problem (4)), [15], [12]. The SDP relaxation of (31), similar to (6), is formulated as

$$\underset{\{\boldsymbol{W}_m, \bar{\boldsymbol{W}}_m, \hat{\boldsymbol{W}}, \lambda_m, \mu_k\}}{\text{minimize}} \quad \sum_{m=1}^{M} \boldsymbol{I} \bullet \boldsymbol{W}_m \tag{32a}$$

$$\text{subject to} \quad \text{(6b), (6c) satisfied,} \tag{32b}$$

$$\begin{bmatrix} -\hat{\boldsymbol{W}} & \boldsymbol{0} \\ \boldsymbol{0} & \eta_k \end{bmatrix} + \mu_k \begin{bmatrix} \boldsymbol{I} & -\bar{\boldsymbol{g}}_k \\ -\bar{\boldsymbol{g}}_k^H & \bar{\boldsymbol{g}}_k^H \bar{\boldsymbol{g}}_k - \epsilon_k^2 \end{bmatrix} \succeq \boldsymbol{0}, \quad k = 1,\dots,K, \tag{32c}$$

$$\hat{\boldsymbol{W}} = \sum_{m=1}^{M} \boldsymbol{W}_m, \tag{32d}$$

$$\lambda_m \geq 0,\ \mu_k \geq 0,\ \boldsymbol{W}_m \succeq \boldsymbol{0}, \tag{32e}$$

which is again obtained via the S-lemma while relaxing the rank-one constraints. In general, the convexity of the problem (31) remains elusive, similar as in the situation of (5); in other words, it is not known yet whether or not the SDP (32) will always have a rank-one optimal solution.

As a compromise, we consider the following robust (conservative) beamforming design (similar to (7)):

$$\begin{aligned} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} \quad & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} \quad & \frac{1}{\sqrt{\gamma_m}} \Re((\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_m) \\ & \geq \sqrt{\sum_{i=1,i\neq m}^{M} \boldsymbol{w}_i^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_i + \sigma_m^2}, \\ & \qquad \forall \boldsymbol{\delta}_m : \|\boldsymbol{\delta}_m\| \leq \epsilon_m,\ m = 1,\dots,M, \\ & \sqrt{(\bar{\boldsymbol{g}}_k + \boldsymbol{\delta}_k)^H \left(\boldsymbol{w}_1\boldsymbol{w}_1^H + \cdots + \boldsymbol{w}_M\boldsymbol{w}_M^H\right)(\bar{\boldsymbol{g}}_k + \boldsymbol{\delta}_k)} \\ & \leq \sqrt{\eta_k},\ \forall \boldsymbol{\delta}_k : \|\boldsymbol{\delta}_k\| \leq \epsilon_k,\ k = 1,\dots,K, \end{aligned} \tag{33}$$

aiming at an equivalent SDP reformulation of the semi-infinite SOCP (33). We only need to handle the robust CR constraints, considering that the robust SINR constraints have LMI representations as shown in Section III. To proceed, let us start by studying a specific class of robust QMIs.

A. Robust Quadratic Matrix Inequalities

The Hilderbrand theorem (cf. Lemma 3.2) presents an LMI description for the set consisting of Lorentz-positive maps (indeed the set is a cone). In contrast, the famous S-lemma (see e.g., ([22], Lemma 3.3)) gives another completely different tool to derive an LMI characterization for a set of nonnegative quadratic functions over a domain defined by a quadratic function. In this subsection, we will extend the complex-valued S-lemma (see [28]) to a Hermitian matrix case. Let us first cite the complex S-lemma (see e.g., [28], [29]) in the following general formulation.

Lemma 4.1: Suppose that the quadratic functions $q_i : \mathbb{C}^N \rightarrow \mathbb{R}$ are given as $q_i(\boldsymbol{z}) = \boldsymbol{z}^H \boldsymbol{F}_i \boldsymbol{z} + 2\Re \boldsymbol{f}_i^H \boldsymbol{z} + f_i$, $i = 0, 1, 2$, where $\boldsymbol{F}_i \in \mathcal{H}^N$ (the set of N-by-N complex Hermitian matrices), $\boldsymbol{f}_i \in \mathbb{C}^N$, and $f_i \in \mathbb{R}$. Suppose that there is $\boldsymbol{z}_0$ such that $q_i(\boldsymbol{z}_0) > 0$, $i = 1, 2$. Then, it holds that

$$q_0(\boldsymbol{z}) \geq 0,\ \forall \boldsymbol{z} : q_1(\boldsymbol{z}) \geq 0,\ q_2(\boldsymbol{z}) \geq 0, \tag{34}$$

if and only if there are $\lambda_i \geq 0$, $i = 1, 2$, such that

$$\begin{bmatrix} \boldsymbol{F}_0 & \boldsymbol{f}_0 \\ \boldsymbol{f}_0^H & f_0 \end{bmatrix} - \lambda_1 \begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{f}_1 \\ \boldsymbol{f}_1^H & f_1 \end{bmatrix} - \lambda_2 \begin{bmatrix} \boldsymbol{F}_2 & \boldsymbol{f}_2 \\ \boldsymbol{f}_2^H & f_2 \end{bmatrix} \succeq \boldsymbol{0} \ (\in \mathcal{H}_+^{N+1}). \tag{35}$$

Note that if Lemma 4.1 is applied to the second set of constraints in (33), then we can turn them into the corresponding QMI constraints, i.e., (32c) with $\hat{\boldsymbol{W}} = \sum_{m=1}^{M} \boldsymbol{w}_m \boldsymbol{w}_m^H$ therein. These are still computationally intractable. Nevertheless, as shall be seen, the second set of constraints in (33) can be transformed into robust LMI (a special type of robust complex QMI) constraints. By employing Lemma 4.1, we will be able to characterize the type of robust complex QMI (in a more general form) via LMI (which is a slight extension of the finite convex representation of real QMI in ([21], Theorem 3.5)).

This leads to equivalent LMI reformulations for the second set of constraints in (33).

To proceed, let us define the following four matrix inequality systems:

$$\begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X} \\ (\boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X})^H & \boldsymbol{H}_4+\boldsymbol{H}_5\boldsymbol{X}+(\boldsymbol{H}_5\boldsymbol{X})^H+\boldsymbol{X}^H\boldsymbol{H}_6\boldsymbol{X}\end{bmatrix}\succeq\boldsymbol{0},\quad \forall\boldsymbol{X}:\boldsymbol{I}\succeq\boldsymbol{X}^H\boldsymbol{D}_1\boldsymbol{X},\mathrm{tr}\,(\boldsymbol{D}_2\boldsymbol{X}\boldsymbol{X}^H)\le 1; \tag{36}$$

$$\begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X} \\ (\boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X})^H & \boldsymbol{H}_4+\boldsymbol{H}_5\boldsymbol{X}+(\boldsymbol{H}_5\boldsymbol{X})^H+\boldsymbol{X}^H\boldsymbol{H}_6\boldsymbol{X}\end{bmatrix}\succeq\boldsymbol{0},\quad \forall\boldsymbol{X}:\boldsymbol{I}\succeq\boldsymbol{X}^H\boldsymbol{D}_i\boldsymbol{X},i=1,2; \tag{37}$$

$$\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\end{bmatrix}^H\begin{bmatrix}\boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{H}_3\\ \boldsymbol{H}_2^H & \boldsymbol{H}_4 & \boldsymbol{H}_5\\ \boldsymbol{H}_3^H & \boldsymbol{H}_5^H & \boldsymbol{H}_6\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\end{bmatrix}\ge 0,\quad \forall\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}:\boldsymbol{y}^H\boldsymbol{y}\ge\boldsymbol{z}^H\boldsymbol{D}_i\boldsymbol{z},i=1,2; \tag{38}$$

$$\exists\lambda_1\ge 0,\lambda_2\ge 0,\text{such that }\begin{bmatrix}\boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{H}_3\\ \boldsymbol{H}_2^H & \boldsymbol{H}_4 & \boldsymbol{H}_5\\ \boldsymbol{H}_3^H & \boldsymbol{H}_5^H & \boldsymbol{H}_6\end{bmatrix} - \lambda_1\begin{bmatrix}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&-\boldsymbol{D}_1\end{bmatrix} - \lambda_2\begin{bmatrix}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&-\boldsymbol{D}_2\end{bmatrix}\succeq\boldsymbol{0}. \tag{39}$$

Note that in the above systems, the parameters $\boldsymbol{H}_i$ and $\boldsymbol{D}_i$ could be of any proper dimensions so that the systems (36)–(39) are well-defined. With the systems defined, we have the following theorem.

Theorem 4.2: It holds that the matrix inequality systems (37)–(39) are equivalent to each other. Furthermore, if $\boldsymbol{D}_2\succeq\boldsymbol{0}$, then the four systems (36)–(39) are equivalent.

Proof: See Appendix B. ■

Based on the proof of the above theorem, we have an immediate corollary as follows.

Corollary 4.3: If $\boldsymbol{D}_i\succeq\boldsymbol{0}$, $i=1,2$, then the following QMI system is equivalent to (37)–(39):

$$\begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X} \\ (\boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X})^H & \boldsymbol{H}_4+\boldsymbol{H}_5\boldsymbol{X}+(\boldsymbol{H}_5\boldsymbol{X})^H+\boldsymbol{X}^H\boldsymbol{H}_6\boldsymbol{X}\end{bmatrix}\succeq\boldsymbol{0},\quad \forall\boldsymbol{X}:\mathrm{tr}\,(\boldsymbol{D}_i\boldsymbol{X}\boldsymbol{X}^H)\le 1,i=1,2. \tag{40}$$

B. Equivalent Convex Representation of Problem (33)

Employing the LMI description (39) for the robust QMI (37) (or (36)), we can reexpress the second set of constraints of (33) into LMIs. In fact, let $\boldsymbol{W}=[\boldsymbol{w}_1\ \dots\ \boldsymbol{w}_M]$, and then the robust interference temperature constraints amount to the formulation:

$$\|\boldsymbol{W}^H(\bar{\boldsymbol{g}}_k+\boldsymbol{\delta}_k)\|\le\sqrt{\eta_k},\ \forall\boldsymbol{\delta}_k:\|\boldsymbol{\delta}_k\|\le\epsilon_k$$

which in turn is tantamount to

$$\begin{bmatrix}\eta_k\boldsymbol{I} & \boldsymbol{W}^H(\bar{\boldsymbol{g}}_k+\boldsymbol{\delta}_k)\\ (\bar{\boldsymbol{g}}_k+\boldsymbol{\delta}_k)^H\boldsymbol{W} & 1\end{bmatrix}\succeq\boldsymbol{0},\forall\boldsymbol{\delta}_k:\boldsymbol{\delta}_k^H\boldsymbol{\delta}_k\le\epsilon_k^2. \tag{41}$$

Setting $\boldsymbol{X}=\boldsymbol{\delta}_k$, $\boldsymbol{D}_1=\frac{1}{\epsilon_k^2}\boldsymbol{I}$, $\boldsymbol{D}_2=\boldsymbol{0}$, $\boldsymbol{H}_1=\eta_k\boldsymbol{I}$, $\boldsymbol{H}_2=\boldsymbol{W}^H\bar{\boldsymbol{g}}_k$, $\boldsymbol{H}_3=\boldsymbol{W}^H$, $\boldsymbol{H}_4=1$, $\boldsymbol{H}_5=\boldsymbol{0}$, and $\boldsymbol{H}_6=\boldsymbol{0}$ in (36), one obtains (41). In other words, (41) is a particular form of (36). It follows from Theorem 4.2 that the LMI representation for (41) is (39) specified to (with $\lambda_{k2}=0$): $\exists\lambda_{k1}\ge 0$ such that

$$\begin{bmatrix}\eta_k\boldsymbol{I} & \boldsymbol{W}^H\bar{\boldsymbol{g}}_k & \boldsymbol{W}^H\\ \bar{\boldsymbol{g}}_k^H\boldsymbol{W} & 1 & \boldsymbol{0}\\ \boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} - \lambda_{k1}\begin{bmatrix}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&1&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&-\frac{1}{\epsilon_k^2}\boldsymbol{I}\end{bmatrix}\succeq\boldsymbol{0}, \tag{42}$$

which is identical to

$$\begin{bmatrix}\eta_k\boldsymbol{I} & \boldsymbol{W}^H & \boldsymbol{W}^H\bar{\boldsymbol{g}}_k\\ \boldsymbol{W} & \frac{\lambda_{k1}}{\epsilon_k^2}\boldsymbol{I} & \boldsymbol{0}\\ \bar{\boldsymbol{g}}_k^H\boldsymbol{W} & \boldsymbol{0} & 1-\lambda_{k1}\end{bmatrix}\succeq\boldsymbol{0}\ (\in\mathcal{H}_+^{M+N+1}). \tag{43}$$

We remark that it is also possible to extend the robust convex quadratic inequality with the unstructured norm-bounded uncertainty ([22], Theorem 3.2) from the real context to the complex context, and apply it to reformulate (41) into (42).

Having (43) together with Proposition 3.4, we establish a convex (SDP) reformulation for the robust beamforming problem (33), as stated below.

Proposition 4.4: The robust MISO cognitive beamforming problem (33) is equivalent to the following linear conic program:

$$\begin{aligned}
&\underset{\{\boldsymbol{w}_m,\boldsymbol{X}_m,\lambda_k\},t}{\text{minimize}} && t\\
&\text{subject to} && \left[t\ \boldsymbol{w}_{11}^T\ \boldsymbol{w}_{12}^T\ \cdots\ \boldsymbol{w}_{M1}^T\ \boldsymbol{w}_{M2}^T\right]^T\in\mathbb{L}^{2MN+1},\\
& && \hat{\boldsymbol{A}}(\boldsymbol{B}_m(\boldsymbol{w}_m,\boldsymbol{W}_{-m}))+\boldsymbol{X}_m\succeq\boldsymbol{0}\\
& && \left(\in\mathcal{S}_+^{2(2M-1)N}\right),\quad m=1,\dots,M,\\
& && \begin{bmatrix}\eta_k\boldsymbol{I} & \boldsymbol{W}^H & \boldsymbol{W}^H\bar{\boldsymbol{g}}_k\\ \boldsymbol{W} & \frac{\lambda_k}{\epsilon_k^2}\boldsymbol{I} & \boldsymbol{0}\\ \bar{\boldsymbol{g}}_k^H\boldsymbol{W} & \boldsymbol{0} & 1-\lambda_k\end{bmatrix}\succeq\boldsymbol{0}\\
& && (\in\mathcal{H}_+^{M+N+1}),\quad k=1,\dots,K,\\
& && \boldsymbol{W}=[\boldsymbol{w}_1\ \dots\ \boldsymbol{w}_M],\\
& && \boldsymbol{X}_m\in\mathcal{L}_{2M-1,2N}^{\perp},\ \lambda_k\ge 0.
\end{aligned} \tag{44}$$

Since (44) is a standard form of SDP, one can employ solver `SeDuMi` to solve it. We note that the optimal value of (44) equals the square root of that of (33).

V. Simulation Results

We consider the simulated scenario with an N-antenna BS serving three single-antenna users ($M=3$). The elements of the nominal channel vectors ($\bar{\boldsymbol{h}}_1,\bar{\boldsymbol{h}}_2,\bar{\boldsymbol{h}}_3$) are the i.i.d. standard complex Gaussian variables. The noise variance is set $\sigma_m^2=0.1$ for each user, and the SINR threshold value for the users is set to a common 12 dB. The bound of the error norm is chosen as $\epsilon_m=\epsilon\|\bar{\boldsymbol{h}}_m\|$ for user m (so that $\epsilon_m\ge\|\boldsymbol{\delta}_m\|$), and all results are averaged over the 2 000 Monte Carlo simulation runs.

Example 1: In this example, we examine the beamforming design problem without external users, and compare the performance of the SDP relaxation problem (6) of (5), i.e., a benchmark, with the performance of the proposed convex equivalent reformulation (28) of the semi-infinite SOCP (7). The two convex problems (i.e., (6) and (28)) are termed "SDR" and "Robust-SOCP", respectively, in the figures. We run simulations for the scenarios with different $N\in\{3,4,5\}$, and 2 000 sets of channel realizations are generated for each given N, and both convex problems are solved respectively for $\epsilon\in\{0.02,0.04,0.06,0.08,0.10,0.12\}$, for each set of channel realization.

Fig. 1 shows the problem feasibility rate versus the relative perturbation bound ϵ for different values of N. As we can see, the feasibility rate of the SDP relaxation problem is only slightly higher than that of (28), and this behavior coincides with the fact

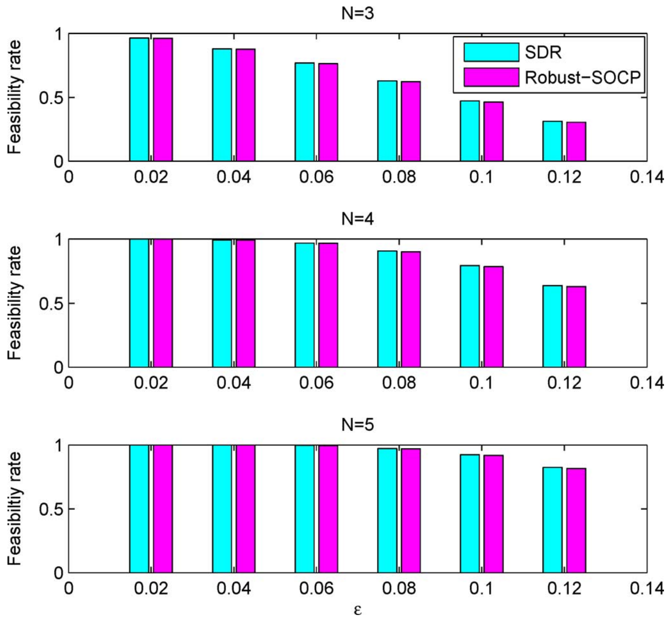

Fig. 1. Feasibility rate versus the perturbation bound ϵ for different values of transmit antennas N ($M = 3$ users).

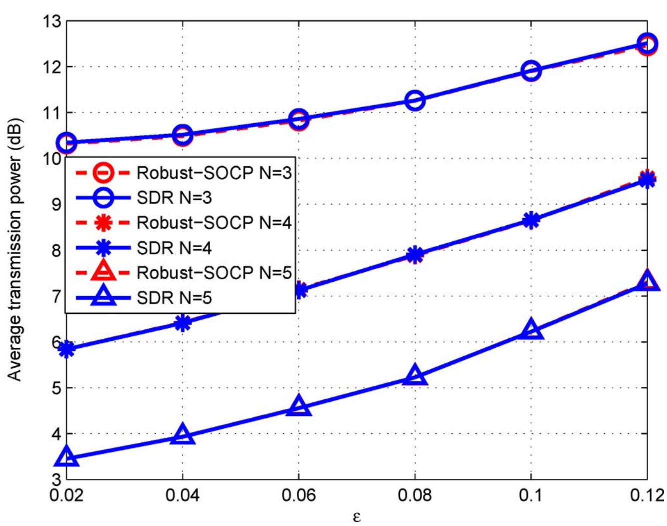

Fig. 2. Average transmission power versus the perturbation bound ϵ for different values of transmit antennas N ($M = 3$ users). (Average over the channels where the SDP (6) is feasible.)

that (7) is a more conservative (but convex) form, but not excessively so. It is observed that the feasibility rate increases when the number of transmit antenna N increases, and that the feasibility rate decreases when the perturbation bound ϵ increases.

Figs. 2 and 3 display the average transmission power versus the error norm bound ϵ for the scenarios with various N. Particularly, in Fig. 2, the transmission power by the SDP relaxation method (by the semi-infinite SOCP method) is averaged over all channel realizations where only the SDP (6) (only the linear conic program reformulation (28)) is feasible, while in Fig. 3 the transmission power is averaged over all channels where both the two convex problems ((6) and (28)) are feasible. The average transmission power by (28) in the both figures is taken over the channels where it is feasible. As expected, the higher transmission power is required to meet the robust QoS constraints for

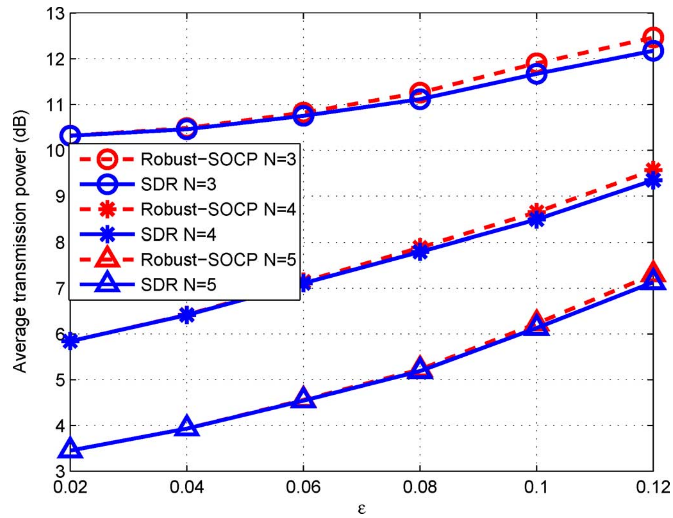

Fig. 3. Average transmission power versus the perturbation bound ϵ for different values of transmit antennas N ($M = 3$ users). (Average only over the channels where both (6) and (28) are feasible.)

the larger bound ϵ of uncertainty, as well as for less transmit antennas. From the simulations results, it is clearly seen that the more conservative model in (7) is sufficiently tight (namely, not too conservative) in practice.

Example 2: In this example, we test the optimal beamforming problem (31) (i.e., with presence of external coexisting users), and compare the performance of the SDP relaxation (32) and the equivalent linear conic program formulation (44) for the conservative design (33). We set $K = 1$ (one external user), in order to avoid low problem feasibility rate, fix $N = 4$ (4-antenna BS), and run simulations for different channel error norm bounds $\epsilon_m = \epsilon_k = \epsilon \in \{0.06, 0.08\}$. A total of 2 000 sets of channel realizations are generated for a given ϵ, and in each set of channel realizations, the two convex problems (32) and (44) are solved respectively for the tolerable interference threshold value $\eta_k = \eta \in \{0, 1, 2, 3, 4\}$ dB.

Fig. 4 shows the problem feasibility rate versus the allowable interference threshold η, for two different uncertainty bound ϵ. It can be seen that the feasibility rate reduces as the η decreases, which is expected since the feasible set is larger when the tolerable interference level is higher. Besides, in contrast to Fig. 1 (with $N = 4$ and $\epsilon \in \{0.06, 0.08\}$), we observe that the feasibility rate dramatically decreases (from about 0.95 to less than 0.8 for $\epsilon = 0.06$, and from about 0.9 to less than 0.6 for $\epsilon = 0.08$), which is accounted as cost for protecting the external user (of course, more antennas at the BS provide more degrees of freedom to deal with additional constraints).

Figs. 5 and 6 show the average transmission power versus the allowable interference upper bound η, for different values of the perturbation $\epsilon \in \{0.06, 0.08\}$. Again, in Fig. 5, the transmission power by the SDP method is obtained by averaging over the channel realizations where (32) is feasible and the transmission by the semi-infinite SOCP method is averaged over the channels where (44) is feasible, while in Fig. 6 the average transmission power is taken over the channels where both the convex problems (32) and (44) are feasible. We report that the SDP relaxation problem (32) always has rank-one solutions in our numerous simulations, as long as it is solvable. As observed, more efforts need to be paid (in term of higher transmission power

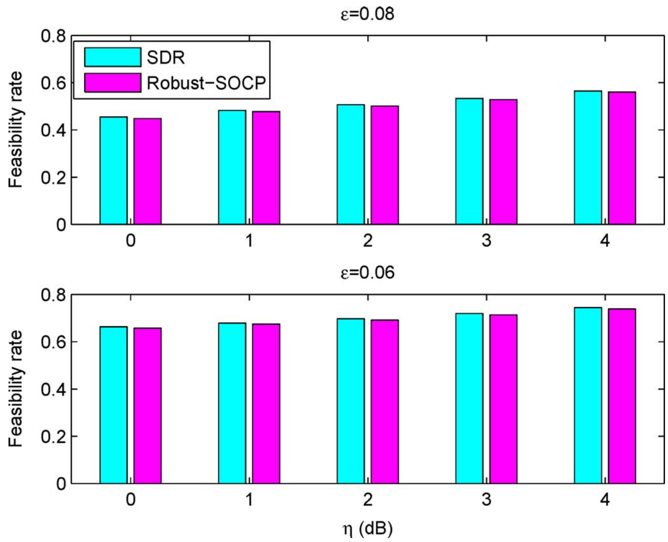

Fig. 4. Feasibility rate versus the tolerable interference bound η for different channel perturbation bounds ϵ ($N = 4$ antennas, $M = 3$ users, $K = 1$ external user).

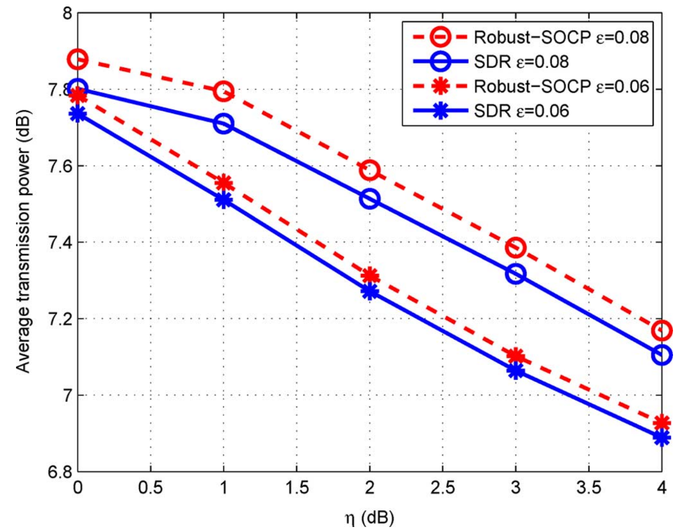

Fig. 6. Average transmission power versus the tolerable interference bound η for different channel perturbation bounds ϵ ($N = 4$ antennas, $M = 3$ users, $K = 1$ external user). (Average only over the channels where both (32) and (44) are feasible.)

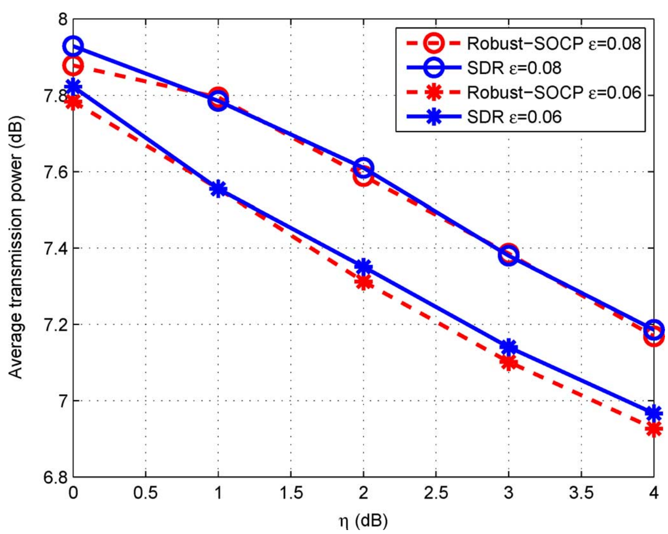

Fig. 5. Average transmission power versus the tolerable interference bound η for different channel perturbation bounds ϵ ($N = 4$ antennas, $M = 3$ users, $K = 1$ external user). (Average over the channels where the SDP (32) is feasible.)

required) to better protect the external user (by lessening the allowable interference level η) while keeping the acceptable QoS for the three internal users. This is consistent with the fact that the feasible sets of the minimization problems are smaller (thus the optimal values become higher) when η decreases. It is noted from Fig. 6 that the conservativeness of the design (33) is marginal, comparing to the SDP relaxation (32) of the original robust design, the same phenomenon we observed in Fig. 3.

VI. Conclusion

In a unicast MISO transmission system, we have considered the robust downlink beamforming problem, which minimizes the total transmission power subject to robust SINR constraints. Given that the convexity of the robust problem remains unknown, we have considered a conservative formulation in the form of a semi-infinite SOCP, and have derived an equivalent convex reformulation. The optimization tool we utilized is the exact LMI description of the cone of Lorentz-positive matrices. The resulting problem reformulation is a standard form of linear conic program, and thus can be implemented in a convenient fashion. Further, we have also considered the optimal transmit beamforming problem with additional robust soft-shaping constraints to protect external co-existing systems, and a conservative design of the robust problem is formulated into another semi-infinite SOCP. It is shown that the semi-infinite SOCP has an equivalent SDP reformulation, by employing an LMI representation for a class of QMIs. The numerical performance shows the conservativeness of the two proposed semi-infinite SOCP formulations is not excessive, compared to the SDP relaxations of the original robust downlink beamforming problems.

Appendix

A. Proof of Proposition 3.3

Proof: Necessity: Since $\boldsymbol{G}$ is Lorentz-positive, it then follows from Lemma 3.2 that there is $\boldsymbol{X} \in \mathcal{A}^{L-1} \otimes \mathcal{A}^{K-1}$ such that $\boldsymbol{Y} := \hat{\boldsymbol{A}}(\boldsymbol{G}) + \boldsymbol{X} \in \mathcal{S}_+^{(K-1)(L-1)}$. Note that $-\boldsymbol{X} \in \mathcal{A}^{L-1} \otimes \mathcal{A}^{K-1} \subseteq \mathcal{L}_{L-1,K-1}^{\perp}$ (22). Therefore, one has $\hat{\boldsymbol{A}}(\boldsymbol{G}) = \boldsymbol{Y} - \boldsymbol{X} \in \mathcal{S}_+^{(K-1)(L-1)} + \mathcal{L}_{L-1,K-1}^{\perp}$.

Sufficiency: Suppose that $\boldsymbol{G} = \sum_{r=1}^{R} \boldsymbol{a}_r \boldsymbol{b}_r^T$ is a singular value decomposition, with $\boldsymbol{a}_r \in \mathbb{R}^L$ and $\boldsymbol{b}_r \in \mathbb{R}^K$. It thus is verified that

$$\hat{\boldsymbol{A}}(\boldsymbol{G}) = \sum_{r=1}^{R} \hat{\boldsymbol{A}}\left(\boldsymbol{a}_r \boldsymbol{b}_r^T\right) = \sum_{r=1}^{R} \boldsymbol{A}(\boldsymbol{a}_r) \otimes \boldsymbol{A}(\boldsymbol{b}_r).$$

Since $\hat{\boldsymbol{A}}(\boldsymbol{G}) \in \mathcal{S}_+^{(K-1)(L-1)} + \mathcal{L}_{L-1,K-1}^{\perp}$, hence

$$\begin{aligned} 0 &\le \hat{\boldsymbol{A}}(\boldsymbol{G}) \bullet \left(\begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix} [1, \boldsymbol{x}^T]\right) \otimes \left(\begin{bmatrix} 1 \\ \boldsymbol{y} \end{bmatrix} [1, \boldsymbol{y}^T]\right) \\ &= 4 \sum_{r=1}^{R} \left(\boldsymbol{a}_r^T \tilde{\boldsymbol{x}}\right) \left(\boldsymbol{b}_r^T \tilde{\boldsymbol{y}}\right) = 4\boldsymbol{G} \bullet (\tilde{\boldsymbol{x}} \tilde{\boldsymbol{y}}^T), \end{aligned} \tag{45}$$

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}^H \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 + \boldsymbol{H}_3\boldsymbol{X} \\ (\boldsymbol{H}_2 + \boldsymbol{H}_3\boldsymbol{X})^H & \boldsymbol{H}_4 + \boldsymbol{H}_5\boldsymbol{X} + (\boldsymbol{H}_5\boldsymbol{X})^H + \boldsymbol{X}^H\boldsymbol{H}_6\boldsymbol{X} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}^H \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{H}_3 \\ \boldsymbol{H}_2^H & \boldsymbol{H}_4 & \boldsymbol{H}_5 \\ \boldsymbol{H}_3^H & \boldsymbol{H}_5^H & \boldsymbol{H}_6 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \geq 0. \quad (47)$$

where $\tilde{\boldsymbol{x}} = [\frac{1+\|\boldsymbol{x}\|^2}{2}, \frac{1-\|\boldsymbol{x}\|^2}{2}, \boldsymbol{x}^T]^T \in \mathbb{R}^L$, and $\tilde{\boldsymbol{y}} = [\frac{1+\|\boldsymbol{y}\|^2}{2}, \frac{1-\|\boldsymbol{y}\|^2}{2}, \boldsymbol{y}^T]^T \in \mathbb{R}^K$, $\boldsymbol{x} \in \mathbb{R}^{L-2}$, $\boldsymbol{y} \in \mathbb{R}^{K-2}$. In other words, $\tilde{\boldsymbol{x}}^T\boldsymbol{G}\tilde{\boldsymbol{y}} \geq 0$. Observe that $\mathbb{L}^L = \mathrm{cl}-\mathrm{cone}(\mathbb{S})$, where $\mathbb{S} = \{[\frac{1+\|\boldsymbol{x}\|^2}{2}, \frac{1-\|\boldsymbol{x}\|^2}{2}, \boldsymbol{x}^T]^T \,|\, \boldsymbol{x} \in \mathbb{R}^{L-2}\}$, and "cl" and "cone" denote the closure and convex cone hull operators, respectively; namely the primary set $\mathbb{S}$ is the parabolic section of the boundary of $\mathbb{L}^L$ (see ([19], p. 722)). Then, (45) implies that $\boldsymbol{G}$ satisfies $\tilde{\boldsymbol{x}}^T\boldsymbol{G}\tilde{\boldsymbol{y}} \geq 0$, $\forall\tilde{\boldsymbol{x}} \in \mathbb{S}$, $\tilde{\boldsymbol{y}} \in \mathbb{T} = \{[\frac{1+\|\boldsymbol{y}\|^2}{2}, \frac{1-\|\boldsymbol{y}\|^2}{2}, \boldsymbol{y}^T]^T \,|\, \boldsymbol{y} \in \mathbb{R}^{K-2}\}$, which amounts to $\boldsymbol{x}^T\boldsymbol{G}\boldsymbol{y} \geq 0$, $\forall\boldsymbol{x} \in \mathbb{L}^L$, $\boldsymbol{y} \in \mathbb{L}^K$ (cf. ([19], Corollary 2.7) and ([21], Lemma 2.2)), i.e., $\boldsymbol{G}$ is Lorentz-positive. ■

B. Proof of Theorem 4.2

Proof: By Lemma 4.1 (S-lemma), it follows that systems (38) and (39) are equivalent to each other. Therefore, we need to show the equivalence between (37) and (38).

"(37)⇒(38)": Suppose that $\boldsymbol{y}$ and $\boldsymbol{z}$ are given such that $\boldsymbol{y}^H\boldsymbol{y} \geq \boldsymbol{z}^H\boldsymbol{D}_i\boldsymbol{z}$, $i = 1, 2$. Suppose $\boldsymbol{y} \neq \boldsymbol{0}$. Let $\boldsymbol{X} = \frac{\boldsymbol{z}\boldsymbol{y}^H}{\boldsymbol{y}^H\boldsymbol{y}}$. It is easily seen that $\boldsymbol{X}$ complies with $\boldsymbol{I} - \boldsymbol{X}^H\boldsymbol{D}_i\boldsymbol{X} \succeq \boldsymbol{0}$, $i = 1, 2$. It follows from (37) that

$$\begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 + \boldsymbol{H}_3\boldsymbol{X} \\ (\boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X})^H & \boldsymbol{H}_4 + \boldsymbol{H}_5\boldsymbol{X} + (\boldsymbol{H}_5\boldsymbol{X})^H + \boldsymbol{X}^H\boldsymbol{H}_6\boldsymbol{X} \end{bmatrix} \succeq \boldsymbol{0}, \quad (46)$$

which implies [see (47) at the top of the page]. Here in the last equality, we use the fact that $\boldsymbol{z} = \boldsymbol{X}\boldsymbol{y}$. Suppose $\boldsymbol{y} = \boldsymbol{0}$. Let $\boldsymbol{X} = \frac{\boldsymbol{z}\boldsymbol{u}^H}{\boldsymbol{u}^H\boldsymbol{u}}$. It is clear again that $\boldsymbol{I} - \boldsymbol{X}^H\boldsymbol{D}_i\boldsymbol{X} \succeq \boldsymbol{0}$, $i = 1, 2$, and it follows from (37) that (46) is true with the just defined $\boldsymbol{X}$. In other words, one has that

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix}^H \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 + \boldsymbol{H}_3\boldsymbol{X} \\ (\boldsymbol{H}_2+\boldsymbol{H}_3\boldsymbol{X})^H & \boldsymbol{H}_4+\boldsymbol{H}_5\boldsymbol{X}+(\boldsymbol{H}_5\boldsymbol{X})^H+\boldsymbol{X}^H\boldsymbol{H}_6\boldsymbol{X} \end{bmatrix} \\ &\times \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \\ \boldsymbol{z} \end{bmatrix}^H \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{H}_3 \\ \boldsymbol{H}_2^H & \boldsymbol{H}_4 & \boldsymbol{H}_5 \\ \boldsymbol{H}_3^H & \boldsymbol{H}_5^H & \boldsymbol{H}_6 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \\ \boldsymbol{z} \end{bmatrix} \\ &+ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \\ \boldsymbol{z} \end{bmatrix}^H \begin{bmatrix} \boldsymbol{0} & \boldsymbol{H}_2 & \boldsymbol{0} \\ \boldsymbol{H}_2^H & \boldsymbol{H}_4 & \boldsymbol{H}_5 \\ \boldsymbol{0} & \boldsymbol{H}_5^H & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \\ \boldsymbol{z} \end{bmatrix} \geq 0, \end{aligned} \quad (48)$$

where $\boldsymbol{X}\boldsymbol{u} = \boldsymbol{z}$ is applied. We can pick up a $\boldsymbol{u}$ with sufficiently small norm so that the first term in (48) is nonnegative, which means (38) is true.

"(38)⇒(37)": Suppose that $\boldsymbol{X}$ is given such that $\boldsymbol{I} \succeq \boldsymbol{X}^H\boldsymbol{D}_i\boldsymbol{X}$, $i = 1, 2$. Then, we have $\boldsymbol{y}^H\boldsymbol{y} - \boldsymbol{z}^H\boldsymbol{D}_i\boldsymbol{z} \geq 0$ with $\boldsymbol{z} = \boldsymbol{X}\boldsymbol{y}$. Thus (38) together with (47) leads to (37). Consequently, the systems (37)–(39) are equivalent.

Now, suppose $\boldsymbol{D}_2 \succeq \boldsymbol{0}$, and we wish to show the systems (36)–(38) are equivalent (noting that (38) and (39) are equivalent due to S-lemma).

"(37)⇒(36)": Since $\boldsymbol{D}_2 \succeq \boldsymbol{0}$, hence $\mathrm{tr}(\boldsymbol{X}^H\boldsymbol{D}_2\boldsymbol{X}) \leq 1$ implies the eigenvalues $0 \leq \lambda_k(\boldsymbol{X}^H\boldsymbol{D}_2\boldsymbol{X}) \leq 1$. which means $\boldsymbol{I} - \boldsymbol{X}^H\boldsymbol{D}_2\boldsymbol{X} \succeq \boldsymbol{0}$. Therefore, the uncertainty set $\{\boldsymbol{X} \,|\, \boldsymbol{I} \succeq \boldsymbol{X}^H\boldsymbol{D}_i\boldsymbol{X}, i = 1, 2\}$ in (37) contains the uncertainty set in (36) and, consequently, (36) follows immediately from (37).

The proof for (36)⇒(38) is the same as that of (37)⇒(38), and thus we omit it. Furthermore, we have shown (38)⇒(37). Thus we conclude that (36)–(38) are equivalent. ■

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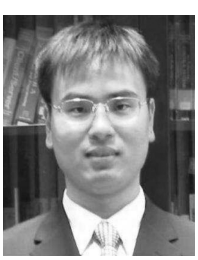

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