Randomized Algorithms for Optimal Solutions of Double-Sided QCQP With Applications in Signal Processing

Yongwei Huang, *Member, IEEE*, and Daniel P. Palomar, *Fellow, IEEE*

***Abstract*—Quadratically constrained quadratic programming (QCQP) with double-sided constraints has plenty of applications in signal processing as have been addressed in recent years. QCQP problems are hard to solve, in general, and they are typically approached by solving a semidefinite programming (SDP) relaxation followed by a postprocessing procedure. Existing postprocessing schemes include Gaussian randomization to generate an approximate solution, rank reduction procedure (the so-called purification), and some specific rank-one matrix decomposition techniques to yield a globally optimal solution. In this paper, we propose several randomized postprocessing methods to output not an approximate solution but a globally optimal solution for some solvable instances of the double-sided QCQP (i.e., instances with a small number of constraints). We illustrate their applicability in robust receive beamforming, radar optimal code design, and broadcast beamforming for multiuser communications. As a byproduct, we derive an alternative (shorter) proof for the Sturm-Zhang rank-one matrix decomposition theorem.**

***Index Terms*—Robust receive beamforming, optimal radar waveform, multicast downlink beamforming, homogeneous quadratically constrained quadratic program (QCQP), semidefinite programming (SDP) relaxation.**

I. Introduction

The problem formulation and effective solution of a quadratically constrained quadratic program (QCQP) has recently obtained ubiquitous applicability in signal processing for wireless communications, radar, microphone array speech processing, etc. An equivalent matrix form of a QCQP is a rank-one constrained semidefinite programming (SDP). It is known that SDP is a major class of convex optimization problems and can be solved polynomially via an interior-point method (e.g., see [1], [2]). However, SDPs with rank-one constraints (i.e., QCQP problems) are, in general, NP-hard (see [3]–[6] and references therein), but they can still be approximated by solving the SDP relaxation plus a postprocessing for finding a rank-one solution in a randomized or deterministic manner (e.g., see the comprehensive surveys [7], [8] and the book chapter [9] from a perspective of signal processing). Interestingly, when the number of constraints in a nonconvex QCQP is not too large, the QCQP can be solved in polynomial time (e.g., see [7], [9]–[20]), namely, its SDP relaxation is tight, and thus such class of QCQPs are convex in a hidden way.

Herein, we further focus on the class of solvable[1] QCQPs and some interesting applications in signal processing, both of which are equally important. A modern approach to solve a QCQP includes a postprocessing scheme to polynomially construct optimal or suboptimal solutions of the rank-one SDP problem, following the resolution of the SDP relaxation problem (which outputs a general-rank solution). For a solvable QCQP, there are several postprocessing methods in the literature to retrieve a rank-one optimal solution from the SDP relaxation. Among them, a classical one is the rank-reduction procedure (also termed "purification") for generating a lower-rank solution from a high-rank candidate (see [11], [3], [9], [18]), while another (more direct) approach to extract a rank-one solution is due to the specific rank-one decomposition theorems (see [13], [15], [17]). For a QCQP with two inhomogeneous constraints, a rank-one solution procedure consisting of solving the quadratic feasibility problems was proposed in [14], [21]. More generally, for a separable QCQP with structured constraints, an indirect rank-one solution approach via the dual solution was derived in [22], [23], [19].

In this paper, we consider a randomized postprocessing procedure to construct a rank-one solution of the SDP relaxation problem, emphasizing the application to interesting problems arising in signal processing. The proposed randomized algorithms are very different from the existing ones (i.e., the rank-one decomposition technique, the rank-one reduction procedure): These existing ones are deterministic as opposed to our probabilistic approach. The proposed methods, which deliver the exact optimal solution, are not to be confused with the existing randomized methods to obtain suboptimal solutions. Some applications of such class of QCQPs include the design of robust receive beamforming, multiuser transmit beamforming (both unicast and multicast), and relay network beamforming [8] (see

Manuscript received July 10, 2013; revised November 27, 2013; accepted December 06, 2013. Date of publication January 02, 2014; date of current version February 06, 2014. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Martin Haardt. This work was supported by the Hong Kong RGC under Research Grant 617911.

Y. Huang is with Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Kowloon, Hong Kong (e-mail: huang@math.hkbu.edu.hk, eeyw@ece.ust.hk).

D. P. Palomar is with Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong (e-mail: palomar@ust.hk).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2013.2297683

[1]By "solvable" we mean that the minimization (maximization) problem is feasible, bounded below (above) and the optimal valued is attained ([2], page 13).

references therein), robust adaptive radar detection [24], design of optimum coded waveform for radar [25], the Capon beamforming problem for microphone array [26], and some other design problems from signal processing and communications listed in [9]. Additionally, as a byproduct of the main result (i.e., the randomized algorithms for solving QCQP optimally), we provide an alternative proof for the Sturm-Zhang rank-one decomposition theorem [13] using a probabilistic method.

The remainder of this paper is organized as follows. Section II is devoted to listing three concrete motivating applications in signal processing. In Section III, we develop the randomized algorithms for a global solution of a class of QCQPs and, in Section IV, we demonstrate how to apply the proposed algorithms to solve the optimization problems in the mentioned applications. In Section V, we provide an alternative proof of the Sturm-Zhang rank-one decomposition theorem in [13]. Numerical examples are presented in Section VI. Finally, conclusions are drawn in Section VII.

Notation: We adopt the notation of using boldface for vectors $\boldsymbol{a}$ (lower case), and matrices $\boldsymbol{A}$ (upper case). The transpose operator and the conjugate transpose operator are denoted by the symbols $(\cdot)^T$ and $(\cdot)^H$ respectively. The notation $\mathrm{tr}(\cdot)$ stands for the trace of the square matrix argument; $\boldsymbol{I}$ and $\boldsymbol{0}$ denote respectively the identity matrix and the matrix (or the row vector or the column vector) with zero entries (their size is determined from the context). The letter j represents the imaginary unit (i.e., $j = \sqrt{-1}$), while the letter i often serves as index in this paper. For any complex number x, we use $\Re(x)$ and $\Im(x)$ to denote respectively the real and the imaginary parts of x, $|x|$ and $\arg(x)$ represent the modulus and the argument of x, and x^* ($\boldsymbol{x}^*$ or $\boldsymbol{X}^*$) stands for the (component-wise) conjugate of x ($\boldsymbol{x}$ or $\boldsymbol{X}$). We employ $\boldsymbol{A} \bullet \boldsymbol{B}$ standing for the inner product $\mathrm{tr}(\boldsymbol{AB})$ between Hermitian or symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. The Euclidean norm (the Frobenius norm) of the vector $\boldsymbol{x}$ (the matrix $\boldsymbol{X}$) is denoted by $\|\boldsymbol{x}\|$ ($\|\boldsymbol{X}\|$). The symbol $\odot$ represents the Hadamard element-wise product. The curled inequality symbol $\succeq$ (and its strict form $\succ$) is used to denote generalized inequality: $\boldsymbol{A} \succeq \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is an Hermitian positive semidefinite matrix ($\boldsymbol{A} \succ \boldsymbol{B}$ for positive definiteness). The space of Hermitian $N \times N$ matrices (the space of real-valued symmetric $N \times N$ matrices) is denoted by $\mathcal{H}^N$ ($\mathcal{S}^N$), and the set of all positive semidefinite matrices in $\mathcal{H}^N$ ($\mathcal{S}^N$) by $\mathcal{H}_+^N$ ($\mathcal{S}_+^N$). $\mathsf{E}[\cdot]$ represents the statistical expectation. The notations $\mathrm{Range}\,(\boldsymbol{X})$ and $\mathrm{Null}\,(\boldsymbol{X})$ stand for the range space and the null space, respectively. $\lambda_{\min}(\boldsymbol{X})$ ($\lambda_{\max}(\boldsymbol{X})$) denotes the minimal (maximal) eigenvalue. Finally, $v^\star(\cdot)$ represents the optimal value of problem $(\cdot)$. $\boldsymbol{X}^{1/2}$ stands for the square root matrix if $\boldsymbol{X} \succeq \boldsymbol{0}$.

II. Motivation and Problem Formulations

In this section, we introduce the considered generic formulation of QCQP, and list several applications arising from different subjects in signal processing.

Consider the following class of QCQP:

$$\begin{aligned} &\underset{\boldsymbol{x}\in\mathbb{F}^N}{\text{minimize}} && \boldsymbol{x}^H\boldsymbol{A}_0\boldsymbol{x} \\ &\text{subject to} && b_i \trianglelefteq_{il} \boldsymbol{x}^H\boldsymbol{A}_i\boldsymbol{x} \trianglelefteq_{iu} c_i, \quad i=1,\ldots,I, \quad (1)\end{aligned}$$

where the variable $\boldsymbol{x}$ could be either real-valued or complex-valued (i.e., $\mathbb{F}$ could be $\mathbb{R}$ or $\mathbb{C}$), $\boldsymbol{A}_i, i = 1,\ldots,I$, are Hermitian (symmetric if $\mathbb{F} = \mathbb{R}$) matrices, b_i, c_i are real numbers, and $\trianglelefteq \in \{\leq, \mathrm{unrestricted}\}$ (the subscripts l and u denote the lower and upper bounds, respectively). Herein, we consider QCQP with number of double-sided constraints not greater than three, i.e., $I \leq 3$, and will show that it is solvable by presenting randomized (polynomial) algorithms for globally optimal solutions.

Note that the form of constraint $b_1 \trianglelefteq_{1l} \boldsymbol{x}^H\boldsymbol{A}_1\boldsymbol{x} \trianglelefteq_{1u} c_1$ is general in the sense that it includes the following types of constraints: (1) no constraint (taking both $\trianglelefteq_{1l}$ and $\trianglelefteq_{1u}$ to be "unrestricted", i.e., the constraint vanishes, and b_1, c_1 become meaningless); (2) one-sided inequality constraint $\boldsymbol{x}^H\boldsymbol{A}_1\boldsymbol{x} \leq c_1$ (taking $\trianglelefteq_{1u}$ to be "$\leq$" and $\trianglelefteq_{1l}$ to be "unrestricted"); (3) equality constraint $\boldsymbol{x}^H\boldsymbol{A}_1\boldsymbol{x} = c_1$ (taking both $\trianglelefteq_{1l}$ and $\trianglelefteq_{1u}$ to be "$\leq$" and $c_1 = b_1$); and (4) double-sided constraint $b_1 \leq \boldsymbol{x}^H\boldsymbol{A}_1\boldsymbol{x} \leq c_1$ (with $b_1 < c_1$).

An example for (1) is the inhomogeneous QCQP with double-sided constraints:

$$\begin{aligned} &\underset{\boldsymbol{x}\in\mathbb{C}^N}{\text{minimize}} && \boldsymbol{x}^H\boldsymbol{A}_0\boldsymbol{x} + 2\Re\left(\boldsymbol{b}_0^H\boldsymbol{x}\right) + c_0 \\ &\text{subject to} && l_i \leq \boldsymbol{x}^H\boldsymbol{A}_i\boldsymbol{x} + 2\Re\left(\boldsymbol{b}_i^H\boldsymbol{x}\right) + c_i \leq u_i, \\ &&& i=1,\ldots,I, \quad (2)\end{aligned}$$

since its equivalent homogenized version with an extra variable t is (cf. [12]):

$$\begin{aligned} &\underset{\boldsymbol{x}\in\mathbb{C}^N, t\in\mathbb{C}}{\text{minimize}} && \boldsymbol{x}^H\boldsymbol{A}_0\boldsymbol{x} + 2\Re\left(\boldsymbol{b}_0^H\boldsymbol{x}t^*\right) + c_0|t|^2 \\ &\text{subject to} && l_i \leq \boldsymbol{x}^H\boldsymbol{A}_i\boldsymbol{x} + 2\Re\left(\boldsymbol{b}_i^H\boldsymbol{x}t^*\right) + c_i|t|^2 \leq u_i, \\ &&& i=1,\ldots,I, \quad |t|^2 = 1. \quad (3)\end{aligned}$$

In particular, we note that a real-valued QCQP with one double-sided constraint was studied in [10] (see also [12]).

We now list some applications of (1) in signal processing.

A. Robust Receive Beamforming

In a design of robust receive beamforming (cf. [8] and references therein), also termed robust adaptive beamforming, the narrowband signal received by a N-antenna array is given by

$$\boldsymbol{y}(t) = s(t)\boldsymbol{a} + \boldsymbol{i}(t) + \boldsymbol{n}(t),$$

where $s(t)\boldsymbol{a}, \boldsymbol{i}(t)$, and $\boldsymbol{n}(t)$ are statistically independent vectors corresponding to the signal of interest (SOI), interference, and noise, respectively, $s(t)$ is the SOI waveform, and $\boldsymbol{a}$ is its steering vector (the actual array response or spatial signature of the SOI). The receive beamformer outputs the signal

$$x(t) = \boldsymbol{w}^H\boldsymbol{y}(t),$$

where $\boldsymbol{w}$ is the $N \times 1$ vector of beamformer complex weight coefficients. The beamforming problem is to find an optimal beamvector $\boldsymbol{w}$ maximizing the beamformer output signal-to-interference-plus-noise ratio (SINR)

$$\mathrm{SINR} = \frac{\sigma_s^2 |\boldsymbol{w}^H \boldsymbol{a}|^2}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}}, \tag{4}$$

where σ_s^2 is the SOI power and $\boldsymbol{R}_{i+n}$ is the interference plus noise covariance. In practical cases, the true covariance $\boldsymbol{R}_{i+n}$ is unavailable and, thus, the data sample estimate $\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{y}(t)\boldsymbol{y}(t)$ is used instead, with T being the number of available snapshots. The SINR maximization problem is equivalent to the following convex problem:

$$\underset{\boldsymbol{w}}{\mathsf{minimize}} \ \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \mathsf{subject\ to}\ |\boldsymbol{w}^H \boldsymbol{a}| = 1,$$

and a solution is

$$\boldsymbol{w}^\star = (\boldsymbol{a}^H \hat{\boldsymbol{R}}^{-1} \boldsymbol{a})^{-1} \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}. \tag{5}$$

This is the well-known sample matrix inversion (SMI) based minimum variance (MV) beamformer, and the corresponding beamformer output power $\mathsf{E}[|\boldsymbol{w}^{\star H}\boldsymbol{y}(t)|^2]$ is written as

$$\boldsymbol{w}^{\star H} \hat{\boldsymbol{R}} \boldsymbol{w}^\star = \frac{1}{\boldsymbol{a}^H \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}}. \tag{6}$$

In practice (e.g., multi-antenna wireless communications and passive source localization), the beamformer suffers from dramatic performance degradation due to mismatch between the actual steering vector $\boldsymbol{a}$ and the presumed steering vector $\hat{\boldsymbol{a}}$. To mitigate the degradation, robust receive beamforming techniques have been proposed in the last decade (e.g., see [27]–[33], and references therein).

A popular robust receive beamformer adopts the beamvector (5) with $\boldsymbol{a}$ therein replaced by an optimal estimate $\hat{\boldsymbol{a}}$ via different methods (in addition to the robust adaptive beamformer based on worst-case strategies, e.g., see [27], [32], and references therein). For example, a recent effective technique of receive beamforming is addressed in [33]. Therein, the optimal estimate $\hat{\boldsymbol{a}}$ of signal steering vector is obtained by maximizing the beamformer output power (6) subject to a constraint of avoiding the convergence of the estimate $\hat{\boldsymbol{a}}$ to the region where the interference steering vectors and their linear combinations are located, as well as a norm constraint. The following formulation is a generalization by allowing the norm of the steering vector to lie on an interval:

$$\begin{aligned} \underset{\boldsymbol{a}}{\mathsf{minimize}} \quad & \boldsymbol{a}^H \hat{\boldsymbol{R}}^{-1} \boldsymbol{a} \\ \mathsf{subject\ to} \quad & l \le \boldsymbol{a}^H \bar{\boldsymbol{C}} \boldsymbol{a} \le u, \\ & N - \eta_1 \le \|\boldsymbol{a}\|^2 \le N + \eta_2, \end{aligned} \tag{7}$$

where $\bar{\boldsymbol{C}} = \int_{\bar{\Theta}} \boldsymbol{d}(\theta)\boldsymbol{d}^H(\theta) d\theta, \boldsymbol{d}(\theta)$ is the steering vector associated with θ that has the structure defined by the antenna array geometry, and $\bar{\Theta}$ is the complement of the prefix angular sector Θ where the SOI is located. Also, the parameters l and u are $\min_{\theta\in\Theta} \boldsymbol{d}^H(\theta)\bar{\boldsymbol{C}}\boldsymbol{d}(\theta)$ and $\max_{\theta\in\Theta} \boldsymbol{d}^H(\theta)\bar{\boldsymbol{C}}\boldsymbol{d}(\theta)$, respectively, and the positive parameters η_1 and η_2 control the perturbation bound. We highlight that the second double-sided constraint accounts for the steering vector gain perturbations caused, e.g., by the sensor amplitude and phase error as well as by the sensor position error (cf. ([30], pp. 2408 and 2414)).

Note that the constraint $l \le \boldsymbol{a}^H \bar{\boldsymbol{C}} \boldsymbol{a}$ in the first double-sided constraint is necessary for the completeness of problem formulation, even in the particular case of $\eta_1 = \eta_2 = 0$. In fact, assume that $\|\boldsymbol{a}\|^2 = N$ is the case. Then the known lower bound for $\boldsymbol{a}^H \bar{\boldsymbol{C}} \boldsymbol{a}$ is $N\lambda_{\min}(\bar{\boldsymbol{C}})$. If $l > N\lambda_{\min}(\bar{\boldsymbol{C}})$ (it is true in Example 2 of [33]), then $l \le \boldsymbol{a}^H \bar{\boldsymbol{C}} \boldsymbol{a}$ provides additional information (otherwise, it is redundant). Evidently, if $\eta_1 = \eta_2 = 0$ and l happens to be equal to $N\lambda_{\min}(\bar{\boldsymbol{C}})$, then (7) reduces to the beamforming problem (23)–(25) in [33]. As for more motivations and interpretations of the first constraint in (7), we refer to ([33], Section III).

Another example of robust receive beamforming (based on steering vector estimation) is the design of a doubly constrained Capon beamformer considered in [30] (later extended in [21]). A more general Capon beamforming optimization problem including a double-sided constraint is formulated as:

$$\begin{aligned} \underset{\boldsymbol{a}}{\mathsf{minimize}} \quad & \boldsymbol{a}^H \hat{\boldsymbol{R}}^{-1} \boldsymbol{a} \\ \mathsf{subject\ to} \quad & \|\boldsymbol{Q}^{1/2}(\boldsymbol{a} - \hat{\boldsymbol{a}})\|^2 \le \epsilon, \\ & N - \eta_1 \le \|\boldsymbol{a}\|^2 \le N + \eta_2, \end{aligned} \tag{8}$$

where $\boldsymbol{Q} \succeq \boldsymbol{0}, \eta_1 \ge 0$, and $\eta_2 \ge$ are given parameters. The formulation in (8) embraces the problems in [21] and [30], and generalizes them by allowing the norm to lie on an interval. In (8), minimizing the objective is identical to maximizing the output power (6), while the first constraint governs the possible steering vectors around the nominal (presumed) $\hat{\boldsymbol{a}}$ and the second describes the norm constraint perturbed possibly by some coarse array calibration procedure.

It is clear that problems (7) and (8) are two particular instances of (1) (also of (2)), and we will be able to solve them using a proposed randomized algorithm via SDP relaxation (see Section IV.A).

In addition, there are other robust receive beamforming problem formulations (cf. [31], [34]), where the resulting optimization problems belong to the class of QCQPs (1). Further, we highlight the QCQP applications to robust adaptive radar detection in the presence of steering vector mismatches (e.g., see [24], and references therein).

B. Optimum Coded Waveform Design for Radar

Radar waveform optimization in the presence of colored disturbance has been addressed for two decades. A recently popular signal design approach relies on the modulation of a pulse train parameters (amplitude, phase, and frequency) in order to synthesize waveforms with some specified properties. This technique is known as radar coding. In [25], the design of optimal coded waveforms in the presence of colored Gaussian disturbance is formulated as the maximization of the detection performance under a control both on the region of achievable values for the Doppler estimation accuracy and on the degree of similarity with a pre-fixed radar code. This last constraint is equivalent to forcing a similarity between the ambiguity functions of the devised waveform and of the pulse train encoded with the pre-fixed sequence.

Consider a vector comprising the N samples of the received signal (after down-conversion and matched filtering) as in [25] (or [35]):

$$\boldsymbol{v} = \alpha \boldsymbol{c} \odot \boldsymbol{p} + \boldsymbol{n}, \tag{9}$$

where α is the complex echo amplitude (accounting for the transmit amplitude, phase, target reflectivity, and channels propagation effects), $\boldsymbol{c} = [c(0), c(1), \ldots, c(N-1)]^T$ is the vector containing the transmitted code elements (to be optimally designed), $\boldsymbol{p} = [1, e^{j2\pi\nu_d}, \ldots, e^{j2\pi(2N-1)\nu_d}]^T$ is the temporal steering vector, ν_d denotes the normalized target Doppler frequency, and $\boldsymbol{n} = [n(0), n(1), \ldots, n(N-1)]^T$ is the vector containing the disturbance samples. We assume that $\boldsymbol{n}$ is a zero-mean complex circular Gaussian vector with known positive definite covariance matrix $\mathsf{E}[\boldsymbol{n}\boldsymbol{n}^H] = \boldsymbol{M}$.

It is known, from [25] and references therein, that the detection probability (describing the detection performance) P_{d} of the generalized likelihood ratio test, for a given value of the false alarm probability P_{fa}, depends on the radar code, the disturbance covariance matrix, and the temporal steering vector only through the signal-to-noise ratio (SNR), defined as

$$\mathrm{SNR} = |\alpha|^2 (\boldsymbol{c} \odot \boldsymbol{p})^H \boldsymbol{M}^{-1} (\boldsymbol{c} \odot \boldsymbol{p}), \tag{10}$$

which is a function of the actual Doppler frequency due to the dependence of $\boldsymbol{p}$ over ν_d. Moreover P_{d} is an increasing function of SNR and as a consequence, the maximization of P_{d} can be obtained maximizing the SNR (10).

In a design of radar waveform, it is of importance to have a reliable measurement of the Doppler frequency [25]. The Doppler accuracy is bounded below by Cramér-Rao Bound (CRB) and CRB-like techniques which provide lower bounds for the variances of unbiased estimates. Constraining the CRB is equivalent to controlling the region $\mathcal{A}$ of achievable Doppler estimation accuracies; particularly, forcing an upper bound to CRB results in a lower bound on the size of the region $\mathcal{A}$. According to this guideline, we focus on the class of radar codes complying with the CRB constraint [25]:

$$(\boldsymbol{c} \odot \boldsymbol{p} \odot \boldsymbol{u})^H \boldsymbol{M}^{-1} (\boldsymbol{c} \odot \boldsymbol{p} \odot \boldsymbol{u}) \geq \delta_a, \tag{11}$$

where $\boldsymbol{u} = [0, j2\pi, \ldots, j2\pi(N-1)]^T$ and δ_a rules the lower bound on the size of $\mathcal{A}$. Thus, adding a similarity constraint with a known unit-norm code $\boldsymbol{c}_0$ (cf. [30], [25]), as well as a norm constraint (termed also energy constraint), we formulate the waveform (radar code) design problem into the following QCQP:

$$\begin{aligned} \underset{\boldsymbol{c} \in \mathbb{C}^N}{\text{maximize}} \quad & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} && \text{(12a)} \\ \text{subject to} \quad & 1 - \eta_1 \leq \boldsymbol{c}^H \boldsymbol{c} \leq 1 + \eta_2, && \text{(12b)} \\ & \boldsymbol{c}^H \boldsymbol{R}_1 \boldsymbol{c} \geq \delta_a, && \text{(12c)} \\ & \|\boldsymbol{c} - \boldsymbol{c}_0\|^2 \leq \epsilon, && \text{(12d)} \end{aligned}$$

where $\boldsymbol{R} = \boldsymbol{M}^{-1} \odot (\boldsymbol{p}\boldsymbol{p}^H)^*$, $\boldsymbol{R}_1 = \boldsymbol{M}^{-1} \odot (\boldsymbol{p}\boldsymbol{p}^H)^* \odot (\boldsymbol{u}\boldsymbol{u}^H)^*$, η_1 and η_2 characterize the allowable perturbation of the code norm and, ϵ controls the size of the similarity region (a practical assumption is that $\epsilon \leq 1$ since all codes under consideration are normalized). Note that the problem formulation (12) is more general than that in [25], where the first constraint (i.e., the energy constraint) is the equality constraint $\boldsymbol{c}^H \boldsymbol{c} = 1$. Clearly, (12) is a particular case of (2) (or equivalently (1)).

Since the feasible region of (12) is larger than that of the optimization problem in [25], hence the performance (in term of SNR, or equivalently probability of detection) of the optimal code obtained herein is better. In Section IV.B, we will show how to obtain a global solution for the problem, via SDP relaxation and a randomized algorithm. For an application of QCQP to radar space-time adaptive processing, we refer to [36], and for applications in radar signal estimation and detection, we refer to [20] and [37].

C. Single-Group Multicast Transmit Beamforming With Soft-Shaping Interference Constraints

Consider a downlink transmission scenario where the base station (BS), equipped with N antennas, sends a common signal $s(t)$ to M single-antenna users (i.e., a multiple-input single-output (MISO) system). Let $\boldsymbol{w} \in \mathbb{C}^N$ denote the beamforming weight vector applied to the N transmit antenna elements. The transmitted signal is given by $\boldsymbol{x}(t) = s(t)\boldsymbol{w}$, where $s(t)$ is assumed to be zero-mean and white with unit variance. Let $\boldsymbol{h}_m \in \mathbb{C}^N$ be the channel vector of user m, and assume that it is randomly fading with known second-order statistics $\boldsymbol{R}_m = \mathsf{E}[\boldsymbol{h}_m \boldsymbol{h}_m^H]$ (cf. [22], [23]). The signal received at user m is given by

$$\boldsymbol{y}_m(t) = \boldsymbol{h}_m^H \boldsymbol{x}(t) + n_m(t),$$

where $n_m(t)$ is additive noise with power σ_m^2. The average SNR can be expressed as

$$\frac{\boldsymbol{w}^H \boldsymbol{R}_m \boldsymbol{w}}{\sigma_m^2}.$$

In the expression of SNR, we may set $\boldsymbol{R}_m = \boldsymbol{h}_m \boldsymbol{h}_m^H$, if the instantaneous channel state information (CSI) is available at the BS, or if the BS employs a uniform linear antenna array under light-of-sight propagation conditions (see e.g., [5], [38]). Therefore, the meaningful design of the beamvector that minimizes the total transmitted power subject to constraints on the received SNR of each user, can be formulated as

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^H \boldsymbol{R}_m \boldsymbol{w} \geq \sigma_m^2 \tau_m, \;\; m = 1, \ldots, M, \end{aligned} \tag{13}$$

where τ_m denotes the prescribed minimum received SNR for user m.

In a modern communication system, one may need to control the amount of interference generated along some particular directions, so as to protect the cochannel and coexisting systems (e.g., see [18], [39]). The caused interference power to user k of a coexisting systems is given by $\boldsymbol{w}^H \boldsymbol{S}_k \boldsymbol{w}$, $k = 1, \ldots, K$, where $\boldsymbol{S}_k = \boldsymbol{g}_k \boldsymbol{g}_k^H$ and $\boldsymbol{g}_k$ is the channel between the BS and external user k. Given both the SNR constraints for the internal users and upper bound constraints on the interference to the external users, the beamforming problem of interest can be written as

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^H \boldsymbol{R}_m \boldsymbol{w} \geq \sigma_m^2 \tau_m, \ m = 1, \ldots, M, \\ & \boldsymbol{w}^H \boldsymbol{g}_k \boldsymbol{g}_k^H \boldsymbol{w} \leq \eta_k, \ k = 1, \ldots, K, \end{aligned} \tag{14}$$

where η_k is a tolerable interference threshold value for external user k. The second set of constraints is termed soft-shaping interference constraints [39]. In particular, if $\eta_k = 0$, they are called null-shaping interference constraints [39], which are more strict; in this case, (14) reduces to the following QCQP:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^H \boldsymbol{R}_m \boldsymbol{w} \geq \sigma_m^2 \tau_m, \ m = 1, \ldots, M, \\ & \boldsymbol{w}^H \boldsymbol{g}_k \boldsymbol{g}_k^H \boldsymbol{w} = 0, \ k = 1, \ldots, K. \end{aligned} \tag{15}$$

The broadcasting (i.e., single multicast group) beamforming problem (14) is discussed for the special scenario of $\boldsymbol{R}_m = \boldsymbol{h}_m \boldsymbol{h}_m^H$ for all m in [41] in the context of an underlay cognitive radio (CR) network. Evidently, (1) embraces (13)–(15). In Section IV.C, we will show how to efficiently solve some scenarios of (15) (e.g., $M \leq 3$ and any K) as well as some instances of (14), provided that their SDP relaxation problems are solvable.

D. Other Applications

Some other applications include the following:

- the crosstalk resilient Capon beamformer design for microphone arrays in [26];
- the multicast transmit beamforming problem in [5], [38] and [40];
- the transmit beamforming problem for a CR network in [41]–[43] and [44];
- the optimal relay network beamforming problems in [45] and [46];
- the optimal fusion designs for distributed networks in [47] and [48];
- the portfolio risk management in financial engineering in [49].

III. Globally Optimal Solutions of QCQP via Randomization

We aim at obtaining randomized algorithms for solving homogeneous QCQP in this section. To be more specific, a global solution of QCQP is generated by randomization techniques following the resolution of the SDP relaxation, which is different from the existing ones (i.e., rank reduction-based technique (e.g., [11] and [18]), the rank-one matrix decomposition theorems (e.g., [13] and [15]), solving the quadratic feasibility problems proposed in [14], [21]).

The random variables that we adopt are bounded (without tails), which are different from commonly used Gaussian random variables in the context of SDP relaxation.[2] Specifically, we consider either the binary distribution (i.e., Bernoulli)

$$\xi = \begin{cases} +1, & \text{with probability } p = 1/2 \\ -1, & \text{with probability } 1-p \end{cases}, \tag{16}$$

or the uniform distribution on the unit circle[3]

$$\xi = e^{j\theta}, \quad \theta \in [0, 2\pi). \tag{17}$$

We will use $\boldsymbol{\xi}$ to denote a vector with independent and identically distributed (i.i.d.) random components having distribution (16) or (17).

To proceed, let us recall the conventional SDP relaxation of (1):

$$\begin{aligned} \underset{\boldsymbol{X}}{\text{minimize}} \quad & \boldsymbol{A}_0 \bullet \boldsymbol{X} \\ \text{subject to} \quad & b_1 \trianglelefteq_{1l} \boldsymbol{A}_1 \bullet \boldsymbol{X} \trianglelefteq_{1u} c_1, \\ & \vdots \\ & b_I \trianglelefteq_{Il} \boldsymbol{A}_I \bullet \boldsymbol{X} \trianglelefteq_{Iu} c_I, \\ & \boldsymbol{X} \succeq \boldsymbol{0}, \end{aligned} \tag{18}$$

and its dual

$$\begin{aligned} \underset{\{x_i, y_i\}, \boldsymbol{Z}}{\text{maximize}} \quad & \sum_{i=1}^{I} (x_i b_i - y_i c_i) \\ \text{subject to} \quad & \boldsymbol{Z} = \boldsymbol{A}_0 - \sum_{i=1}^{I} (x_i - y_i) \boldsymbol{A}_i \succeq \boldsymbol{0}, \\ & x_i \trianglelefteq_{il}^* 0, \ y_i \trianglelefteq_{iu}^* 0, \ i = 1, \ldots, I, \end{aligned} \tag{19}$$

where the dual operation $\trianglelefteq^*$ is defined by

$$\trianglelefteq^* \text{ is} \begin{cases} \geq, & \text{if } \trianglelefteq \text{ is } \leq \\ =, & \text{if } \trianglelefteq \text{ is unrestricted} \end{cases}. \tag{20}$$

Note that the operator $\bullet$ represents the inner product, namely $\boldsymbol{A}_0 \bullet \boldsymbol{X} = \operatorname{tr}(\boldsymbol{A}_0 \boldsymbol{X})$. Assume that $(\boldsymbol{X}; \boldsymbol{Z}, \{x_i, y_i\})$ is a primal-dual feasible pair, it then follows from the strong duality theorem (cf. ([2], Theorem 1.4.2)) that it is a pair of optimal solutions (respectively for the primal and dual SDPs) if and only if the complementary conditions are satisfied:

$$\boldsymbol{Z} \bullet \boldsymbol{X} = \boldsymbol{A}_0 \bullet \boldsymbol{X} - \sum_{i=1}^{I} (x_i - y_i) \boldsymbol{A}_i \bullet \boldsymbol{X} = 0, \tag{21}$$

$$x_i (\boldsymbol{A}_i \bullet \boldsymbol{X} - b_i) = 0, \tag{22}$$

$$y_i (\boldsymbol{A}_i \bullet \boldsymbol{X} - c_i) = 0, \quad i = 1, \ldots, I. \tag{23}$$

Let us start with the case of homogeneous QCQP with one double-sided constraint.

[2]In optimization and signal processing literature, Gaussian random variables are often considered to generate an approximate solution for an NP-hard optimization problem (e.g., an NP-hard QCQP problem) based on SDP relaxation techniques. The key to the analysis of approximation bounds (to measure the approximation quality) is based on the mathematical estimate of tail probabilities of the distribution of the quadratic form of a Gaussian random vector (e.g., see Lemmas 3.1 and 5.1 in [4]). However, the uniform or binary random variables are bounded, which is a key to allow us to yield a globally optimal solution for a solvable QCQP problem.

[3]We note that such random variables have been adopted to generate approximate solutions for an optimal transmit beamforming problem in [5]. Herein the distribution will be used instead to yield an optimal solution.

A. Homogeneous QCQP With One Double-Sided Constraint

Consider the QCQP:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^H \boldsymbol{A}_0 \boldsymbol{x} \\ \text{subject to} \quad & b_1 \trianglelefteq_{1l} \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \trianglelefteq_{1u} c_1, \end{aligned} \tag{24}$$

where $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{A}_i \in \mathcal{H}^N, i = 0, 1$. For QCQP (24) with real-valued parameters and variables, the discussion is similar. Suppose that the SDP relaxation (18) (with only one double-sided constraint) of (24) is solvable; to this end, it suffices to assume that the corresponding dual (19) is bounded above and strictly feasible (from the strong duality theorem ([2], Theorem 1.4.2)). Upon an optimal solution for the relaxed formulation (18), we are about to construct a randomized optimal solution for (24).

Proposition 3.1: Suppose that $\boldsymbol{X}^\star$ is an optimal solution for the relaxed problem (18) in the real- or complex-valued case with $I = 1$ (one double-sided constraint). Consider the eigenvalue decomposition $\boldsymbol{X}^{\star 1/2} \boldsymbol{A}_1 \boldsymbol{X}^{\star 1/2} = \boldsymbol{U}_1 \boldsymbol{\Lambda}_1 \boldsymbol{U}_1^H$.[4] Then, $\boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \boldsymbol{\xi}$ is an optimal solution for (24), where $\boldsymbol{\xi}$ is any vector with its components $|\xi_i| = 1$.

Proof: See Appendix A. ■

Algorithm 1 summarizes the procedure (as stated in the proof, see Appendix A) producing a globally optimal solution for Problem (24).

Algorithm 1 Randomization procedure for homogeneous QCQP (24)

Input: $\boldsymbol{A}_0, \boldsymbol{A}_1, b_1, c_1, \trianglelefteq_{1l}, \trianglelefteq_{1u}$;

Output: An optimal solution $\boldsymbol{x}^\star$ of problem (24);

1: solve the SDP (18) with one constraint only ($I = 1$), finding $\boldsymbol{X}^\star$;
2: implement the eigen-decomposition $\boldsymbol{X}^{\star 1/2} \boldsymbol{A}_1 \boldsymbol{X}^{\star 1/2} = \boldsymbol{U}_1 \boldsymbol{\Lambda}_1 \boldsymbol{U}_1^H$;
3: generate a random $\boldsymbol{\xi}$ with i.i.d. components having binary distribution (16);
4: output $\boldsymbol{x}^\star = \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \boldsymbol{\xi}$.

If multiple optimal solutions are required, then one may have to perform step 3 in Algorithm 1 multiple times. We note that vector $\boldsymbol{\xi}$ in step 3 could be chosen randomly or deterministically as a vector with components $|\xi_i| = 1$ (e.g., simply as the all-one vector). The computational complexity of Algorithm 1 is dominated by solving the SDP (18) which is of a worst-case complexity $O(N^{4.5} \log(1/\zeta))$ with a prefixed solution accuracy ζ (cf. [7]).

We note that if the binary distribution (16) is changed to the uniform distribution (17) over the unit circle in step 3 of Algorithm 1 (this only applies to the complex-valued version of QCQP (24)), then it can be shown in the same way that the random vector $\boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \boldsymbol{\xi}$ is also optimal.

[4]Here, $\boldsymbol{X}^{\star 1/2}$ denotes the unique Hermitian/symmetric square root matrix $\boldsymbol{Y}$ such that $\boldsymbol{Y}^2 = \boldsymbol{X}^\star$.

It is noteworthy that Problem (24) could be possibly solved with the existing algorithms, namely, solving the SDP relaxation problem followed by the rank reduction technique (cf. [11] and [18]), the rank-one matrix decomposition theorem (cf. [13]), and solving the quadratic feasibility problem proposed in [14]. The difference between Algorithm 1 and the three aforementioned approaches is twofold: (1) Our proposed algorithm contains an intrinsically probabilistic step (i.e., step 3, even though in practice it can be performed deterministically) while the other ones are all deterministic; (2) when the SDP relaxation has a high-rank solution, our algorithm appears quick and clean (see steps 2 and 3) to generate a rank-one solution of the SDP relaxation problem (whereas the other ones are of a more iterative nature; for example, the rank reduction technique is iterative, the specific rank-one matrix decomposition theorem includes rotation steps). In any case, despite the differences of the different approaches, all of them can give an optimal solution to the problem.

B. Homogeneous QCQP With Two Double-Sided Constraints

Consider now the following homogeneous QCQP:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^H \boldsymbol{A}_0 \boldsymbol{x} \\ \text{subject to} \quad & b_1 \trianglelefteq_{1l} \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \trianglelefteq_{1u} c_1, \\ & b_2 \trianglelefteq_{2l} \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x} \trianglelefteq_{2u} c_2, \end{aligned} \tag{25}$$

where $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{A}_i \in \mathcal{H}^N, i = 0, 1, 2$. The conventional SDP relaxation of (25) and its dual are (18) and (19) with $I = 2$, respectively.

Proposition 3.2: Suppose that $(\boldsymbol{X}^\star; \boldsymbol{Z}^\star, \{x_i^\star, y_i^\star\})$ is a primal-dual optimal pair for (18)–(19) in the real- or complex-valued case with $I = 2$. Then, there is a randomized solution (see Algorithm 2) such that it is globally optimal for (25).

Proof: See Appendix B. ■

Algorithm 2 Randomization procedure for homogeneous QCQP (25)

Input: $\boldsymbol{A}_0, \boldsymbol{A}_i, b_i, c_i, \trianglelefteq_{il}, \trianglelefteq_{iu}, i = 1, 2$;

Output: An optimal solution $\boldsymbol{x}^\star$ of problem (25);

1: solve the SDP (18) finding $\boldsymbol{X}^\star$; let $\delta_i = \boldsymbol{A}_i \bullet \boldsymbol{X}^\star, i = 1, 2$ (suppose $\delta_2 \neq 0$ w.l.o.g.);
2: implement the eigen-decomposition $\boldsymbol{X}^{\star 1/2}(\boldsymbol{A}_1 - \frac{\delta_1}{\delta_2}\boldsymbol{A}_2)\boldsymbol{X}^{\star 1/2} = \boldsymbol{U}_1 \boldsymbol{\Lambda}_1 \boldsymbol{U}_1^H$;
3: draw a realization $\bar{\boldsymbol{\xi}}$ of the random vector with i.i.d. components having the binary distribution (16), such that $\bar{\boldsymbol{\xi}}^T \boldsymbol{U}_1^H \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_2 \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}} \geq (\leq)\, \delta_2$ is satisfied if $\delta_2 > (<)\, 0$;
4: let $\lambda = \bar{\boldsymbol{\xi}}^T \boldsymbol{U}_1^H \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_2 \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}} / \delta_2$, and output $\boldsymbol{x}^\star = \frac{1}{\sqrt{\lambda}} \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}}$;

It is noteworthy that we resort to the complementary conditions (21)–(23) to show the randomized solution is optimal but we cannot ensure that the randomized solution is optimal with probability one (i.e., in step 3 of Algorithm 2, one may have to draw multiple realizations until the condition is satisfied), unlike Proposition 3.1 where any randomly or deterministically constructed vector is optimal. In this sense, Proposition 3.1 is stronger, due to the reduced number of constraints.

Note also that the elements of $\bar{\boldsymbol{\xi}}$ in step 3 of Algorithm 2 could be drawn randomly from (17) (accordingly, $\bar{\boldsymbol{\xi}}^T$ is changed to $\bar{\boldsymbol{\xi}}^H$ in steps 3 and 4) if the optimization variables of (25) are complex-valued, and the proof of Proposition 3.2 remains the same (in addition to the change of $\bar{\boldsymbol{\xi}}^T$ into $\bar{\boldsymbol{\xi}}^H$). As before, the computational complexity of the algorithm consists mainly of solving the SDP relaxation.

C. Homogeneous QCQP With Three Double-Sided Constraints

Consider the following QCQP:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^H \boldsymbol{A}_0 \boldsymbol{x} \\ \text{subject to} \quad & b_1 \trianglelefteq_{1l} \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \trianglelefteq_{1u} c_1, \\ & b_2 \trianglelefteq_{2l} \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x} \trianglelefteq_{2u} c_2, \\ & b_3 \trianglelefteq_{3l} \boldsymbol{x}^H \boldsymbol{A}_3 \boldsymbol{x} \trianglelefteq_{3u} c_3, \end{aligned} \tag{26}$$

where $\boldsymbol{x} \in \mathbb{C}^N$, and $\boldsymbol{A}_i \in \mathcal{H}^N, i = 0,1,2,3$. Suppose that both the primal SDP (18) and the dual SDP (19) are solvable with $I = 3$. Based on a solution pair of the primal and dual SDPs, we are about to retrieve an optimal solution $\boldsymbol{x}^\star$ of (26), in a randomized way. To proceed, let us first state the following lemma.

Lemma 3.3: Suppose that $\boldsymbol{\Lambda}$ is an $N \times N$ real-valued diagonal matrix and $\boldsymbol{Q}$ belongs to $\mathcal{H}^N$. Then there is a vector $\boldsymbol{v} \in \mathbb{C}^N$ such that

$$\boldsymbol{\Lambda} \bullet \boldsymbol{v}\boldsymbol{v}^H = \boldsymbol{\Lambda} \bullet \boldsymbol{I}, \quad \boldsymbol{Q} \bullet \boldsymbol{v}\boldsymbol{v}^H = \boldsymbol{Q} \bullet \boldsymbol{I}, \quad \boldsymbol{v}^H \boldsymbol{v} = N. \tag{27}$$

Proof: See Appendix C. ■

The proof is constructive, and we summarize the randomization procedure into Algorithm 3.

Algorithm 3 Randomization procedure for generating a complex vector satisfying (27)

Input: $\boldsymbol{\Lambda}$ (real diagonal matrix), $\boldsymbol{Q} \in \mathcal{H}^N$ (with $\boldsymbol{Q} = \boldsymbol{Q}_1 + j\boldsymbol{Q}_2$)

Output: A complex-valued vector $\boldsymbol{v}$ such that $\boldsymbol{v}^H \boldsymbol{\Lambda} \boldsymbol{v} = \boldsymbol{\Lambda} \bullet \boldsymbol{I}, \boldsymbol{v}^H \boldsymbol{Q} \boldsymbol{v} = \boldsymbol{Q} \bullet \boldsymbol{I} = \delta_2$ and $\boldsymbol{v}^N \boldsymbol{v} = N$;

1: take two random vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, with i.i.d. components having the binary distribution (16), such that $(\boldsymbol{\xi}^T \boldsymbol{Q}_1 \boldsymbol{\xi} - \delta_2)(\boldsymbol{\eta}^T \boldsymbol{Q}_1 \boldsymbol{\eta} - \delta_2) \leq 0$;
2: **if** $\boldsymbol{\xi}^T \boldsymbol{Q}_1 \boldsymbol{\xi} = \delta_2$ or $\boldsymbol{\eta}^T \boldsymbol{Q}_1 \boldsymbol{\eta} = \delta_2$ **then**
3: output $\boldsymbol{v} = \boldsymbol{\xi}$ or $\boldsymbol{v} = \boldsymbol{\eta}$ such that $\boldsymbol{v}^T \boldsymbol{Q}_1 \boldsymbol{v} = \delta_2$;
4: **else**
5: set $\gamma_0 = \frac{\boldsymbol{\xi}^T \boldsymbol{Q}_2 \boldsymbol{\eta} + \sqrt{(\boldsymbol{\eta}^T \boldsymbol{Q}_2 \boldsymbol{\xi})^2 - (\boldsymbol{\eta}^T \boldsymbol{Q}_1 \boldsymbol{\eta} - \delta_2)(\boldsymbol{\xi}^T \boldsymbol{Q}_1 \boldsymbol{\xi} - \delta_2)}}{\boldsymbol{\xi}^T \boldsymbol{Q}_1 \boldsymbol{\xi} - \delta_2}$, and output $\boldsymbol{v} = \frac{\gamma_0}{\sqrt{1+\gamma_0^2}} \boldsymbol{\xi} + j \frac{1}{\sqrt{1+\gamma_0^2}} \boldsymbol{\eta}$.
6: **end**

We remark that from (57) in the proof in Appendix C, one can only find a complex vector $\boldsymbol{v}$ such that (27) is satisfied and the result does not extend to the real-valued case. For some $\boldsymbol{\Lambda}$ and $\boldsymbol{Q}$, there is not a real $\boldsymbol{v}$ complying with (27); for example, it is verified that it is the case with

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

It is noteworthy that Algorithm 3 can be used to solve the complex version of the QCQP with two double-sided constraints in (25)[5] (i.e., with $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{A}_i \in \mathcal{H}^N$). Indeed, obtaining an optimal solution $\boldsymbol{X}^\star$ for the SDP relaxation (18) of (25) and an eigen-decomposition $\boldsymbol{U}_1 \boldsymbol{\Lambda} \boldsymbol{U}_1^H = \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_1 \boldsymbol{X}^{\star 1/2}$, we call Algorithm 3 (with the inputs $\boldsymbol{\Lambda}$ and $\boldsymbol{Q} = \boldsymbol{U}_1^H \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_2 \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1$) to output $\boldsymbol{v}$. It then is easily verified that $\boldsymbol{w} = \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \boldsymbol{v}$ is feasible and optimal for problem (25).[6]

Now let us see how to obtain an optimal solution for the QCQP with three double-sided constraints in (26) using the randomization procedure of Algorithm 3.

Proposition 3.4: Suppose that $(\boldsymbol{X}^\star; \boldsymbol{Z}^\star, \{x_i^\star, y_i^\star\})$ is a primal-dual optimal pair for (18)–(19) for the complex-valued case with $I = 3$. Then, there is a randomized solution (see Algorithm 4) such that it is globally optimal for (26).

Proof: See Appendix D. ■

Algorithm 4 Randomization procedure for complex-valued homogeneous QCQP (26)

Input: $\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3$;

Output: A solution $\boldsymbol{x}^\star$;

1: solve the SDP (18) finding $\boldsymbol{X}^\star$; evaluate $R = \text{Rank}(\boldsymbol{X}^\star)$; let $\delta_i = \boldsymbol{A}_i \bullet \boldsymbol{X}^\star, i = 1,2,3$ (suppose that $\delta_3 \neq 0$);
2: **while** $R \geq 2$ **do**
3: decompose $\boldsymbol{X}^\star = \boldsymbol{X}_1 \boldsymbol{X}_1^H$, where $\boldsymbol{X}_1 \in \mathbb{C}^{N \times R}$;
4: implement the eigen-decomposition $\boldsymbol{X}_1^H (\boldsymbol{A}_1 - \frac{\delta_1}{\delta_3} \boldsymbol{A}_3) \boldsymbol{X}_1 = \boldsymbol{U}_1 \boldsymbol{\Lambda}_1 \boldsymbol{U}_1^H$; let $\boldsymbol{Q}_1 = \boldsymbol{U}_1^H \boldsymbol{X}_1^H (\boldsymbol{A}_2 - \frac{\delta_2}{\delta_3} \boldsymbol{A}_3) \boldsymbol{X}_1 \boldsymbol{U}_1$;
5: $\boldsymbol{v}_1 = \text{Algorithm3}(\boldsymbol{\Lambda}_1, \boldsymbol{Q}_1)$, and set $\boldsymbol{w}_1 = \boldsymbol{X}_1 \boldsymbol{U}_1 \boldsymbol{v}_1$ and $\sigma_1 = \boldsymbol{w}_1^H \boldsymbol{A}_3 \boldsymbol{w}_1 / \delta_3$;
6: **if** $\sigma_1 > 0$ **then**
7: output $\boldsymbol{x}^\star = \frac{1}{\sqrt{\sigma_1}} \boldsymbol{w}_1$; terminate;
8: **else**
9: let $\boldsymbol{X}^\star = \frac{1}{1-\sigma_1/R} (\boldsymbol{X}_1 \boldsymbol{X}_1^H - \frac{1}{R} \boldsymbol{w}_1 \boldsymbol{w}_1^H)$ and $R = R - 1$;
10: **end**
11: **end while**
12: return $\boldsymbol{x}^\star$ such that $\boldsymbol{x}^\star \boldsymbol{x}^{\star H} = \boldsymbol{X}^\star$.

As can be seen, the computational load includes solving the SDP relaxation problem and some eigen-decompositions, and thus the total computational complexity of the algorithms is dominated by solving the SDP.

We note that the real homogeneous QCQP with three double-sided constraints or more cannot be solved to achieve the global

[5]In contrast, Algorithm 2 is applicable to both $\boldsymbol{x} \in \mathbb{R}^N$ and $\mathbb{C}^N$.

[6]The proof is similar to that of Proposition 3.2.

optimality in general (cf. [16]) and the same thing happens for the complex homogeneous QCQP with four double-sided constraints or more (cf. [17]); in other words, the SDP relaxation is not tight for any of the two cases, and the duality gap between the primal QCQP and its dual is positive. Nevertheless, for the case of complex QCQP (1) with $I = 4$ (four double-sided constraints), some mild sufficient conditions could be added to ensure the QCQP is convex (similar comments are valid for the real homogeneous QCQP with three constraints). For example, if there is a nonzero $\boldsymbol{\alpha} \in \mathbb{R}^4$ such that $\alpha_1 \boldsymbol{A}_1 + \alpha_2 \boldsymbol{A}_2 + \alpha_3 \boldsymbol{A}_3 + \alpha_4 \boldsymbol{A}_4 \succ \boldsymbol{0}$, and an optimal solution $\boldsymbol{Z}^\star$ ($\succeq \boldsymbol{0}$) of the dual (19) of the SDP relaxation has the null space of dimension more than two (i.e., ≥ 3), then the QCQP is solvable (cf. [17]). For general number of one-sided constraints, it is known that the QCQP is NP-hard, and some interesting randomized approximate solution schemes with provable approximation quality have been well studied in optimization literature (e.g., see [3], [4], and [9]).

D. Some Extensions

In this subsection, we handle some generalizations of the previous solvable QCQPs. Suppose that $\boldsymbol{A}_{ip}, \boldsymbol{B}_p \in \mathcal{H}^{N_p}, i = 0,1,2,3, p = 1,\ldots,P$, are given with $\boldsymbol{B}_p \succeq \boldsymbol{0}, \forall p$. Consider then the following separable homogenous QCQP:

$$\begin{array}{ll} \underset{\{\boldsymbol{x}_p\}}{\text{minimize}} & \boldsymbol{x}_1^H \boldsymbol{A}_{01} \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_P^H \boldsymbol{A}_{0P} \boldsymbol{x}_P \\ \text{subject to} & b_1 \trianglelefteq_{1l} \boldsymbol{x}_1^H \boldsymbol{A}_{11} \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_P^H \boldsymbol{A}_{1P} \boldsymbol{x}_P \trianglelefteq_{1u} c_1, \\ & b_2 \trianglelefteq_{2l} \boldsymbol{x}_1^H \boldsymbol{A}_{21} \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_P^H \boldsymbol{A}_{2P} \boldsymbol{x}_P \trianglelefteq_{2u} c_2, \\ & b_3 \trianglelefteq_{3l} \boldsymbol{x}_1^H \boldsymbol{A}_{31} \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_P^H \boldsymbol{A}_{3P} \boldsymbol{x}_P \trianglelefteq_{3u} c_3, \\ & \boldsymbol{x}_p^H \boldsymbol{B}_p \boldsymbol{x}_p = 0, \ p = 1,\ldots,P. \end{array} \tag{28}$$

Specially, if $\boldsymbol{B}_p = \boldsymbol{0}, \forall p$, and $\boldsymbol{A}_{ip} = \boldsymbol{0}, p = 2,\ldots,P$, the above QCQP reduces to (26); and if $\boldsymbol{B}_p = \boldsymbol{0}$, then the separable QCQP is identical to (26) with block-diagonal matrix parameters $\boldsymbol{A}_i, i = 0,1,2,3$. In a similar way, we can deal with (28) with one or two double-sided constraints.

Let us see how Algorithm 4 is applied to solve (28). Suppose that the SDP relaxation of (28)

$$\begin{array}{ll} \underset{\{\boldsymbol{X}_p \in \mathcal{H}^{N_p}\}}{\text{minimize}} & \boldsymbol{A}_{01} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{0P} \bullet \boldsymbol{X}_P \\ \text{subject to} & b_1 \trianglelefteq_{1l} \boldsymbol{A}_{11} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{1P} \bullet \boldsymbol{X}_P \trianglelefteq_{1u} c_1, \\ & b_2 \trianglelefteq_{2l} \boldsymbol{A}_{21} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{2P} \bullet \boldsymbol{X}_P \trianglelefteq_{2u} c_2, \\ & b_3 \trianglelefteq_{3l} \boldsymbol{A}_{31} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{3P} \bullet \boldsymbol{X}_P \trianglelefteq_{3u} c_3, \\ & \boldsymbol{B}_p \bullet \boldsymbol{X}_p = 0, \ p = 1,\ldots,P, \\ & \boldsymbol{X}_p \succeq \boldsymbol{0}, \end{array} \tag{29}$$

and its dual

$$\begin{array}{ll} \underset{\{x_i, y_i, z_p, \boldsymbol{Z}_p\}}{\text{maximize}} & \sum_{i=1}^{3} (x_i b_i - y_i c_i) \\ \text{subject to} & \boldsymbol{Z}_p = \boldsymbol{A}_{0p} - \sum_{i=1}^{3} (x_i - y_i) \boldsymbol{A}_{ip} - z_p \boldsymbol{B}_p \succeq \boldsymbol{0}, \\ & \qquad p = 1,\ldots,P, \\ & x_i \trianglelefteq_{il}^* 0, \ y_i \trianglelefteq_{iu}^* 0, \ i = 1,2,3, \ z_p \in \mathbb{R}, \end{array} \tag{30}$$

are solvable, and let $(\{\boldsymbol{X}_p^\star\}; \{x_i^\star, y_i^\star\}, \{z_p^\star, \boldsymbol{Z}_p^\star\})$ be an optimal primal-dual pair. Then they satisfy the complementary conditions (similar to (21)–(23)): (i) $\boldsymbol{Z}_p \bullet \boldsymbol{X}_p = 0, \forall p$; (ii) $x_i(\sum_{p=1}^P \boldsymbol{A}_{ip} \bullet \boldsymbol{X}_p - b_i) = 0, \forall i$; (3) $y_i(\sum_{p=1}^P \boldsymbol{A}_{ip} \bullet \boldsymbol{X}_p - c_i) = 0, \forall i$.

We can invoke Algorithm 4 P times to generate an optimal solution for (28): (i) solve the primal and dual SDPs (29)–(30) getting an optimal solution $(\{\boldsymbol{X}_p^\star\}; \{x_i^\star, y_i^\star\}, \{z_p^\star, \boldsymbol{Z}_p^\star\})$; (ii) let $\boldsymbol{x}_p^\star = \text{Algorithm 4}$ with the input $(\boldsymbol{A}_{0p}, \boldsymbol{A}_{1p}, \boldsymbol{A}_{2p}, \boldsymbol{A}_{3p}), \forall p$, where for each p, step 1 in Algorithm 4 should be adapted and changed to: evaluate $R = \text{Rank}\,(\boldsymbol{X}_p^\star)$ and let $\delta_i = \boldsymbol{A}_{ip} \bullet \boldsymbol{X}_p^\star, i = 1,2,3$ (suppose that $\delta_3 \neq 0$). It is not hard to show that the obtained tuple $\{\boldsymbol{x}_p^\star \boldsymbol{x}_p^{\star H}\}$ is optimal for (29)[7], and thus $\{\boldsymbol{x}_p^\star\}$ is optimal for (28).

More generally, let us consider the following separable SDP:

$$\begin{array}{ll} \underset{\{\boldsymbol{X}_p \in \mathcal{H}^{N_p}\}}{\text{minimize}} & \boldsymbol{A}_{01} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{0P} \bullet \boldsymbol{X}_P \\ \text{subject to} & b_1 \trianglelefteq_{1l} \boldsymbol{A}_{11} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{1P} \bullet \boldsymbol{X}_P \trianglelefteq_{1u} c_1, \\ & \qquad \vdots \\ & b_I \trianglelefteq_{Il} \boldsymbol{A}_{I1} \bullet \boldsymbol{X}_1 + \cdots + \boldsymbol{A}_{IP} \bullet \boldsymbol{X}_P \trianglelefteq_{Iu} c_I, \\ & \boldsymbol{B}_p \bullet \boldsymbol{X}_p = 0, \ p = 1,\ldots,P, \\ & \boldsymbol{X}_p \succeq \boldsymbol{0}, \end{array} \tag{31}$$

where all $\boldsymbol{A}_{ip}$ are Hermitian, and $\boldsymbol{B}_p \succeq \boldsymbol{0}$ for all p. One may conclude that a solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_P^\star)$ with

$$\sum_{p=1}^{P} \text{Rank}^2(\boldsymbol{X}_p^\star) \leq I$$

for the above SDP can be found in polynomial time, under some mild conditions (cf. ([18], Theorem 3.2)).

IV. Application of the Proposed Randomized Algorithms to Solve Problems (8), (12), and (14)

In this section, we demonstrate how to employ the randomized algorithms previously designed to the three signal processing applications formulated in Section II.

A. Solving the Robust Receive Beamforming Problems (7) and (8)

The robust receive beamforming problem (7) is a particular instance of (1) with two double-sided constraints and, thus, randomized Algorithm 2 can be immediately employed to solve it.

The generalized doubly constrained Capon beamforming problem (8) has the equivalent form of homogenous QCQP:

$$\begin{array}{ll} \underset{\boldsymbol{x} \in \mathbb{C}^{N+1}}{\text{minimize}} & \boldsymbol{x}^H \boldsymbol{A}_0 \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \leq 0, \\ & N - \eta_1 \leq \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x} \leq N + \eta_2, \\ & \boldsymbol{x}^H \boldsymbol{A}_3 \boldsymbol{x} = 1, \end{array} \tag{32}$$

[7]It is seen that the solution $\{\boldsymbol{x}_p^\star\}$ is feasible for the constraint $\boldsymbol{x}_p^H \boldsymbol{B}_p \boldsymbol{x}_p = 0$, due to $\boldsymbol{B}_p \succeq \boldsymbol{0}$ and the obtained $\boldsymbol{x}_p^\star \in \text{Range}\,(\boldsymbol{X}_p^\star) \subseteq \text{Null}\,(\boldsymbol{B}_p)$, and that $(\{\boldsymbol{x}_p^\star \boldsymbol{x}_p^{\star H}\}; \{x_i^\star, y_i^\star\}, \{z_p^\star, \boldsymbol{Z}_p^\star\})$ is a primal-dual pair satisfying the complementary conditions for (29)–(30), i.e., an optimal pair.

where the matrix parameters are

$$\boldsymbol{A}_0 = \begin{bmatrix} \hat{\boldsymbol{R}}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{Q}\hat{\boldsymbol{a}} \\ \hat{\boldsymbol{a}}^H\boldsymbol{Q} & \hat{\boldsymbol{a}}^H\boldsymbol{Q}\hat{\boldsymbol{a}} - \epsilon \end{bmatrix},$$
$$\boldsymbol{A}_2 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}.$$

In words, if $\boldsymbol{x}^\star = [\boldsymbol{x}_1^{\star T}, t^\star]^T \in \mathbb{C}^{N+1}$ solves (32), then $\boldsymbol{x}_1^\star/t^\star$ solves (8) (conversely, if $\boldsymbol{x}_1^\star$ solves (8), then $[\boldsymbol{x}_1^{\star T}, 1]^T$ is an optimal solution for (32)). Also, the optimal values are equal: $v^\star((32)) = v^\star((8))$. Since (32) is in a special form of (26), hence Algorithm 4 is applicable to (32).

B. Solving the Radar Code Maximization Problem (12)

Like (32), the optimal radar code problem (12) has the following equivalent homogeneous QCQP reformulation with four constraints:

$$\begin{aligned} \underset{\boldsymbol{x}=[\boldsymbol{c}^T,t]^T}{\text{maximize}} \quad & \boldsymbol{Q}_0 \bullet \boldsymbol{X} && \text{(33a)} \\ \text{subject to} \quad & 1 - \eta_1 \le \boldsymbol{Q}_1 \bullet \boldsymbol{X} \le 1 + \eta_2, && \text{(33b)} \\ & \boldsymbol{Q}_2 \bullet \boldsymbol{X} \ge \delta_a, && \text{(33c)} \\ & \boldsymbol{Q}_3 \bullet \boldsymbol{X} \le 0, && \text{(33d)} \\ & \boldsymbol{Q}_4 \bullet \boldsymbol{X} = 1, && \text{(33e)} \\ & \boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^H,\ \boldsymbol{x} = [\boldsymbol{c}^T, t]^T, && \text{(33f)} \end{aligned}$$

where matrices $\boldsymbol{Q}_i$ are defined as follows:

$$\boldsymbol{Q}_0 = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \boldsymbol{Q}_1 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \boldsymbol{Q}_2 = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \tag{34}$$

$$\boldsymbol{Q}_3 = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{c}_0 \\ -\boldsymbol{c}_0^H & 1-\epsilon \end{bmatrix}, \boldsymbol{Q}_4 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}. \tag{35}$$

It is seen that the optimal values of (12) and (33) are equal, i.e.,

$$v^\star((12)) = v^\star((33)). \tag{36}$$

The conventional SDP relaxation for (33) is as follows:

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{H}^{N+1}}{\text{maximize}} \quad & \boldsymbol{Q}_0 \bullet \boldsymbol{X} \\ \text{subject to} \quad & \text{(33b)} - \text{(33e) satisfied}, \\ & \boldsymbol{X} \succeq \boldsymbol{0}, \end{aligned} \tag{37}$$

which is also an SDP relaxation for (12), considering (36).

Note that (33) does not belong to the class of (28). Nonetheless, we show that the SDP relaxation (37) is tight (i.e., $v^\star((37)) = v^\star((33))$), and it turns out that the proof is based on exploiting the structure that the constraint functions in (12b) and (12d) share the same Hessian $\boldsymbol{I}$. In other words, the QCQP (12) is hidden convex and can be solved polynomially. The proof can be found in Appendix E. According to the proof, we summarize how to solve (12): (i) solve (37) getting a solution $\boldsymbol{X}^\star$ (as in (67)), (ii) apply Algorithm 4 to solve (71) in which $\delta_1 = \boldsymbol{Q}_1 \bullet \boldsymbol{X}^\star$ is set[8], obtaining solution $\boldsymbol{c}^\star$, and (iii) return $\boldsymbol{c}^\star e^{j\arg(\boldsymbol{c}^{\star H}\boldsymbol{c}_0)}$ (which is optimal for (12)).

C. Solving the Single-Group Multicast Downlink Beamforming Problem (14)

It is clear that the beamforming problem (14) is a QCQP with $M+K$ one-sided constraints. Thus, Algorithm 2 is applicable if $M = K = 1$, and Algorithm 4 can be employed to solve it if $M + K = 3$ (e.g., $M = 2$ and $K = 1$). Furthermore, for the beamforming problem (15) with null-shaping interference constraints, it is solvable for M up to three and any number of null-shaping constraints, provided that the SDP relaxation of it is solvable. As a matter of fact, (15) is tantamount to the following QCQP:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H\boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^H\boldsymbol{R}_m\boldsymbol{w} \ge \sigma_m^2\tau_m,\ m = 1,\ldots,M, \\ & \boldsymbol{w}^H\boldsymbol{S}\boldsymbol{w} = 0, \end{aligned} \tag{38}$$

where $\boldsymbol{S} = \sum_{k=1}^K \boldsymbol{g}_k\boldsymbol{g}_k^H$. When $M = 3$, Problem (38) belongs to the class of QCQP (28) (with $P = 1$). Therefore the algorithm proposed in Section III.D for (28) can be utilized to solve (38) (or equivalently (15)), as long as its SDP relaxation problem (cf. (29)) is solvable.

V. An Alternative Proof of Sturm-Zhang Rank-One Matrix Decomposition Theorem

In [13], a specific rank-one decomposition for a positive semidefinite matrix was proposed, and it turns out that the matrix decomposition technique is useful and fundamental as it can be used to prove some convexity results of joint numerical range and S-lemma (cf. [13], [15], and [17]). We start by quoting the rank-one decomposition theorem, and then we present an alternative (shorter) proof using probabilistic methods as done in the previous section. This result is a byproduct of this paper. Like the advantages of the rank-one decomposition theorems, one can exploit the probabilistic methods to show the convexity for some joint numerical ranges, and S-lemma. Now, let us start with the case of real matrices.

Lemma 5.1 ([13]): Suppose that $\boldsymbol{X}$ is a $N \times N$ real positive semidefinite matrix of rank R and $\boldsymbol{A}$ is a $N \times N$ symmetric matrix. Then, there is a rank-one decomposition $\boldsymbol{X} = \sum_{r=1}^R \boldsymbol{x}_r\boldsymbol{x}_r^H$ such that

$$\boldsymbol{x}_r^T\boldsymbol{A}\boldsymbol{x}_r = \frac{\boldsymbol{X}\bullet\boldsymbol{A}}{R}, \quad r = 1,\ldots,R. \tag{39}$$

Proof: First we claim that there is a vector $\boldsymbol{x}_1 \in \text{Range}(\boldsymbol{X})$ such that $\boldsymbol{X} - \boldsymbol{x}_1\boldsymbol{x}_1^T \succeq \boldsymbol{0}$ and $\boldsymbol{A}\bullet\boldsymbol{x}_1\boldsymbol{x}_1^T = \frac{1}{R}\boldsymbol{A}\bullet\boldsymbol{X}$. In fact, let $\boldsymbol{X} = \boldsymbol{X}_1\boldsymbol{X}_1^T$ and $\boldsymbol{X}_1^T\boldsymbol{A}\boldsymbol{X}_1 = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$. It thus follows that $\boldsymbol{A}\bullet\boldsymbol{X} = \boldsymbol{\Lambda}\bullet\boldsymbol{I}$. Take any vector $\boldsymbol{\xi}$ such that its components $|\xi_i| = 1$. Alternatively, draw a random vector $\boldsymbol{\xi} \in \mathbb{R}^R$

[8]When implementing Algorithm 4, we particularly solve the SDP relaxation (73) of (71) in step 1 of Algorithm 4, obtaining a solution $\boldsymbol{C}^\star$; however, we do not have to solve (73) in that step, because the block $\boldsymbol{C}^\star$ of $\boldsymbol{X}^\star$ (the solution of (37) just obtained) is optimal for (73), due to the last remark in Appendix E.

according to the distribution (16). And set $\boldsymbol{x}_1 = \frac{1}{\sqrt{R}}\boldsymbol{X}_1\boldsymbol{U}\boldsymbol{\xi}$. It is verified that $\boldsymbol{x}_1$ is the desired vector.

Let $\boldsymbol{X}_1 = \boldsymbol{X} - \boldsymbol{x}_1\boldsymbol{x}_1^T$, and it is seen that $\mathrm{Rank}\,(\boldsymbol{X}_1) = R-1$. We repeat the above procedure, getting vector $\boldsymbol{x}_2$ so that $\boldsymbol{x}_2^T\boldsymbol{A}\boldsymbol{x}_2 = \frac{1}{R-1}\boldsymbol{A}\bullet\boldsymbol{X}_1 = \frac{1}{R}\boldsymbol{A}\bullet\boldsymbol{X}$. Repeating the procedure iteratively, we obtain vectors $\boldsymbol{x}_3,\ldots,\boldsymbol{x}_R$ such that they satisfy (39). ■

Likewise, capitalizing on Lemma 3.3, we can provide another proof of the rank-one decomposition theorem for Hermitian matrices in [15].

Lemma 5.2 ([15]): Suppose that $\boldsymbol{X}$ is a $N\times N$ complex Hermitian positive semidefinite matrix of rank R, and $\boldsymbol{A}$, $\boldsymbol{B}$ are two $N\times N$ given Hermitian matrices. Then, there is a rank-one decomposition $\boldsymbol{X} = \sum_{r=1}^{R}\boldsymbol{x}_r\boldsymbol{x}_r^H$ such that

$$\boldsymbol{x}_r^H\boldsymbol{A}\boldsymbol{x}_r = \frac{\boldsymbol{X}\bullet\boldsymbol{A}}{R} \quad \text{and}$$
$$\boldsymbol{x}_r^H\boldsymbol{B}\boldsymbol{x}_r = \frac{\boldsymbol{X}\bullet\boldsymbol{B}}{R}, \quad r=1,\ldots,R.$$

Proof: We claim that there is a $\boldsymbol{x}\in\mathrm{Range}\,(\boldsymbol{X})$ such that

$$\begin{aligned}&\boldsymbol{X}-\boldsymbol{x}\boldsymbol{x}^H \succeq \boldsymbol{0},\\ &\boldsymbol{A}\bullet\boldsymbol{x}\boldsymbol{x}^H = \frac{1}{R}\boldsymbol{A}\bullet\boldsymbol{X},\ \boldsymbol{B}\bullet\boldsymbol{x}\boldsymbol{x}^H = \frac{1}{R}\boldsymbol{B}\bullet\boldsymbol{X}.\end{aligned} \tag{40}$$

In fact, let $\boldsymbol{X} = \boldsymbol{X}_1\boldsymbol{X}_1^H$ ($\boldsymbol{X}_1\in\mathbb{C}^{N\times R}$), $\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H = \boldsymbol{X}_1^H\boldsymbol{A}\boldsymbol{X}_1$ ($\boldsymbol{U}$ and $\boldsymbol{\Lambda}$ are $R\times R$), and $\boldsymbol{Q} = \boldsymbol{U}^H\boldsymbol{X}_1^H\boldsymbol{B}\boldsymbol{X}_1\boldsymbol{U}$. It follows from Lemma 3.3 that there is a randomized vector $\boldsymbol{v}\in\mathbb{C}^R$ such that $\boldsymbol{\Lambda}\bullet\boldsymbol{v}\boldsymbol{v}^H = \boldsymbol{\Lambda}\bullet\boldsymbol{I} = \boldsymbol{A}\bullet\boldsymbol{X}, \boldsymbol{Q}\bullet\boldsymbol{v}\boldsymbol{v}^H = \boldsymbol{Q}\bullet\boldsymbol{I} = \boldsymbol{B}\bullet\boldsymbol{X}$, and $\boldsymbol{I} - \frac{1}{R}\boldsymbol{v}\boldsymbol{v}^H \succeq \boldsymbol{0}$. It is easily verified that $\boldsymbol{x}_1 = \frac{1}{\sqrt{R}}\boldsymbol{X}_1\boldsymbol{U}\boldsymbol{v}$ is the intended vector satisfying (40).

Like the second paragraph in the proof of Lemma 5.1, we can obtain $\boldsymbol{x}_2,\ldots,\boldsymbol{x}_R$ such that $\boldsymbol{x}_r^H\boldsymbol{A}\boldsymbol{x}_r = \frac{1}{R}\boldsymbol{A}\bullet\boldsymbol{X}$ and $\boldsymbol{x}_r^H\boldsymbol{B}\boldsymbol{x}_r = \frac{1}{R}\boldsymbol{B}\bullet\boldsymbol{X}, r=2,\ldots,R$. This completes the proof. ■

We remark that the rank-one decomposition Lemma 5.2 can be utilized to show Lemma 3.3. However, the difference lies in that the proof of Lemma 5.2 in [15] (cf. [13]) includes some deterministic rotation procedures to generate the desired vectors, while the selection of a vector in the proof of Lemma 3.3 is based on some randomization steps.

VI. Numerical Examples

In this section, we present some numerical examples to illustrate the performance of the algorithms. The numerical examples are from both robust receive beamforming and optimal downlink transmit beamforming.

A. Simulation Examples: Robust Receive Beamforming Problem (7)

In the simulations, we assume a uniform linear array with $N=10$ omni-directional sensors spaced half a wavelength and with spatially and temporally white Gaussian noise whose covariance is given by $\boldsymbol{I}$. Two equal-power interferers with the interference-to-noise ratio (INR) of 30 dB are assumed to impinge on the array from the angles $\theta_1 = 40^\circ$ and $\theta_1 = 60^\circ$ with respect to the array broadside. The angular sector of interest Θ is preset to $[\hat{\theta}-5^\circ, \hat{\theta}+5^\circ]$ with $\hat{\theta} = 5^\circ$. The actual SOI is assumed to impinge on the array from the direction $\theta_0 = 1^\circ$ (which could be viewed as a 4° mismatch in the signal look direction from the presumed $\hat{\theta}$), and it is always present in the training data with training sample size $T=50$. The norm perturbation parameters both η_1 and η_2 are set to $0.1N$. All results are averaged over 200 simulation runs. In each run, both the beamforming problem (7) and the beamforming problem (23)–(25) in [33] are solved for different SNR$= -20:5:20$ dB. The two beamforming problems are termed "Proposed Beamformer" and "KVH Beamformer" respectively in the following figures.

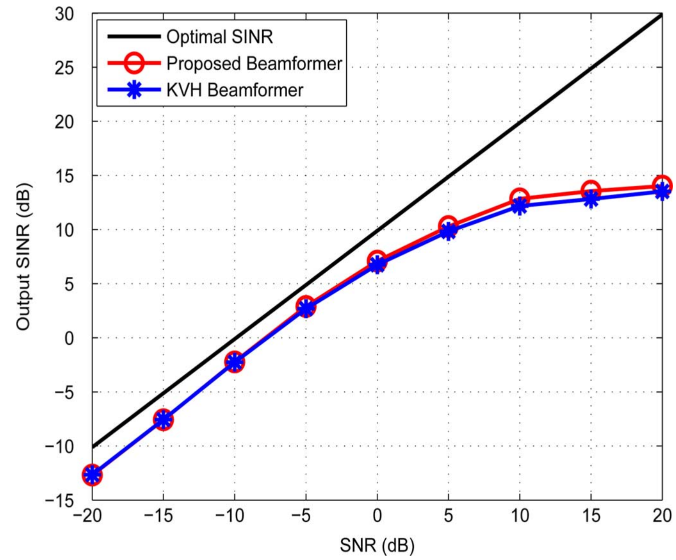

Fig. 1. Average output array SINR versus SNR.

Further, we assume that the SOI steering vector from the angle θ_0 is distorted by wave propagation effects in the way that independent-increment phase distortions are accumulated by the components of the SOI steering vector (starting from zero), and assume that the phase increments are independent Gaussian variables with zero mean and standard deviation 0.08, and they are randomly generated and remain frozen in each simulation run.

Simulation Example 1: We evaluate the performance in terms of both beamformer output SINR (4) and output power (6). Fig. 1 shows the output array SINR versus the SNR, and Fig. 2 displays the beamformer output power curves versus SNR. As we can see, the output SINR by (7) is only slightly better than that by the beamforming problem in [33], while the former beamformer has higher output power than the latter one. It is expected, since the double-sided constraint describing the norm perturbation in (7) allows the larger feasible region for searching the optimal steering vector.

B. Simulation Examples: Optimal Multicast Downlink Beamforming Problem (15)

In the subsection, we test several simulated scenarios for the single-group multicast downlink beamforming problem (15) with null-shaping interference constraints described in Sections II.C and IV.C.

We consider a simulated scenario with a 16-antenna base station transmitting a common data stream to three single-antenna users, i.e., $N=16$ and $M=3$ in Problem (14). The users are located at $\theta_1 = -10^\circ, \theta_2 = 5^\circ$ and $\theta_3 = 20^\circ$ relative to the

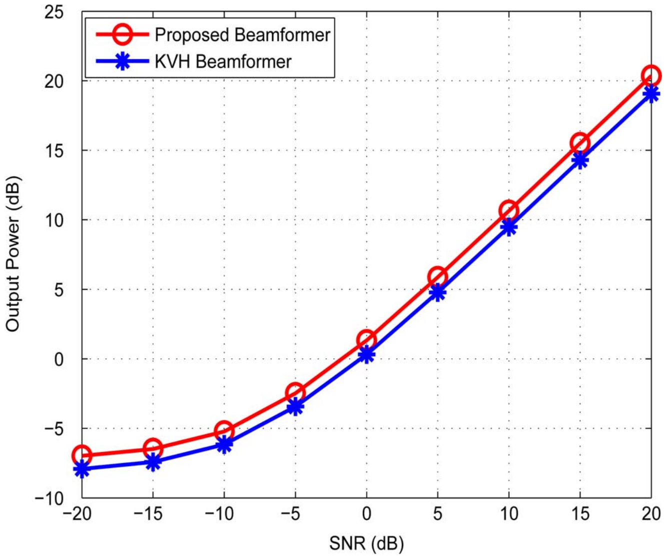

Fig. 2. Average beamformer output power versus SNR.

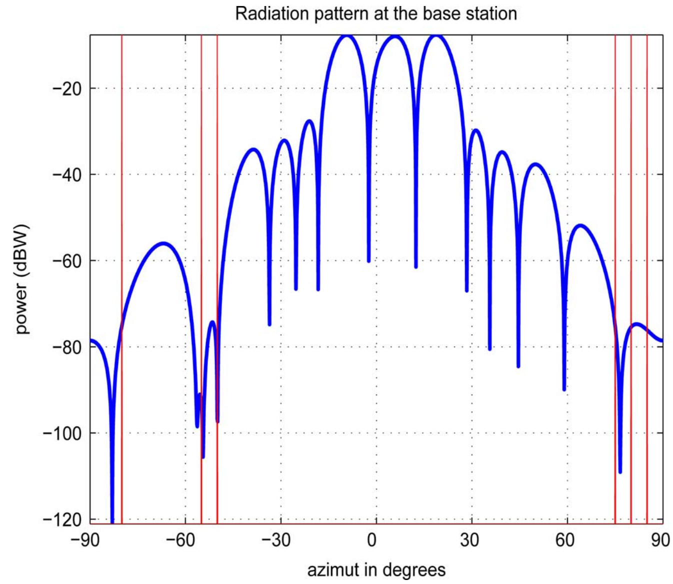

Fig. 3. Radiation pattern of the base station, for the problem using the general-rank channel covariance matrices for the internal users, with six null-shaping interference constraints and $N = 16$ transmit antennas. (The required transmission power is 14.92 dBm.)

array broadside of the base station. The transmit antenna array is linear and has the elements spaced half a wavelength. The channel covariance matrix for users $m = 1, 2, 3$ is generated according to (see [22])

$$[\boldsymbol{R}_m(\theta_m)]_{pq} = e^{j\pi(p-q)\sin\theta_m} e^{-(\pi(p-q)\sigma_\theta \cos\theta_m)^2/2}, \quad (41)$$

$p, q \in \{1, \ldots, N\}$, where σ_θ is the angular spread of local scatters surrounding user m. The noise variance is set to $\sigma_m^2 = 0.1$ and the SNR threshold value $\tau_m = 1$ for each internal user m. The channel between the base station and external user k (located at $\hat{\theta}_k$ relative to the base station) is given by

$$\boldsymbol{h}(\hat{\theta}_k) = \left[1 \; e^{j\phi_k} \cdots e^{j(N-1)\phi_k}\right]^T, \quad (42)$$

with $\phi_k = 2\pi d \sin(\hat{\theta}_k)/\lambda$ and $d/\lambda = 1/2$. The tolerable value η_k of the soft-shaping constraint k in (14) is set to zero; in words, the stricter version (15) (or (38)) with null interference constraints are considered in the simulation.

Simulation Example 2: In this example, we present simulation results for the beamforming problem (38) where six null interference constraints (i.e., $K = 6$) for six external users at $\{-80°, -55°, -50°, 75°, 80°, 85°\}$, together with the three internal users, are involved. In order to illustrate the effect of the null interference constraints, we evaluate the power radiation pattern of the base station, for $\theta \in [-90°, 90°]$, according to

$$P(\theta) = \boldsymbol{h}(\theta)\boldsymbol{h}^H(\theta) \bullet \boldsymbol{w}^\star \boldsymbol{w}^{\star H} \quad (43)$$

where $\boldsymbol{w}^\star$ is an optimal beamvector and $\boldsymbol{h}(\theta)$ is defined in (42). We report that the SDP relaxation of (38) gives a high-rank solution in our simulation experiment, and thus Algorithm 4 is applied to generate a rank-one solution. Fig. 3 displays the radiation pattern of the base station with the channel covariance matrix (high-rank) (41) employed (the minimal required transmission power is 14.92 dBm).

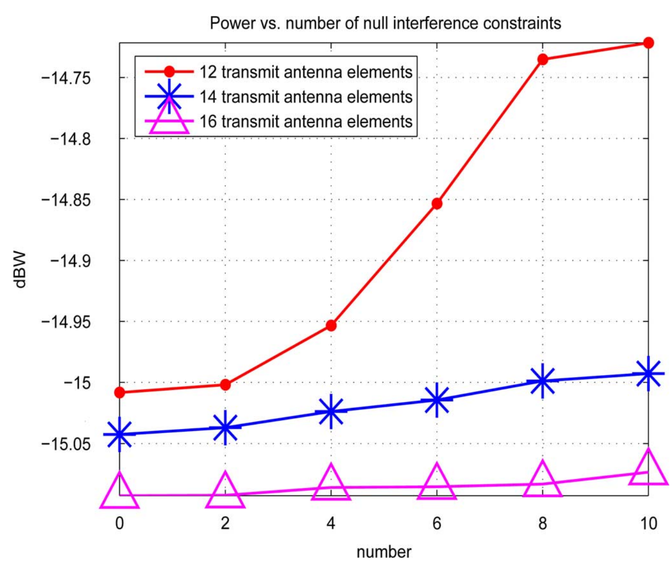

Fig. 4. Minimal transmission power versus the number of null-shaping interference constraints, with different numbers of transmit antennas, adopting the general-rank channel covariance matrices for the internal users.

Simulation Example 3: This example shows how the minimal transmission power in (15) is affected by the number of null-shaping interference constraints and by the number of transmit antennas. Apart from the three internal users, we consider ten external users located at $\hat{\theta}_k \in \Omega = \{-85°, -65°, -60°, -55°, -35°, -25°, 35°, 50°, 70°, 80°\}$. By saying k null interference constraints, we mean the k external users located at $\hat{\theta}_k$ less than or equal to the first kth smallest element of $\Omega, k = 0, \ldots, 10$. Fig. 4 illustrates the minimal transmission power versus the number of null interference constraints. As can be seen, higher and higher transmission power is required to comply with null interference constraints for more and more external users, as well as less and less transmit antennas. The latter observation is due to the fact that given K, the beamforming problem (15) with larger N (number of transmit antennas) possesses bigger search space (thus the minimal value is smaller).

VII. Conclusion

In this paper, we have considered the QCQP with double-sided constraints. Although the problem is typically hard to solve in general, we have proposed efficient algorithms for some instances of the problem with a small number of constraints. The presented algorithms are mainly composed of simple randomization steps following the resolution of the SDP relaxation of the problem. The optimization problems arising in three signal processing applications have been formulated in a general QCQP form and it has been demonstrated how to solve them (achieving global optimality) by the randomized algorithms. As a byproduct, we have provided a shorter proof for the Sturm-Zhang rank-one matrix decomposition theorem using the probabilistic method. Numerical examples have been conducted to show the effectiveness of the proposed algorithms for the QCQP in the context of both the robust receive beamforming problem and the broadcast MISO downlink beamforming problem.

Appendix

A. Proof of Proposition 3.1

Proof: Since any $\boldsymbol{\xi}$ with its components $|\xi_i| = 1$ can be always obtained by taking a random vector whose components are i.i.d. variables having the distribution (16), hence it suffices for us to show that $\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}$ is an optimal solution for (24), where that $\boldsymbol{\xi}$ is a vector whose components are i.i.d. random variables ξ_i following the binary distribution (16).

Let $\boldsymbol{x}^\star = \boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}$. It follows that

$$\begin{aligned}\boldsymbol{x}^{\star H}\boldsymbol{A}_1\boldsymbol{x}^\star &= \boldsymbol{\xi}^H\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_1\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}\\ &= \boldsymbol{\Lambda}_1 \bullet \boldsymbol{\xi}\boldsymbol{\xi}^H = \boldsymbol{\Lambda}_1 \bullet \boldsymbol{I} = \boldsymbol{A} \bullet \boldsymbol{X}^\star \end{aligned}\tag{44}$$

(note that $\boldsymbol{\xi}^H = \boldsymbol{\xi}^T$ when it is real-valued) and

$$\begin{aligned}\mathsf{E}[\boldsymbol{x}^{\star H}\boldsymbol{A}_0\boldsymbol{x}^\star] &= \boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_0\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1 \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^H]\\ &= \boldsymbol{A}_0 \bullet \boldsymbol{X}^\star = v^\star, \end{aligned}\tag{45}$$

where $v^\star = v^\star((18))$. We further claim that

$$\boldsymbol{x}^{\star H}\boldsymbol{A}_0\boldsymbol{x}^\star = \boldsymbol{\xi}^H\boldsymbol{U}_1^T\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_0\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi} = v^\star. \tag{46}$$

In fact, assume that there is a random vector $\boldsymbol{\xi}_1$ whose components are i.i.d. random variables following the distribution (16), such that

$$v_1 = \boldsymbol{\xi}_1^H\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_0\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}_1 < v^\star.$$

It is seen that $\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}_1(\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}_1)^H$ is feasible for Problem (18) (with one double-sided constraint), and v_1 thus is better than the optimal value, which is an evident contradiction. Therefore, $\mathsf{Prob}_{\boldsymbol{\xi}}\{\boldsymbol{\xi}^H\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_0\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi} < v^\star\} = 0$. This together with (45) yields

$$\mathsf{Prob}_{\boldsymbol{\xi}}\left\{\boldsymbol{\xi}^H\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_0\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi} > v^\star\right\} = 0. \tag{47}$$

Therefore we conclude that the vector $\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}$ is an optimal solution of QCQP (24). ■

We remark that the proof can alternatively be done in a deterministic way, namely, the optimal solution $\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}$ can be proved always within the null space of the optimal dual solution (i.e., the complementary condition (21)), where $\boldsymbol{\xi}$ is a vector satisfying the proposition. (However, we do not have to consider the optimal dual solution in the probabilistic proof.)

B. Proof of Proposition 3.2

Proof: Let $\delta_i = \boldsymbol{A}_i \bullet \boldsymbol{X}^\star, i = 1, 2$. If $\delta_1 = \delta_2 = 0$, then the optimal value $v^\star((18))$ (note that $I = 2$ in (18)) is equal to zero, and $\boldsymbol{x}^\star = \boldsymbol{0}$ is optimal. Otherwise, $v^\star((18)) \neq 0$, then it is easily seen that $v^\star((18)) = -\infty$, which contradicts to the premise of SDP's solvability. Thus we consider $\delta_2 \neq 0$, without loss of generality, in what follows.

Observe that $(\boldsymbol{A}_1 - \frac{\delta_1}{\delta_2}\boldsymbol{A}_2) \bullet \boldsymbol{X}^\star = 0$, which amounts to $\boldsymbol{X}^{\star 1/2}(\boldsymbol{A}_1 - \frac{\delta_1}{\delta_2}\boldsymbol{A}_2)\boldsymbol{X}^{\star 1/2} \bullet \boldsymbol{I} = 0$. Compute a eigen-decomposition $\boldsymbol{X}^{\star 1/2}(\boldsymbol{A}_1 - \frac{\delta_1}{\delta_2}\boldsymbol{A}_2)\boldsymbol{X}^{\star 1/2} = \boldsymbol{U}_1\boldsymbol{\Lambda}_1\boldsymbol{U}_1^H$, and let $\boldsymbol{\xi}$ be a vector whose components are i.i.d. random variables with the distribution (16). Then, it is verified that with probability one,

$$\begin{aligned}\boldsymbol{\xi}^T\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\left(\boldsymbol{A}_1 - \frac{\delta_1}{\delta_2}\boldsymbol{A}_2\right)\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi} &= \boldsymbol{\Lambda}_1 \bullet \boldsymbol{\xi}\boldsymbol{\xi}^T\\ &= \boldsymbol{\Lambda}_1 \bullet \boldsymbol{I} = 0, \end{aligned}\tag{48}$$

where we employ the fact that $\boldsymbol{\xi}\boldsymbol{\xi}^T$ has diagonal elements one. It is verified also that

$$\begin{aligned}&\mathsf{E}\left[\boldsymbol{\xi}^T\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_2\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}\right]\\ &\quad= \boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_2\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1 \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T] = \boldsymbol{A}_2 \bullet \boldsymbol{X}^\star = \delta_2,\end{aligned}$$

where we use the fact that $\mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T] = \boldsymbol{I}$. It follows that there is a realization $\bar{\boldsymbol{\xi}}$ of the random vector $\boldsymbol{\xi}$ (with i.i.d. random components of the distribution (16)) such that

$$\bar{\boldsymbol{\xi}}^T\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_2\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\bar{\boldsymbol{\xi}}\begin{cases}\geq \delta_2, & \text{if } \delta_2 > 0\\ \leq \delta_2, & \text{if } \delta_2 < 0\end{cases}. \tag{49}$$

Here, we note that the probability $p = \mathsf{Prob}\{\eta \geq \mathsf{E}[\eta]\}$ is always positive for the case of $\delta_2 > 0$ (discussion is similar to the case of $\delta_2 < 0$), where $\eta = \boldsymbol{\xi}^T\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_2\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\boldsymbol{\xi}$ and $\mathsf{E}[\eta] = \delta_2$. Thus the probability that after L independent trials there is no $\bar{\boldsymbol{\xi}}$ satisfying (49) is $(1-p)^L$, which equals 0.0066 for $p = 0.01$ and $L = 500$.

Regarding (48), one may try to find a $\bar{\boldsymbol{\xi}}$ in a deterministic way (as long as N is not large and $\boldsymbol{\xi}$ adopted in (49) is of i.i.d. components according to (16), instead of (17)), although a sufficient (but not excessive) number of randomizations can be generated to find a desired $\bar{\boldsymbol{\xi}}$.

Let

$$\lambda = \frac{\bar{\boldsymbol{\xi}}^T\boldsymbol{U}_1^H\boldsymbol{X}^{\star 1/2}\boldsymbol{A}_2\boldsymbol{X}^{\star 1/2}\boldsymbol{U}_1\bar{\boldsymbol{\xi}}}{\delta_2}, \tag{50}$$

and it is clear that $\lambda \geq 1$. From (48) it follows that

$$\bar{\boldsymbol{\xi}}^T \boldsymbol{U}_1^H \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_1 \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}} = \frac{\delta_1}{\delta_2} \bar{\boldsymbol{\xi}}^T \boldsymbol{U}_1^H \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_2 \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}}. \quad (51)$$

Evidently, (50) and (51) yield

$$\bar{\boldsymbol{\xi}}^T \boldsymbol{U}_1^H \boldsymbol{X}^{\star 1/2} \boldsymbol{A}_i \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}} = \lambda \delta_i,\ i = 1, 2, \quad (52)$$

Define $\boldsymbol{w} = \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}}$, and (52) is identical to

$$\boldsymbol{A}_i \bullet \frac{1}{\sqrt{\lambda}} \boldsymbol{w} \frac{1}{\sqrt{\lambda}} \boldsymbol{w}^H = \delta_i, \quad i = 1, 2. \quad (53)$$

In other words, $\frac{1}{\lambda} \boldsymbol{w}\boldsymbol{w}^H$ (where we note $\bar{\boldsymbol{\xi}}^T = \bar{\boldsymbol{\xi}}^H$ in $\boldsymbol{w}$) is feasible for (18), and complies with the complementary conditions (22)–(23) together with the dual optimal solution $(x_1^\star, y_1^\star, x_2^\star, y_2^\star)$ (according to the assumption). Besides, note that

$$\boldsymbol{w} = \boldsymbol{X}^{\star 1/2} \boldsymbol{U}_1 \bar{\boldsymbol{\xi}} \in \mathrm{Range}\,(\boldsymbol{X}^{\star 1/2}) = \mathrm{Range}\,(\boldsymbol{X}^\star) \subseteq \mathrm{Null}\,(\boldsymbol{Z}^\star), \quad (54)$$

where the last inclusion relation is due to (21)[9]. This means $\boldsymbol{Z}^\star \bullet \boldsymbol{w}\boldsymbol{w}^H = 0$, i.e., $\frac{1}{\lambda}\boldsymbol{w}\boldsymbol{w}^H$ and $\boldsymbol{Z}^\star$ fulfill the complementary condition (21). Therefore we conclude that $\frac{1}{\lambda}\boldsymbol{w}\boldsymbol{w}^H$ is optimal for the SDP (18); that is, $\frac{1}{\sqrt{\lambda}}\boldsymbol{w}$ is an optimal solution of the QCQP (25). ■

C. Proof of Lemma 3.3

Proof: Let $\delta_1 = \boldsymbol{\Lambda} \bullet \boldsymbol{I}$ and $\delta_2 = \boldsymbol{Q} \bullet \boldsymbol{I}$. Take i.i.d. random variables ξ_i of the distribution (16) for $i = 1, \ldots, N$. Thus $\delta_1 = \boldsymbol{\Lambda} \bullet \boldsymbol{I} = \boldsymbol{\Lambda} \bullet \boldsymbol{\xi}\boldsymbol{\xi}^T$ with probability one, and $\delta_2 = \boldsymbol{Q} \bullet \boldsymbol{I} = \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]$. We proceed the proof in two cases: (i) $\mathsf{Prob}_{\boldsymbol{\xi}}\{\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} = \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]\} = 1$ (note that it is the case if $\boldsymbol{Q}$ is a diagonal matrix); (ii) $\mathsf{Prob}_{\boldsymbol{\xi}}\{\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} = \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]\} < 1$.

Case (i). Since $\boldsymbol{\xi}^T\boldsymbol{\xi} = N$ with probability one, hence any realization of the random vector $\boldsymbol{\xi}$ satisfies (27).

Case (ii). Let $p_1 = \mathsf{Prob}_{\boldsymbol{\xi}}\{\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} < \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]\}$ and $p_2 = \mathsf{Prob}_{\boldsymbol{\xi}}\{\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} > \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]\}$. Assume that $p_1 p_2 > 0$. Otherwise, if $p_1 = 0$, it then follows that $p_2 = 0$ and $\mathsf{Prob}_{\boldsymbol{\xi}}\{\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} = \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]\} = 1$, which is case (i) we have just shown. Let us take a random vector $\boldsymbol{\eta} \in \mathbb{R}^N$ which is independent from $\boldsymbol{\xi}$ and is with i.i.d. components having the distribution (16). Observe that $\mathsf{Prob}_{\boldsymbol{\xi},\boldsymbol{\eta}}\{(\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} - \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T])(\boldsymbol{\eta}^T\boldsymbol{Q}\boldsymbol{\eta} - \boldsymbol{Q} \bullet \mathsf{E}[\boldsymbol{\eta}\boldsymbol{\eta}^T]) < 0\} = 2p_1p_2 > 0$. Therefore there are two realizations $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\eta}}$ of the random vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ respectively, such that[10]

$$(\bar{\boldsymbol{\xi}}^T\boldsymbol{Q}\bar{\boldsymbol{\xi}} - \boldsymbol{Q} \bullet \boldsymbol{I})(\bar{\boldsymbol{\eta}}^T\boldsymbol{Q}\bar{\boldsymbol{\eta}} - \boldsymbol{Q} \bullet \boldsymbol{I}) < 0. \quad (55)$$

The next step is to form another vector $\boldsymbol{v}$ such that $\boldsymbol{Q} \bullet \boldsymbol{v}\boldsymbol{v}^H = \boldsymbol{Q} \bullet \boldsymbol{I}$ using $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\eta}}$.

[9]To be more specific, for any element $\boldsymbol{X}^\star\boldsymbol{y} \in \mathrm{Range}\,(\boldsymbol{X}^\star)$, it has the property $\boldsymbol{Z}^\star\boldsymbol{X}^\star\boldsymbol{y} = \boldsymbol{0}$, since $\boldsymbol{Z}^\star\boldsymbol{X}^\star = \boldsymbol{0}$, which is due to $\boldsymbol{Z}^\star \bullet \boldsymbol{X}^\star = 0$ (i.e., (21)), $\boldsymbol{X}^\star \succeq \boldsymbol{0}$ and $\boldsymbol{Z}^\star \succeq \boldsymbol{0}$. Thus it follows that $\boldsymbol{X}^\star\boldsymbol{y} \in \mathrm{Null}\,(\boldsymbol{Z}^\star)$.

[10]We note that the probability of the event that no such $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\eta}}$ are generated after L independent trials is at most $(1 - 2p_1p_2)^L$, which equals 0.0059 for $p_1p_2 = 0.025$ and $L = 100$, for example. Thus such $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ requires relatively few trials to generate, and the number L of trials is independent from the size N of $\boldsymbol{\xi}$.

For easy notation, we remove the bars over $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in the remaining proof. Let $\boldsymbol{Q} = \boldsymbol{Q}_1 + j\boldsymbol{Q}_2$, and set

$$\gamma_0 = \frac{\boldsymbol{\xi}^T\boldsymbol{Q}_2\boldsymbol{\eta} + \sqrt{(\boldsymbol{\eta}^T\boldsymbol{Q}_2\boldsymbol{\xi})^2 - (\boldsymbol{\eta}^T\boldsymbol{Q}_1\boldsymbol{\eta} - \delta_2)(\boldsymbol{\xi}^T\boldsymbol{Q}_1\boldsymbol{\xi} - \delta_2)}}{\boldsymbol{\xi}^T\boldsymbol{Q}_1\boldsymbol{\xi} - \delta_2}, \quad (56)$$

and

$$\boldsymbol{v} = \frac{\gamma_0}{\sqrt{1+\gamma_0^2}}\boldsymbol{\xi} + j\frac{1}{\sqrt{1+\gamma_0^2}}\boldsymbol{\eta}. \quad (57)$$

It is seen that γ_0 is a root of

$$\gamma^2(\boldsymbol{\xi}^T\boldsymbol{Q}_1\boldsymbol{\xi} - \delta_2) - 2\gamma\boldsymbol{\xi}^T\boldsymbol{Q}_2\boldsymbol{\eta} + \boldsymbol{\eta}^T\boldsymbol{Q}_1\boldsymbol{\eta} - \delta_2 = 0. \quad (58)$$

In (58), we note that $\boldsymbol{\xi}^T\boldsymbol{Q}\boldsymbol{\xi} = \boldsymbol{\xi}^T\boldsymbol{Q}_1\boldsymbol{\xi}$ and $\boldsymbol{\eta}^T\boldsymbol{Q}_2\boldsymbol{\xi} = -\boldsymbol{\xi}^T\boldsymbol{Q}_2\boldsymbol{\eta}$ (recall that $\boldsymbol{Q}_1$ is symmetric and $\boldsymbol{Q}_2$ is skew-symmetric), and that the (58) does have a root, due to (55). It is further verified that the so-generated $\boldsymbol{v}$ fulfills $\boldsymbol{v}^H\boldsymbol{v} = N, \boldsymbol{v}^H\boldsymbol{\Lambda}\boldsymbol{v} = \boldsymbol{\Lambda} \bullet \boldsymbol{I}$, and $\boldsymbol{v}^H\boldsymbol{Q}\boldsymbol{v} = \boldsymbol{Q} \bullet \boldsymbol{I}$, where the last equality amounts to (58). ■

D. Proof of Proposition 3.4

Proof: Assume that $\delta_i = \boldsymbol{A}_i \bullet \boldsymbol{X}^\star, i = 1, 2, 3$, and $v^\star = \boldsymbol{A}_0 \bullet \boldsymbol{X}^\star$. Let $R = \mathrm{Rank}\,(\boldsymbol{X}^\star) \geq 2$. Suppose that one of δ_i is nonzero (if all δ_i are zero, then the optimal value $v^\star = 0$ and $\boldsymbol{x}^\star = \boldsymbol{0}$ is optimal), say $\delta_3 \neq 0$.

It follows that

$$\left(\boldsymbol{A}_i - \frac{\delta_i}{\delta_3}\boldsymbol{A}_3\right) \bullet \boldsymbol{X}^\star = \boldsymbol{X}_1^H\left(\boldsymbol{A}_i - \frac{\delta_i}{\delta_3}\boldsymbol{A}_3\right)\boldsymbol{X}_1 \bullet \boldsymbol{I} = 0, \quad i = 1, 2, \quad (59)$$

where $\boldsymbol{X}^\star = \boldsymbol{X}_1\boldsymbol{X}_1^H, \boldsymbol{X}_1 \in \mathbb{C}^{N \times R}$. Let

$$\boldsymbol{X}_1^H\left(\boldsymbol{A}_1 - \frac{\delta_1}{\delta_3}\boldsymbol{A}_3\right)\boldsymbol{X}_1 = \boldsymbol{U}_1\boldsymbol{\Lambda}_1\boldsymbol{U}_1^H, \quad (60)$$

$$\boldsymbol{Q}_1 = \boldsymbol{U}_1^H\boldsymbol{X}_1^H\left(\boldsymbol{A}_2 - \frac{\delta_2}{\delta_3}\boldsymbol{A}_3\right)\boldsymbol{X}_1\boldsymbol{U}_1. \quad (61)$$

Then (59) is equivalent to $\boldsymbol{\Lambda}_1 \bullet \boldsymbol{I} = 0$, and $\boldsymbol{Q}_1 \bullet \boldsymbol{I} = 0$. Applying the randomization procedure described in the proof of Lemma 3.3 (cf. Algorithm 3), we can generate a random vector $\boldsymbol{v}_1 \in \mathbb{R}^R$ such that

$$\boldsymbol{\Lambda}_1 \bullet \boldsymbol{v}_1\boldsymbol{v}_1^H = 0, \quad \boldsymbol{Q}_1 \bullet \boldsymbol{v}_1\boldsymbol{v}_1^H = 0, \quad \boldsymbol{I} \bullet \boldsymbol{v}_1\boldsymbol{v}_1^H = R. \quad (62)$$

Letting $\boldsymbol{w}_1 = \boldsymbol{X}_1\boldsymbol{U}_1\boldsymbol{v}_1$, it follows from (60)–(62) that

$$\left(\boldsymbol{A}_i - \frac{\delta_i}{\delta_3}\boldsymbol{A}_3\right) \bullet \boldsymbol{w}_1\boldsymbol{w}_1^H = 0, \quad i = 1, 2, 3. \quad (63)$$

Set $\sigma_1 = \boldsymbol{A}_3 \bullet \boldsymbol{w}_1\boldsymbol{w}_1^H/\delta_3$. If $\sigma_1 > 0$, then it is verified that

$$\boldsymbol{x}^\star = \boldsymbol{w}_1/\sqrt{\sigma_1} = \frac{1}{\sqrt{\sigma_1}}\boldsymbol{X}_1\boldsymbol{U}_1\boldsymbol{v}_1 \quad (64)$$

satisfies $\boldsymbol{A}_i \bullet \boldsymbol{x}^\star\boldsymbol{x}^{\star H} = \delta_i,\ i = 1, 2, 3$, and $\boldsymbol{x}^\star \in \mathrm{Null}\,(\boldsymbol{Z}^\star)$[11]. It follows from the complementary conditions (21)–(23) that $\boldsymbol{x}^\star\boldsymbol{x}^{\star H}$ is optimal for the SDP (18). Then we end up with the randomized optimal solution $\boldsymbol{x}^\star$ for (26).

[11]Since $\boldsymbol{X}_1\boldsymbol{X}_1^H \bullet \boldsymbol{Z}_1 = 0 \Rightarrow \boldsymbol{I} \bullet \boldsymbol{X}_1^H\boldsymbol{Z}^\star\boldsymbol{X}_1 = 0 \Rightarrow \boldsymbol{X}_1^H\boldsymbol{Z}^\star\boldsymbol{X}_1 = \boldsymbol{0}$, hence $\boldsymbol{Z}^{\star\frac{1}{2}}\boldsymbol{X}_1\boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{Z}^\star\boldsymbol{X}_1\boldsymbol{y} = \boldsymbol{0}$ and for any $\boldsymbol{y}$. That is, $\boldsymbol{X}_1\boldsymbol{y} \in \mathrm{Null}\,(\boldsymbol{Z}^\star)$ for any $\boldsymbol{y}$. This implies that $\boldsymbol{x}^\star \in \mathrm{Null}\,(\boldsymbol{Z}^\star)$.

If $\sigma_1 \le 0$, then we see that the matrix

$$\frac{1}{1-\sigma_1/R}\left(\boldsymbol{X}_1\boldsymbol{X}_1^H - \frac{1}{R}\boldsymbol{w}_1\boldsymbol{w}_1^H\right) = \frac{1}{1-\sigma_1/R}\boldsymbol{X}_1\boldsymbol{U}_1\left(\boldsymbol{I} - \frac{1}{R}\boldsymbol{v}_1\boldsymbol{v}_1^H\right)\boldsymbol{U}_1^H\boldsymbol{X}_1^H \succeq 0 \quad (65)$$

is of rank $R-1$, due to $\boldsymbol{v}_1^H\boldsymbol{v}_1 = R$ (see (62)). Let

$$\boldsymbol{X}_2\boldsymbol{X}_2^H := \frac{1}{1-\sigma_1/R}\left(\boldsymbol{X}_1\boldsymbol{X}_1^H - \frac{1}{R}\boldsymbol{w}_1\boldsymbol{w}_1^H\right),$$

where $\boldsymbol{X}_2 \in \mathbb{C}^{N\times(R-1)}$. It is easily verified that $\boldsymbol{A}_i \bullet \boldsymbol{X}_2\boldsymbol{X}_2^H = \delta_i,\ i=1,2,3$, and $\boldsymbol{X}_2\boldsymbol{X}_2^H \bullet \boldsymbol{Z}^\star = 0$ (since $\boldsymbol{X}_1^H\boldsymbol{Z}^\star\boldsymbol{X}_1 = \boldsymbol{0}$). In other words, the randomized matrix $\boldsymbol{X}_2\boldsymbol{X}_2^H$ is feasible for (18) and satisfies the complementary conditions (21)–(23), and thus optimal with rank reduced to $R-1$.

Let $R := R-1$, and repeat the previous procedure while R is more than or equal to two. In fact, when $R=2$, we repeat the same procedure and obtain either a randomized optimal solution for (26) (like (64)), corresponding to $\sigma_1 > 0$, or a rank-reduced solution (i.e., a rank-one solution) for the SDP (18) (like (65)), corresponding to $\sigma_1 \le 0$. ■

E. The SDP Relaxation Tightness for (33)

Proof: We wish to show that the optimal value of SDP relaxation problem (37) is equal to that of radar code problem (33), namely,

$$v^\star((37)) = v^\star((33)).$$

Since it is evident that

$$v^\star((37)) \ge v^\star((33)), \quad (66)$$

hence we need only to show the reverse inequality. To proceed, let

$$\boldsymbol{X}^\star = \begin{bmatrix} \boldsymbol{C}^\star & \boldsymbol{x}^\star \\ \boldsymbol{x}^{\star H} & 1 \end{bmatrix} \quad (67)$$

is an optimal solution for (37), and let

$$\delta_1 = \boldsymbol{Q}_1 \bullet \boldsymbol{X}^\star (= \boldsymbol{I} \bullet \boldsymbol{C}^\star). \quad (68)$$

Let us formulate the new code design problem (specifying the code norm):

$$\begin{aligned} \underset{\boldsymbol{c}\in\mathbb{C}^N}{\text{maximize}} \quad & \boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c} \\ \text{subject to} \quad & \boldsymbol{c}^H\boldsymbol{c} = \delta_1, \\ & (12\text{c}), (12\text{d}) \text{ satisfied}. \end{aligned} \quad (69)$$

Since $\delta_1 \in [1-\eta_1, 1+\eta_2]$, hence we have

$$v^\star((12)) \ge v^\star((69)), \quad (70)$$

The problem (69) can be transformed equivalently into the homogeneous QCQP

$$\begin{aligned} \underset{\boldsymbol{c}\in\mathbb{C}^N}{\text{maximize}} \quad & \boldsymbol{R} \bullet \boldsymbol{C} & (71\text{a}) \\ \text{subject to} \quad & \boldsymbol{I} \bullet \boldsymbol{C} = \delta_1, & (71\text{b}) \\ & \boldsymbol{R}_1 \bullet \boldsymbol{C} \ge \delta_a, & (71\text{c}) \\ & \boldsymbol{c}_0\boldsymbol{c}_0^H \bullet \boldsymbol{C} \ge ((1+\delta_1-\epsilon)/2)^2, & (71\text{d}) \\ & \boldsymbol{C} = \boldsymbol{c}\boldsymbol{c}^H, & (71\text{e}) \end{aligned}$$

through certain phase rotation (cf. ([25], page 5621)), i.e.,

$$v^\star((69)) = v^\star((71)). \quad (72)$$

Further, if $\boldsymbol{c}^\star$ is optimal for (71), then $\boldsymbol{c}^\star e^{j\arg(\boldsymbol{c}^{\star H}\boldsymbol{c}_0)}$ is optimal for (69). Now Problem (71) can be solved by Algorithm 4, and it can be claimed that the SDP relaxation problem for (71)

$$\begin{aligned} \underset{\boldsymbol{C}\in\mathcal{H}^N}{\text{maximize}} \quad & \boldsymbol{R} \bullet \boldsymbol{C} \\ \text{subject to} \quad & (71\text{b}) - (71\text{d}) \text{ satisfied}, \\ & \boldsymbol{C} \succeq \boldsymbol{0}, \end{aligned} \quad (73)$$

is tight, namely,

$$v^\star((71)) = v^\star((73)). \quad (74)$$

It follows from (66), (36), (70), (72), and (74) that

$$\begin{aligned} v^\star((37)) &\ge v^\star((33)) = v^\star((12)) \\ &\ge v^\star((69)) = v^\star((71)) = v^\star((73)). \end{aligned}$$

Thus in order to show that all the optimal values are equal (consequently $v^\star((33)) = v^\star((37))$), it suffices to prove that

$$v^\star((73)) \ge v^\star((37)). \quad (75)$$

Indeed, since any optimal solution $\boldsymbol{X}^\star$ of (37) (as in (67)) is feasible for the problem itself, hence the block $\boldsymbol{C}^\star$ (of $\boldsymbol{X}^\star$) complies with (71b), (71c), and $\boldsymbol{X}^\star$ satisfies (33d). It follows that $\Re(\boldsymbol{x}^{\star H}\boldsymbol{c}_0) \ge (\delta_1 + 1 - \epsilon)/2$. Observe that $\boldsymbol{C}^\star \succeq \boldsymbol{x}^\star\boldsymbol{x}^{\star H}$ (due to $\boldsymbol{X}^\star \succeq \boldsymbol{0}$), and it follows that

$$\boldsymbol{C}^\star \bullet \boldsymbol{c}_0\boldsymbol{c}_0^H \ge |\boldsymbol{x}^{\star H}\boldsymbol{c}_0|^2 \ge ((\delta_1+1-\epsilon)/2)^2,$$

which implies that $\boldsymbol{C}^\star$ also fulfills (71d). Therefore, $\boldsymbol{C}^\star$ is feasible for (73). This means that any optimal solution for (37) gives a feasible solution for (73), and hence we conclude (75).■

We remark that since $v^\star((73)) = v^\star((37))$, hence the block $\boldsymbol{C}^\star$ of an optimal solution $\boldsymbol{X}^\star$ (as in (67)) for (37) is optimal for (73).

References

[1] S. Boyd and L. Vandenberghe, *Convex Optimzation*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[2] A. Ben-Tal and A. Nemirovski, "Lectures on modern convex optimization" Georgia Inst. Technol., Atlanta, GA, USA, 2013 [Online]. Available: http://www2.isye.gatech.edu/nemirovs/Lect_ModConvOpt.pdf

[3] A. Nemirovski, K. Roos, and T. Terlaky, "On maximization of quadratic form over intersection of ellipsoids with common center," *Math. Program.*, ser. A, vol. 86, pp. 463–473, 1999.

[4] Z.-Q. Luo, N. D. Sidiropoulos, P. Tseng, and S. Zhang, “Approximation bounds for quadratic optimization with homogeneous quadratic constraints,” *SIAM J. Optimiz.*, vol. 18, no. 1, pp. 1–28, 2007.

[5] N. Sidiropoulos, T. Davidson, and Z.-Q. Luo, “Transmit beamforming for physical-layer multicasting,” *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 2239–2251, Jun. 2006.

[6] Y. Huang and S. Zhang, “Approximation algorithms for indefinite complex quadratic maximization problems,” *Sci. China Series A: Math.*, vol. 53, no. 10, pp. 2697–2708, Oct. 2010.

[7] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, “Semidefinite relaxation of quadratic optimization problems: From its practical deployments and scope of applicability to key theoretical results,” *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 20–34, May 2010.

[8] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson, and B. Ottersten, “Convex optimization-based beamforming: From receive to transmit and network designs,” *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 62–75, May 2010.

[9] Z.-Q. Luo and T.-H. Chang, , D. P. Palomar and Y. Eldar, Eds., “SDP relaxation of homogeneous quadratic optimization: Approximation bounds and applications,” in *Convex Optimization in Signal Processing and Communications*. Cambridge, U.K.: Cambridge Univ. Press, 2010, ch. 4.

[10] A. Ben-Tal and M. Teboulle, “Hidden convexity in some nonconvex quadratically constrained quadratic programming,” *Math. Program.*, vol. 72, pp. 51–63, 1996.

[11] G. Pataki, “On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues,” *Math. Operations Res.*, vol. 23, no. 2, pp. 339–358, 1998.

[12] Y. Ye and S. Zhang, “New results on quadratic minimization,” *SIAM J. Optimiz.*, vol. 14, no. 1, pp. 245–267, 2003.

[13] J. Sturm and S. Zhang, “On cones of nonnegative quadratic functions,” *Math. Operations Res.*, vol. 28, no. 2, pp. 246–267, 2003.

[14] A. Beck and Y. Eldar, “Strong duality in nonconvex quadratic optimization with two quadratic constraints,” *SIAM J. Optimiz.*, vol. 17, no. 3, pp. 884–860, 2006.

[15] Y. Huang and S. Zhang, “Complex matrix decomposition and quadratic programming,” *Math. Operations Res.*, vol. 32, no. 3, pp. 758–768, 2007.

[16] W. Ai and S. Zhang, “Strong duality for the CDT subproblem: A necessary and sufficient condition,” *SIAM J. Optimiz.*, vol. 19, no. 4, pp. 1735–1756, 2009.

[17] W. Ai, Y. Huang, and S. Zhang, “New results on Hermitian matrix rank-one decomposition,” *Math. Program.: Series A*, vol. 128, no. 1–2, pp. 253–283, June 2011.

[18] Y. Huang and D. P. Palomar, “Rank-constrained separable semidefinite programming with applications to optimal beamforming,” *IEEE Trans. Signal Process.*, vol. 58, no. 2, pp. 664–678, Feb. 2010.

[19] Y. Huang and D. P. Palomar, “A dual perspective on separable semidefinite programming with applications to optimal downlink beamforming,” *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4254–4271, Aug. 2010.

[20] A. De Maio, Y. Huang, D. P. Palomar, S. Zhang, and A. Farina, “Fractional QCQP with applications in ML steering direction estimation for radar detection,” *IEEE Trans. Signal Process.*, vol. 59, no. 1, pp. 172–185, Jan. 2011.

[21] A. Beck and Y. Eldar, “Doubly constrained robust Capon beamformer with ellipsoidal uncertainty sets,” *IEEE Trans. Signal Process.*, vol. 55, no. 2, pp. 753–758, Feb. 2007.

[22] M. Bengtsson and B. Ottersten, “Optimal transmit beamforming using semidefinite optimization,” in *Proc. 37th Annu. Allerton Conf. Commun., Contr., Comput.*, Sept. 1999, pp. 987–996.

[23] M. Bengtsson and B. Ottersten, , L. C. Godara, Ed., “Optimal and suboptimal transmit beamforming,” in *Handbook of Antennas in Wireless Communications*. Boca Raton, FL: CRC Press, Aug. 2001, ch. 18.

[24] O. Besson, “Adaptive detection with bounded steering vectors mismatch angle,” *IEEE Trans. Signal Process.*, vol. 55, no. 4, pp. 1560–1564, Apr. 2007.

[25] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, “Code design to optimize radar detection performance under accuracy and similarity constraints,” *IEEE Trans. Signal Process.*, vol. 56, no. 11, pp. 5618–5629, Nov. 2008.

[26] W.-K. Ma and P. C. Ching, “Crosstalk resilient interference cancellation in microphone arrays using Capon beamforming,” *IEEE Trans. Speech Audio Process.*, vol. 12, no. 5, pp. 468–477, Sep. 2004.

[27] S. A. Vorobyov, A. B. Gershman, and Z.-Q. Luo, “Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem,” *IEEE Trans. Signal Process.*, vol. 51, no. 2, pp. 313–324, Feb. 2003.

[28] R. G. Lorenz and S. P. Boyd, “Robust minimum variance beamforming,” *IEEE Trans. Signal Process.*, vol. 53, no. 5, pp. 1684–1696, May 2005.

[29] J. Li and P. Stoica, *Robust Adaptive Beamforming*. Hoboken, NJ, USA: Wiley, 2006.

[30] J. Li, P. Stoica, and Z. Wang, “Doubly constrained robust Capon beamformer,” *IEEE Trans. Signal Process.*, vol. 52, no. 9, pp. 2407–2423, Sep. 2004.

[31] C.-Y. Chen and P. P. Vaidyanathan, “Quadratically constrained beamforming robust against direction-of-arrival mismatch,” *IEEE Trans. Signal Process.*, vol. 55, no. 8, pp. 4139–4150, Aug. 2007.

[32] S.-J. Kim, A. Magnani, A. Mutapcic, S. P. Boyd, and Z.-Q. Luo, “Robust beamforming via worst-case SINR maximization,” *IEEE Trans. Signal Process.*, vol. 56, no. 4, pp. 1359–1547, Apr. 2008.

[33] A. Khabbazibasmenj, S. A. Vorobyov, and A. Hassanien, “Robust adaptive beamforming based on steering vector estimation with as little as possible prior information,” *IEEE Trans. Signal Process.*, vol. 60, no. 6, pp. 2974–2987, Jun. 2012.

[34] A. Hassanien, S. A. Vorobyov, and K. M. Wong, “Robust adaptive beamforming using sequential quadratic programming: An iterative solution to the mismatch problem,” *IEEE Signal Process. Lett.*, vol. 15, pp. 733–736, 2008.

[35] Y. Huang, A. De Maio, and S. Zhang, , D. P. Palomar and Y. Eldar, Eds. , “Semidefinite programming, matrix decomposition, and radar code design,” in *Convex Optimization in Signal Processing and Communications*. Cambridge, U.K.: Cambridge Univ. Press, 2010, ch. 6.

[36] A. De Maio, S. De Nicola, Y. Huang, D. P. Palomar, S. Zhang, and A. Farina, “Code design for radar STAP via optimization theory,” *IEEE Trans. Signal Process.*, vol. 58, no. 2, pp. 679–694, Feb. 2010.

[37] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, “Adaptive detection and estimation in the presence of useful signal and interference mismatches,” *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 436–450, Feb. 2009.

[38] E. Karipidis, N. Sidiropoulos, and Z.-Q. Luo, “Far-field multicast beamforming for uniform linear antenna arrays,” *IEEE Trans. Signal Process.*, vol. 55, no. 10, pp. 4916–4927, Oct. 2007.

[39] G. Scutari, D. P. Palomar, and S. Barbarossa, “Cognitive MIMO radio,” *IEEE Signal Process. Mag.*, vol. 25, no. 6, pp. 46–59, Nov. 2008.

[40] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, “Convex transmit beamforming for downlink multicasting to multiple co-channel groups,” in *Proc. IEEE ICASSP*, 2006, pp. 973–976.

[41] K. T. Phan, S. A. Vorobyov, N. D. Sidiropoulos, and C. Tellambura, “Spectrum sharing in wireless networks via QoS-aware secondary multicast beamforming,” *IEEE Trans. Signal Process.*, vol. 57, no. 6, pp. 2323–2335, Jun. 2009.

[42] Y. J. Zhang and A. M.-C. So, “Optimal spectrum sharing in MIMO cognitive radio networks via semidefinite programming,” *IEEE J. Sel. Areas Commun.*, vol. 29, no. 2, pp. 362–373, Feb. 2011.

[43] Y. Huang, Q. Li, W.-K. Ma, and S. Zhang, “Robust multicast beamforming for spectrum sharing-based cognitive radios,” *IEEE Trans. Signal Process.*, vol. 60, no. 1, pp. 527–533, Jan. 2012.

[44] G. Zheng, K.-K. Wong, and B. Ottersten, “Robust cognitive beamforming with bounded channel uncertainties,” *IEEE Trans. Signal Process.*, vol. 57, no. 12, pp. 4871–4881, Dec. 2009.

[45] S. Fazeli-Dehkordy, S. Shahbazpanahi, and S. Gazor, “Multiple peer-to-peer communications using a network of relays,” *IEEE Trans. Signal Process.*, vol. 57, no. 8, pp. 3053–3062, Aug. 2009.

[46] R. Wang, M. Tao, and Y. Huang, “Linear precoding designs for amplify-and-forward multiuser two-way relay systems,” *IEEE Trans. Wireless Commun.*, vol. 11, no. 12, pp. 4457–4469, Dec. 2012.

[47] Z. Quan, W.-K. Ma, S. Cui, and A. H. Sayed, “Optimal linear fusion for distributed detection via semidefinite programming,” *IEEE Trans. Signal Process.*, vol. 58, no. 4, pp. 2431–2436, Apr. 2010.

[48] D. Ramírez, J. Vía, and I. Santamaría, “Coherent fusion of information for optimal detection in sensor networks,” in *Proc. 2009 IEEE/SP 15th Workshop Statist. Signal Process.*, Cardiff, U.K., Aug.–Sept. 2009, pp. 793–796.

[49] M. U. Torun, A. N. Akansu, and M. Avellaneda, “Portfolio risk in multiple frequencies,” *IEEE Signal Process. Mag.*, vol. 28, no. 5, pp. 61–71, Sep. 2011.

Yongwei Huang (M'09) received the B.Sc. degree in information and computation science in 1998 and the M.Sc. degree in operations research in 2000, both from Chongqing University. In 2005, he received the Ph.D. degree in operations research from the Chinese University of Hong Kong.

He is a Research Assistant Professor in the Department of Mathematics, the Hong Kong Baptist University, Hong Kong. Prior to the current position, he was a Postdoctoral Fellow with the Department of Systems Engineering and Engineering Management, the Chinese University of Hong Kong, and a Visiting Researcher of the Department of Biomedical, Electronic, and Telecommunication Engineering, University of Naples "Federico II", Italy, and a Research Associate with the Department of Electronic and Computer Engineering, the Hong Kong University of Science and Technology. His research interests are related to optimization theory and algorithms, and their applications in signal processing for radar and wireless communications.

Daniel P. Palomar (S'99–M'03–SM'08–F'12) received the Electrical Engineering and Ph.D. degrees from the Technical University of Catalonia (UPC), Barcelona, Spain, in 1998 and 2003, respectively.

He is an Associate Professor in the Department of Electronic and Computer Engineering at the Hong Kong University of Science and Technology (HKUST), Hong Kong, which he joined in 2006. Since 2013 he is a Fellow of the Institute for Advance Study (IAS) at HKUST. He had previously held several research appointments, namely, at King's College London (KCL), London, UK; Technical University of Catalonia (UPC), Barcelona; Stanford University, Stanford, CA; Telecommunications Technological Center of Catalonia (CTTC), Barcelona; Royal Institute of Technology (KTH), Stockholm, Sweden; University of Rome "La Sapienza", Rome, Italy; and Princeton University, Princeton, NJ. His current research interests include applications of convex optimization theory, game theory, and variational inequality theory to financial systems and communication systems.

Dr. Palomar is an IEEE Fellow, a recipient of a 2004/06 Fulbright Research Fellowship, the 2004 Young Author Best Paper Award by the IEEE Signal Processing Society, the 2002/03 best Ph.D. prize in information technologies and communications by the Technical University of Catalonia (UPC), the 2002/03 Rosina Ribalta first prize for the Best Doctoral Thesis in Information Technologies and Communications by the Epson Foundation, and the 2004 prize for the best Doctoral Thesis in Advanced Mobile Communications by the Vodafone Foundation and COIT.

He serves as an Associate Editor of IEEE TRANSACTIONS ON INFORMATION THEORY, and has been an Associate Editor of IEEE TRANSACTIONS ON SIGNAL PROCESSING, a Guest Editor of the *IEEE Signal Processing Magazine* 2010 Special Issue on "Convex Optimization for Signal Processing," the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS 2008 Special Issue on "Game Theory in Communication Systems," and the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS 2007 Special Issue on "Optimization of MIMO Transceivers for Realistic Communication Networks."