Robust Estimation of Structured Covariance Matrix for Heavy-Tailed Elliptical Distributions

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***Abstract*—This paper considers the problem of robustly estimating a structured covariance matrix with an elliptical underlying distribution with a known mean. In applications where the covariance matrix naturally possesses a certain structure, taking the prior structure information into account in the estimation procedure is beneficial to improving the estimation accuracy. We propose incorporating the prior structure information into Tyler's M-estimator and formulating the problem as minimizing the cost function of Tyler's estimator under the prior structural constraint. First, the estimation under a general convex structural constraint is introduced with an efficient algorithm for finding the estimator derived based on the majorization-minimization (MM) algorithm framework. Then, the algorithm is tailored to several special structures that enjoy a wide range of applications in signal processing related fields, namely, sum of rank-one matrices, Toeplitz, and banded Toeplitz structure. In addition, two types of non-convex structures, i.e., the Kronecker structure and the spiked covariance structure, are also discussed, where it is shown that simple algorithms can be derived under the guidelines of MM. The algorithms are guaranteed to converge to a stationary point of the problems. Furthermore, if the constraint set is geodesically convex, such as the Kronecker structure set, then the algorithm converges to a global minimum. Numerical results show that the proposed estimator achieves a smaller estimation error than the benchmark estimators at a lower computational cost.**

***Index Terms*—Majorization-minimization, robust estimation, structural constraint, Tyler's *M*-estimator.**

I. Introduction

ESTIMATING the covariance matrix is a ubiquitous problem that arises in various fields such as signal processing, wireless communication, bioinformatics, and financial engineering [2]–[4]. It has been noticed that the covariance matrix in some applications naturally possesses some special structures. Exploiting the structure information in the estimation process usually implies a reduction in the number of parameters to be estimated, and thus is beneficial to improving the estimation accuracy [5]. Various types of structures have been studied. For example, the Toeplitz structure with applications in time series analysis and array signal processing was considered in [5]–[7]. A sparse graphical model was studied in [8], where sparsity was imposed on the inverse of the covariance matrix. Banding or tapering the sample covariance matrix was proposed in [3]. A spiked covariance structure, which is closely related to the problem of component analysis and subspace estimation, was introduced in [9]. Other structures such as group symmetry and the Kronecker structure were considered in [10]–[12].

While the previously mentioned works have shown that enforcing a prior structure on the covariance estimator improves its performance in many applications, most of them either assume that the samples follow a Gaussian distribution or attempt to regularize the sample covariance matrix. It has been realized that the sample covariance matrix, which turns out to be the maximum likelihood estimator of the covariance matrix when the samples are assumed to be independent identically normally distributed, performs poorly in many real-world applications. A major factor that causes the problem is that the distribution of a real-world data set is often heavy-tailed or contains outliers. In this case, a single erroneous observation can lead to a completely unreliable estimate [13].

A way to address the aforementioned problem is to find a robust structured covariance matrix estimator that performs well even if the underlying distribution deviates from the Gaussian assumption. One approach is to refer to the minimax principle and seek the "best" estimate of the covariance for the worst case noise. To be precise, the underlying probability distribution of the samples $f(\cdot)$ is assumed to belong to an uncertainty set of functions $\mathcal{F}$ that contains the Gaussian distribution, and the desired minimax robust estimator is the one whose maximum asymptotic variance over the set $\mathcal{F}$ is less than that of any other estimator. Two types of uncertainty sets $\mathcal{F}$, namely the ε-contamination and the Kolmogorov class, were considered in [14], where a structured maximum likelihood type estimate (*M*-estimate) was derived as the solution of a constrained optimization problem. For the elliptically symmetric distributions that we are interested in this paper, it was proved in [15] that given a number of N K-dimensional independent and identically distributed (*i.i.d.*) samples, the Tyler's estimator defined as the solution to the fixed-point equation

$$\mathbf{R} = \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^T}{\mathbf{x}_i^T\mathbf{R}^{-1}\mathbf{x}_i}, \tag{1}$$

is a minimax robust estimator. Additionally, it is "distribution-free" in the sense that its distribution does not depend on the parametric form of the underlying distribution. The estimator has been extended to the complex field in the works [16]–[19].

The problem of obtaining a structured Tyler's estimator was investigated in the recent works [20] and [21]. In particular, the

Manuscript received May 18, 2015; revised October 28, 2015; accepted March 9, 2016. Date of publication March 23, 2016; date of current version May 24, 2016. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Marco Lops. This work was supported by the Hong Kong RGC 16207814 research grant. Part of the results in this paper were preliminary presented at the IEEE International Conference on Acoustics, Speech and Signal Processing (ICCASP) 2015 [1].

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Digital Object Identifier 10.1109/TSP.2016.2546222

authors of [20] focused on the group symmetry structure and proved that it is a geodesically convex set. As the Tyler's estimator can be defined alternatively as the minimizer of a cost function that is also geodesically convex, it is concluded that any local minimum of the cost function on a group symmetry constraint set is a global minimum. A numerical algorithm was also proposed to solve the constrained minimization problem. In [21], a convex structural constraint set was studied and a generalized method of moments type covariance estimator, COCA, was proposed. A numerical algorithm was also provided based on semidefinite relaxation. It was proved that COCA is an asymptotically consistent estimator. However, the algorithm suffers from the drawback that the computational cost increases as either N or K grows.

In this paper, we formulate the structured covariance estimation problem as the minimization of Tyler's cost function under the structural constraint. Our work generalizes [20] by considering a much larger family of structures, which includes the group symmetry structure. Instead of attempting to obtain a global optimal solution, which is a challenging task due to the non-convexity of the objective function, we focus on devising algorithms that converge to a stationary point of the problem. We first work out an algorithm framework for the general convex structural constraint based on the majorization-minimization (MM) framework, where a sequence of convex programming is required to be solved. Then we consider several special cases that appear frequently in practical applications. By exploiting specific problem structures, the algorithm is particularized, significantly reducing the computational load. We further discuss in the end two types of widely studied non-convex structures that turn out to be computationally tractable under the MM framework; one of them being the Kronecker structure and the other one being the spiked covariance structure. Under the assumption that the objective function goes to infinity whenever the variable tends to a singular limit, which is guaranteed when the population distribution is continuous and the number of samples N is larger than their dimension K, the sequence generated by the algorithm converges to a stationary point. It is worth mentioning that the Tyler's cost function was shown to be geodesically convex in [22], therefore the iterates generated by the algorithm converge to a global minimum when the constraint set is also geodesically convex [20], [23].

The paper is organized as follows. In Section II, we formulate the robust covariance estimation problem and introduce the MM algorithm framework. In Section III, we derive a majorization-minimization based algorithm framework for the general convex structure. Several special cases are considered in Section IV, where the algorithm is particularized obtaining higher efficiency by considering the specific form of the structure. Section V discusses the Kronecker structure and the spiked covariance structure, which are non-convex but algorithmically tractable. Numerical results are presented in Section VI and we conclude in Section VII.

Notation: Italic letters denote scalars, lower case boldface letters denote vectors, and upper case boldface letters denote matrices. $\mathbb{R}$ and $\mathbb{C}$ denote the real field and the complex field, respectively. $\left(\mathbb{R}_+^n, \mathbb{R}_{++}^n\right) \mathbb{R}^n$ denotes the set of (non-negative, positive) real vectors of dimension n. $\mathbb{C}^{m\times n}$ denotes complex-valued matrices of size $m \times n$. $\mathbb{S}_+^K$ $\left(\mathbb{S}_{++}^K\right)$ denotes the set of symmetric (real field) and Hermitian (complex field) positive semidefinite (definite) matrices of size $K \times K$. The superscripts $(\cdot)^*, (\cdot)^T, (\cdot)^H, (\cdot)^{-1}$ denote complex conjugate, transpose, conjugate transpose and matrix inversion, respectively. $\mathrm{Tr}(\cdot)$ and $\det(\cdot)$ denote the trace and determinant of a matrix. $\mathrm{vec}\,(\mathbf{X})$ denote a vector constructed by stacking the columns of $\mathbf{X}$. $\mathrm{diag}\,(\mathbf{x})$ denotes a diagonal matrix with $\mathbf{x}$ being its principal diagonal. $\mathbf{A} \succeq (\succ) \, \mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is positive semidefinite (definite). $\|\cdot\|_p$ denote the ℓ_p-norm. $\|\cdot\|_F$ denotes the Frobenius norm. $\mathrm{E}(\cdot), \mathrm{Var}(\cdot)$, and $\mathrm{Cov}(\cdot)$ stand for expectation, variance, and covariance, respectively.

II. Tyler's Estimator With Structural Constraint

Consider a number of N samples $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ in $\mathbb{C}^K$ drawn independently from a complex elliptically symmetric (CES) distribution with probability density function (pdf) as follows:

$$f(\mathbf{x}) = c \det(\mathbf{R}_0)^{-1} \psi\left(\left(\mathbf{x} - \mu_0\right)^H \mathbf{R}_0^{-1}\left(\mathbf{x} - \mu_0\right)\right), \qquad (2)$$

where $\mathbf{R}_0 \in \mathbb{S}_{++}^K$ is the scatter parameter, $\mu_0 \in \mathbb{C}^K$ is the location parameter, $\psi(\cdot)$ is the density generator, and $c > 0$ is a normalizing constant. It can be proved that if the mean $\mathrm{E}(\mathbf{x})$ exists, then it is equal to μ_0; and if the covariance matrix $\mathrm{E}\left(\mathbf{x}\mathbf{x}^H\right)$ exists, then it is proportional to $\mathbf{R}_0$ [24]. In the rests of this paper, we assume μ_0 is known and equal to the zero vector without loss of generality, which is a common assumption in the signal processing literature [25]. In practice, μ_0 can be substituted in prior by an estimate $\hat{\mu}$ if the zero mean assumption is violated.

Tyler's estimator for $\mathbf{R}_0$ is defined as the solution of the following fixed-point equation:

$$\mathbf{R} = \frac{K}{N} \sum_{i=1}^{N} \frac{\mathbf{x}_i \mathbf{x}_i^H}{\mathbf{x}_i^H \mathbf{R}^{-1} \mathbf{x}_i}, \qquad (3)$$

which can be interpreted as a weighted sum of rank one matrices $\mathbf{x}_i\mathbf{x}_i^H$ with the weight decreasing as $\mathbf{x}_i$ getting farther from the center. It is known that if $\mathbf{x}$ is elliptically distributed, then the normalized random variable $\mathbf{s} = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$ follows an angular central Gaussian distribution with the pdf taking the form

$$f(\mathbf{s}) \propto \det(\mathbf{R}_0)^{-1} \left(\mathbf{s}^H \mathbf{R}_0^{-1} \mathbf{s}\right)^{-K}. \qquad (4)$$

Tyler's estimator is the maximum likelihood estimator (MLE) of $\mathbf{R}_0$ by fitting the normalized samples $\{\mathbf{s}_i\}$ to $f(\mathbf{s})$. In other words, the estimator $\widehat{\mathbf{R}}$ is the minimizer of the following cost function

$$L(\mathbf{R}) = \log\det(\mathbf{R}) + \frac{K}{N} \sum_{i=1}^{N} \log\left(\mathbf{x}_i^H \mathbf{R}^{-1} \mathbf{x}_i\right) \qquad (5)$$

on the positive definite cone $\mathbb{S}_{++}^K$.[1] The estimator with a normalized trace is proved to be consistent and asymptotically normal with the variance independent of $\psi(\cdot)$ [4]. It can be verified that $L(\mathbf{R})$ is scale invariant, in the sense that

[1]The cost function (5) also applies to real-valued $\mathbf{x}_i$'s.

$L(\mathbf{R}) = L(r\mathbf{R}), \forall r > 0$. Consequently, Tyler's estimator $\widehat{\mathbf{R}}$ estimates $\mathbf{R}_0$ up to a positive scaling factor. We focus on applications such as the direction-of-arrival (DOA) finding problem and minimum variance portfolio problem, where obtaining an estimator of the covariance matrix up to a scaling factor is sufficient. The problem of estimating the scaling factor is beyond the scope of this paper.

It has been noticed that in some applications, the covariance matrix possesses a certain structure and taking account this information into the estimation yields a better estimate of $\mathbf{R}_0$ [10]–[12], [14]. Motivated by this idea, we consider the problem of including prior structure information into the Tyler's estimator to improve its estimation accuracy. To formulate the problem, we assume that $\mathbf{R}_0$ is constrained in a non-empty set $\mathcal{S}$ that is the intersection of a closed set, which characterizes the covariance structure, and the positive semidefinite cone $\mathbb{S}_+^K$, and then proceed to solve the optimization problem:

$$\begin{aligned} \underset{\mathbf{R}}{\text{minimize}} \quad & \log\det(\mathbf{R}) + \frac{K}{N}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i\right) \\ \text{subject to} \quad & \mathbf{R} \in \mathcal{S}. \end{aligned} \tag{6}$$

The minimizer $\widehat{\mathbf{R}}$ of the above problem is the one in the structural set $\mathcal{S}$ that maximizes the likelihood of the normalized samples $\{\mathbf{s}_i\}$.

Throughout the paper, we make the following assumption.

Assumption 1: The cost function $L(\mathbf{R}_t) \to +\infty$ when the sequence $\{\mathbf{R}_t\}$ tends to a singular limit point of the constraint set $\mathcal{S}$.

Under this assumption, the case that $\mathbf{R}$ is singular can be excluded in the analysis of the algorithms hereafter.

Note that the assumption

Assumption 2: $f(\mathbf{x})$ is a continuous probability distribution, and $N > K$, implies $L(\mathbf{R}_t) \to +\infty$ whenever $\mathbf{R}_t$ tends to the boundary of the positive semidefinite cone $\mathbb{S}_+^K$ with probability one [4]. It is therefore also a sufficient condition for the assumption to be held as $\mathcal{S} \subseteq \mathbb{S}_+^K$.

Problem (6) is difficult to solve for two reasons. First, the constraint set $\mathcal{S}$ is too general to tackle. Second, even if $\mathcal{S}$ possesses a nice property such as convexity, the objective function is still non-convex. Instead of trying to find the global minimizer, which appears to be too ambitious for the reasons pointed out above, we aim at devising efficient algorithms that are capable of finding a stationary point of (6). We rely on the MM framework to derive the algorithms, which is briefly stated next for completeness.

A. The Majorization-Minimization Algorithm

For a general optimization problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & h(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{7}$$

where $\mathcal{X}$ is a closed convex set, the MM algorithm finds a stationary point of (7) by successively solving a sequence of simpler optimization problems. The iterative algorithm starts at some arbitrary feasible initial point $\mathbf{x}_0$, and at the $(t+1)$-th iteration the update of $\mathbf{x}$ is given by

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x}|\mathbf{x}_t), \tag{8}$$

with the surrogate function $g(\mathbf{x}|\mathbf{x}_t)$ satisfying the following assumptions:

$$\begin{aligned} h(\mathbf{x}_t) &= g(\mathbf{x}_t|\mathbf{x}_t), \ \forall \mathbf{x}_t \in \mathcal{X} \\ h(\mathbf{x}) &\le g(\mathbf{x}|\mathbf{x}_t), \ \forall \mathbf{x}, \mathbf{x}_t \in \mathcal{X} \\ h'(\mathbf{x}_t;\mathbf{d}) &= g'(\mathbf{x}_t;\mathbf{d}|\mathbf{x}_t), \ \forall \mathbf{x}_t + \mathbf{d} \in \mathcal{X}, \end{aligned} \tag{9}$$

where $h'(\mathbf{x};\mathbf{d})$ stands for the directional derivative of $h(\cdot)$ at $\mathbf{x}$ along the direction $\mathbf{d}$, and $g(\mathbf{x}|\mathbf{x}_t)$ is continuous in both $\mathbf{x}$ and $\mathbf{x}_t$.

It is proved in [26] that any limit point of the sequence $\{\mathbf{x}_t\}$ generated by the MM algorithm is a stationary point of problem (7). If it is further assumed that the initial level set $\{\mathbf{x}|h(\mathbf{x}) \le h(\mathbf{x}_0)\}$ is compact, then a stronger statement, as follows, can be made:

$$\lim_{t\to+\infty} d(\mathbf{x}_t, \mathcal{X}^\star) = 0,$$

where $\mathcal{X}^\star$ stands for the set of all stationary points of (7), and $d(\mathbf{x}_t, \mathcal{X}^\star)$ is the distance between point $\mathbf{x}_t$ and the set $\mathcal{X}^\star$, defined as

$$d(\mathbf{x}_t, \mathcal{X}^\star) = \inf_{\mathbf{x}\in\mathcal{X}^\star} \| \mathbf{x}_t - \mathbf{x} \|_2. \tag{10}$$

The idea of majorizing $h(\mathbf{x})$ by a surrogate function can also be applied blockwise. Specifically, $\mathbf{x}$ is partitioned into m blocks as $\mathbf{x} = \left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right)$, where each n_i-dimensional block $\mathbf{x}^{(i)} \in \mathcal{X}_i$ and $\mathcal{X} = \prod_{i=1}^m \mathcal{X}_i$.

At the $(t+1)$-th iteration, $\mathbf{x}^{(i)}$ is updated by solving the following problem:

$$\begin{aligned} \underset{\mathbf{x}^{(i)}}{\text{minimize}} \quad & g_i\left(\mathbf{x}^{(i)}|\mathbf{x}_t\right) \\ \text{subject to} \quad & \mathbf{x}^{(i)} \in \mathcal{X}_i \end{aligned} \tag{11}$$

with $i = (t \bmod m) + 1$ and the continuous surrogate function $g_i\left(\mathbf{x}^{(i)}|\mathbf{x}_t\right)$ satisfying the following properties:

$$\begin{aligned} h(\mathbf{x}_t) &= g_i\left(\mathbf{x}_t^{(i)}|\mathbf{x}_t\right), \\ h\left(\mathbf{x}_t^{(1)}, \ldots, \mathbf{x}^{(i)}, \ldots, \mathbf{x}_t^{(m)}\right) &\le g_i\left(\mathbf{x}^{(i)}|\mathbf{x}_t\right) \ \forall \mathbf{x}^{(i)} \in \mathcal{X}_i, \\ h'\left(\mathbf{x}_t;\mathbf{d}_i^0\right) &= g_i'\left(\mathbf{x}_t^{(i)};\mathbf{d}_i|\mathbf{x}_t\right) \\ & \qquad \forall \mathbf{x}_t^{(i)} + \mathbf{d}_i \in \mathcal{X}_i, \\ \mathbf{d}_i^0 &\triangleq (\mathbf{0};\ldots;\mathbf{d}_i;\ldots;\mathbf{0}). \end{aligned}$$

In short, at each iteration, the block MM applies the ordinary MM algorithm to one block while keeping the value of the other blocks fixed. The blocks are updated in cyclic order.

In the rest of this paper, we are going to derive the specific form of the surrogate function $g(\mathbf{R}|\mathbf{R}_t)$ based on a detailed characterization of various kinds of $\mathcal{S}$. In addition, we are going to show how the algorithm can be particularized at a lower computational cost with a finer structure of $\mathcal{S}$ available. Before moving to the algorithmic part, we first compare our formulation with several related works in the literature.

B. Related Works

In [14], the authors derived a minimax robust covariance estimator assuming that $f(\mathbf{x})$ is a corrupted Gaussian distribution with noise that belongs to the ε-contamination class and the Kolmogorov class. The estimator is defined as the solution of a constrained optimization problem similar to (6), but with a different cost function. Apart from the distinction that the family of distributions we consider is the set of elliptical distributions, the focus of our work, which completely differs from [14], is on developing efficient numerical algorithms for different types of structural constraint set $\mathcal{S}$.

Two other closely related works are [20] and [21]. In [20], the authors have investigated a special case of (6), where $\mathcal{S}$ is the set of all positive semidefinite matrices with group symmetry structure. It has been shown that both $L(\mathbf{R})$ and the group symmetry constraint are geodesically convex, therefore any local minimizer of (6) is global. Several examples, including the circulant and persymmetry structure, have been proven to be a special case of the group symmetry constraint. A numerical algorithm has also been provided that decreases the cost function monotonically. Our work includes the group symmetry structure as a special case since the constraint is linear, and provides an alternative algorithm to solve the problem.

In [21], the authors have considered imposing convex constraint on Tyler's estimator. A generalized method of moment type estimator based on semidefinite relaxation defined as the solution of the following problem:

$$\begin{aligned} \underset{\mathbf{R}\in\mathcal{S},d_i}{\text{minimize}} \quad & \left\| \mathbf{R} - \frac{1}{N}\sum_{i=1}^{N} d_i \mathbf{x}_i\mathbf{x}_i^H \right\| \\ \text{subject to} \quad & \mathbf{R} \succeq \frac{1}{K} d_i \mathbf{x}_i\mathbf{x}_i^H,\ \forall i = 1,\ldots,N, \\ & d_i > 0,\ \forall i = 1,\ldots,N, \end{aligned} \tag{12}$$

was proposed and proved to be asymptotically consistent. Nevertheless, the number of constraints grows linearly in N and as it was pointed out in the paper, the algorithm becomes computationally demanding either when the problem dimension K or the number of samples N is large. On the contrary, our algorithm based on formulation (6) is less affected by the number of samples N and is therefore more computationally tractable.

III. Tyler's Estimator With Convex Structural Constraint

In this section, we are going to derive a general algorithm for problem (6) with $\mathcal{S}$ being a closed convex subset of $\mathbb{S}_+^K$, which enjoys a wide range of applications. For instance, the Toeplitz structure can be imposed on the covariance matrix of the received signal in DOA problems. Banding is also considered as a way of regularizing a covariance matrix whose entries decay fast as they get far away from the main diagonal.

Algorithm 1: Robust covariance estimation under convex structure.

1: Set $t = 0$, initialize $\mathbf{R}_t$ to be any positive definite matrix.
2: **repeat**
3: Compute $\mathbf{M}_t = \frac{K}{N}\sum_{i=1}^{N} \frac{\mathbf{x}_i\mathbf{x}_i^H}{\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i}$.
4: Update $\mathbf{R}_{t+1}$ as

$$\widetilde{\mathbf{R}}_{t+1} = \arg\min_{\mathbf{R}\in\mathcal{S}} \operatorname{Tr}\left(\mathbf{R}_t^{-1}\mathbf{R}\right) + \operatorname{Tr}\left(\mathbf{M}_t\mathbf{R}^{-1}\right) \tag{18}$$

$$\mathbf{R}_{t+1} = \widetilde{\mathbf{R}}_{t+1}/\operatorname{Tr}\left(\widetilde{\mathbf{R}}_{t+1}\right). \tag{19}$$

5: $t \leftarrow t + 1$.
6: **until** Some convergence criterion is met

Since $\mathcal{S}$ is closed and convex, constructing a convex surrogate function $g(\mathbf{R}|\mathbf{R}_t)$ for $L(\mathbf{R})$ turns out to be a natural idea since then $\mathbf{R}_{t+1}$ can be found via

$$\mathbf{R}_{t+1} = \arg\min_{\mathbf{R}\in\mathcal{S}}\ g(\mathbf{R}|\mathbf{R}_t), \tag{13}$$

which is a convex problem.

Proposition 1: At any $\mathbf{R}_t \succ \mathbf{0}$, the objective function $L(\mathbf{R})$ can be upperbounded by the convex surrogate function

$$g(\mathbf{R}|\mathbf{R}_t) = \operatorname{Tr}\left(\mathbf{R}_t^{-1}\mathbf{R}\right) + \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i}{\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i} + \text{const.} \tag{14}$$

with equality achieved at $\mathbf{R}_t$. ■

Proof: Since log det$(\cdot)$ is concave, log det$(\mathbf{R})$ can be upperbounded by its first order Taylor expansion at $\mathbf{R}_t$:

$$\log\det(\mathbf{R}) \leq \log\det(\mathbf{R}_t) + \operatorname{Tr}\left(\mathbf{R}_t^{-1}\mathbf{R}\right) - K \tag{15}$$

with equality achieved at $\mathbf{R}_t$.

Also, by the concavity of the $\log(\cdot)$ function we have

$$\log(x) \leq \log a + \frac{x}{a} - 1,\ \forall a > 0, \tag{16}$$

which leads to the bound

$$\log\left(\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i\right) \leq \frac{\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i}{\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i} + \log\left(\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i\right) - 1$$

with equality achieved at $\mathbf{R} = \mathbf{R}_t$. ■

The variable $\mathbf{R}$ then can be updated as (13) with a surrogate function (14).

By the convergence result of the MM algorithm, it can be concluded that every limit point of the sequence $\{\mathbf{R}_t\}$ is a stationary point of problem (6). Note that for all of the structural constraints that we are going to consider in this work, the set $\mathcal{S}$ possesses the property that

$$\mathbf{R} \in \mathcal{S} \text{ iff } r\mathbf{R} \in \mathcal{S},\ \forall r \geq 0. \tag{17}$$

In other words, $\mathcal{S}$ is a cone. Since the cost function $L(\mathbf{R})$ is scale-invariant in the sense that $L(\mathbf{R}) = L(r\mathbf{R})$, we can add a trace normalization step after the update of $\mathbf{R}_t$ without affecting the value of the objective function. The algorithm for a general convex structural set is summarized in Algorithm 1.

Proposition 2: If the set $\mathcal{S}$ satisfies (17), then the sequence $\{\mathbf{R}_t\}$ generated by Algorithm 1 satisfies

$$\lim_{t\to\infty} d(\mathbf{R}_t, \mathcal{S}^\star) = 0, \tag{20}$$

where $\mathcal{S}^\star$ is the set of stationary points of problem (6).

Proof: Since the objective function $L(\mathbf{R})$ is scale-invariant, and the constraint set satisfies (17), solving (6) is equivalent to solving

$$\begin{aligned} &\underset{\mathbf{R}\in\mathcal{S}}{\text{minimize}} && \log\det(\mathbf{R}) + \frac{K}{N}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i\right) \\ &\text{subject to} && \operatorname{Tr}(\mathbf{R}) = 1. \end{aligned}$$

The conclusion follows by a similar argument to Proposition 17 in [27]. ■

A. General Linear Structure

In this subsection we further assume that the set $\mathcal{S}$ is the intersection of $\mathbb{S}_+^K$ and an affine set $\mathcal{A}$. The following lemma shows that in this case, the update of $\mathbf{R}$ (Eq. (18)) can be recast as a semidefinite programming (SDP).

Lemma 3: Problem (18) is equivalent to

$$\begin{aligned} &\underset{\mathbf{S},\mathbf{R}\in\mathcal{S}}{\text{minimize}} && \operatorname{Tr}\left(\mathbf{R}_t^{-1}\mathbf{R}\right) + \operatorname{Tr}(\mathbf{M}_t\mathbf{S}) \\ &\text{subject to} && \begin{bmatrix}\mathbf{S} & \mathbf{I}\\ \mathbf{I} & \mathbf{R}\end{bmatrix} \succeq \mathbf{0}, \end{aligned} \tag{21}$$

in the sense that if $(\mathbf{S}^\star, \mathbf{R}^\star)$ solves (21), then $\mathbf{R}^\star$ solves (18).

Proof: Problem (18) can be written equivalently as

$$\begin{aligned} &\underset{\mathbf{S},\mathbf{R}\in\mathcal{S}}{\text{minimize}} && \operatorname{Tr}\left(\mathbf{R}_t^{-1}\mathbf{R}\right) + \operatorname{Tr}(\mathbf{M}_t\mathbf{S}) \\ &\text{subject to} && \mathbf{S} = \mathbf{R}^{-1}. \end{aligned}$$

Now we relax the constraint $\mathbf{S} = \mathbf{R}^{-1}$ as $\mathbf{S} \succeq \mathbf{R}^{-1}$. By the Schur complement lemma for a positive semidefinite matrix, if $\mathbf{R} \succ \mathbf{0}$, then $\mathbf{S} \succeq \mathbf{R}^{-1}$ is equivalent to

$$\begin{bmatrix}\mathbf{S} & \mathbf{I}\\ \mathbf{I} & \mathbf{R}\end{bmatrix} \succeq \mathbf{0}.$$

Therefore (21) is a convex relaxation of (18).

The relaxation is tight since $\operatorname{Tr}(\mathbf{M}_t\mathbf{S}) \geq \operatorname{Tr}\left(\mathbf{M}_t\mathbf{R}^{-1}\right)$ if $\mathbf{M}_t \succeq \mathbf{0}$ and $\mathbf{S} \succeq \mathbf{R}^{-1}$. ■

Lemma 3 reveals that for linear structural constraint, Algorithm 1 can be particularized as solving a sequence of SDPs.

An application is the case that $\mathbf{R}$ can be parametrized as

$$\mathbf{R} = \sum_{j=1}^{L} a_j\mathbf{B}_j \tag{22}$$

with $a_j \in \mathbb{C}$ being the variable and $\mathbf{B}_j \in \mathbb{C}^{K\times K}$ being the corresponding given basis matrix, and $\mathbf{R}$ is constrained to be in $\mathbb{S}_+^K$. Using expression (22), the minimization problem (21) can be simplified as

$$\begin{aligned} &\underset{\mathbf{S},\{a_j\}}{\text{minimize}} && \sum_{j=1}^{L} a_j \operatorname{Tr}\left(\mathbf{R}_t^{-1}\mathbf{B}_j\right) + \operatorname{Tr}(\mathbf{M}_t\mathbf{S}) \\ &\text{subject to} && \begin{bmatrix}\mathbf{S} & \mathbf{I}\\ \mathbf{I} & \sum_{j=1}^{L} a_j\mathbf{B}_j\end{bmatrix} \succeq \mathbf{0}. \end{aligned} \tag{23}$$

IV. Tyler's Estimator With Special Convex Structures

Having introduced the general algorithm framework for a convex structure in the previous section, we are going to discuss in detail some convex structures that arise frequently in signal processing related fields, and show that by exploiting the problem structure the algorithm can be particularized with a significant reduction in the computational load.

A. Sum of Rank-One Matrices Structure

The structure set $\mathcal{S}$ that we study in this part is

$$\mathcal{S} = \left\{\mathbf{R} \middle| \mathbf{R} = \sum_{j=1}^{L} p_j\mathbf{a}_j\mathbf{a}_j^H,\ p_j \geq 0\right\}, \tag{24}$$

where the $\mathbf{a}_j$'s are known vectors in $\mathbb{C}^K$. The matrix $\mathbf{R}$ can be interpreted as a weighted sum of given matrices $\mathbf{a}_j\mathbf{a}_j^H$.

As an example application where structure (24) appears, consider the following signal model

$$\mathbf{x} = \mathbf{A}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{25}$$

where $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_L]$ is some given matrix. Assuming that the signal $\boldsymbol{\beta}$ and noise $\boldsymbol{\varepsilon}$ are zero-mean random variables and any two elements of them are uncorrelated, then the covariance matrix of $\mathbf{x}$ takes the form

$$\operatorname{Cov}(\mathbf{x}) = \sum_{j=1}^{L} p_j\mathbf{a}_j\mathbf{a}_j^H + \boldsymbol{\Sigma}, \tag{26}$$

where $p_j = \operatorname{Var}(\beta_j)$ is the signal variance and $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_K)$ is the noise covariance matrix.

Define $\mathbf{p} = [p_1, \ldots p_L]^H$ and $\mathbf{P} = \operatorname{diag}(\mathbf{p})$, then $\mathbf{R}$ can be written compactly as $\mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \boldsymbol{\Sigma}$. Further define

$$\begin{aligned} \widetilde{\mathbf{P}} &= \operatorname{diag}(p_1, \ldots, p_L, \sigma_1, \ldots, \sigma_K) \\ \widetilde{\mathbf{A}} &= [\mathbf{A}, \mathbf{I}] \end{aligned} \tag{27}$$

then $\mathbf{R} = \widetilde{\mathbf{A}}\widetilde{\mathbf{P}}\widetilde{\mathbf{A}}^H$. Therefore, without loss of generality, we can focus on the expression $\mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H$, assuming that every K columns of $\mathbf{A}$ are linearly independent and $L > K$.

Recall that the problem to be solved takes the form

$$\begin{aligned} &\underset{\mathbf{R},\mathbf{P}\succeq\mathbf{0}}{\text{minimize}} && \log\det(\mathbf{R}) + \frac{K}{N}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i\right) \\ &\text{subject to} && \mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H. \end{aligned} \tag{28}$$

Since $\mathbf{R}$ is linear in the p_j's, Algorithm 1 can be applied. In the following, we are going to provide a more efficient algorithm by substituting $\mathbf{R} = \mathbf{APA}^H$ into the objective function $L(\mathbf{R})$ and applying the MM procedure with $\mathbf{P}$ being the variable.

Proposition 4: At any $\mathbf{P}_t \succ \mathbf{0}$, the objective function

$$L(\mathbf{P}) = \log\det\left(\mathbf{APA}^H\right) + \frac{K}{N}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^H\left(\mathbf{APA}^H\right)^{-1}\mathbf{x}_i\right) \tag{29}$$

can be upperbounded by the surrogate function

$$g(\mathbf{P}|\mathbf{P}_t) = \mathbf{w}_t^H\mathbf{p} + \mathbf{d}_t^H\mathbf{p}^{-1} + \text{const.} \tag{30}$$

with equality achieved at $\mathbf{P} = \mathbf{P}_t$, where $\mathbf{p}^{-1}$ stands for the element-wise inverse of $\mathbf{p}$, and

$$\begin{aligned}
\mathbf{R}_t &= \mathbf{AP}_t\mathbf{A}^H \\
\mathbf{M}_t &= \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^H}{\mathbf{x}_i^T\mathbf{R}_t^{-1}\mathbf{x}_i} \\
\mathbf{w}_t &= \operatorname{diag}\left(\mathbf{A}^H\mathbf{R}_t^{-1}\mathbf{A}\right) \\
\mathbf{d}_t &= \operatorname{diag}\left(\mathbf{P}_t\mathbf{A}^H\mathbf{R}_t^{-1}\mathbf{M}_t\mathbf{R}_t^{-1}\mathbf{AP}_t\right).
\end{aligned} \tag{31}$$

Proof: First, observe that inequalities (15) and (16) imply that

$$L(\mathbf{P}) \le \mathbf{w}_t^H\mathbf{p} + \operatorname{Tr}\left(\mathbf{M}_t\mathbf{R}^{-1}\right) + \text{const.} \tag{32}$$

with equality achieved at $\mathbf{P} = \mathbf{P}_t$.

Assume that $\mathbf{P} \succ \mathbf{0}$, from the identity

$$\begin{aligned}
\mathbf{S} &= \begin{bmatrix} \mathbf{R}_t^{-1}\mathbf{AP}_t\mathbf{P}^{-1}\mathbf{P}_t\mathbf{A}^H\mathbf{R}_t^{-1} & \mathbf{I} \\ \mathbf{I} & \mathbf{APA}^H \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_t^{-1}\mathbf{AP}_t\mathbf{P}^{-1/2} \\ \mathbf{AP}^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1/2}\mathbf{P}_t\mathbf{A}^H\mathbf{R}_t^{-1} & \mathbf{P}^{1/2}\mathbf{A}^H \end{bmatrix},
\end{aligned}$$

we know that $\mathbf{S} \succeq \mathbf{0}$. By the Schur complement, $\mathbf{S} \succeq \mathbf{0}$ is equivalent to

$$\mathbf{R}_t^{-1}\mathbf{AP}_t\mathbf{P}^{-1}\mathbf{P}_t\mathbf{A}^H\mathbf{R}_t^{-1} \succeq \left(\mathbf{APA}^H\right)^{-1}. \tag{33}$$

Since $\mathbf{M}_t \succeq \mathbf{0}$, we have

$$\operatorname{Tr}\left(\mathbf{M}_t\mathbf{R}^{-1}\right) \le \operatorname{Tr}\left(\mathbf{M}_t\mathbf{R}_t^{-1}\mathbf{AP}_t\mathbf{P}^{-1}\mathbf{P}_t\mathbf{A}^H\mathbf{R}_t^{-1}\right) \tag{34}$$

with equality achieved at $\mathbf{P} = \mathbf{P}_t$.

Since $\mathbf{R} \succ \mathbf{0}$, the left hand side of (34) is finite. Therefore (34) is also valid for $\mathbf{P} \succeq \mathbf{0}$. Substituting (34) into (32) yields the surrogate function (30). ∎

Note that both $\mathbf{w}_t$ and $\mathbf{d}_t$ are real-valued, the update of $\mathbf{P}$ then can be found in closed-form as

$$(p_j)_{t+1} = \sqrt{(d_j)_t/(w_j)_t}. \tag{35}$$

The algorithm is summarized in Algorithm 2.

Compared to Algorithm 1, in which the minimization problem (13) has no closed-form solution and typically requires an iterative algorithm, the new algorithm only requires a single loop iteration in $\mathbf{p}$ and is expected to converge faster.

Algorithm 2: Robust covariance estimation under sum of rank-one matrices structure.

Set $t = 0$, initialize $\mathbf{p}_t$ to be any positive vector.
repeat
  $\widetilde{\mathbf{R}}_t = \mathbf{AP}_t\mathbf{A}^H, \mathbf{R}_t = \widetilde{\mathbf{R}}_t/\operatorname{Tr}\left(\widetilde{\mathbf{R}}_t\right)$.
  Compute $\mathbf{M}_t, \mathbf{w}_t, \mathbf{d}_t$ with (31)
  $(p_j)_{t+1} = \sqrt{(d_j)_t/(w_j)_t}$
  $t \leftarrow t + 1$.
until some convergence criterion is met

B. Toeplitz Structure

Consider the constraint set being the class of real-valued positive semidefinite Toeplitz matrices T_K. If $\mathbf{R} \in T_K$, then it can be determined by its first row $[r_0, \ldots, r_{K-1}]$.[2]

In this subsection, we are going to show that based on the technique of circulant embedding, Algorithm 2 can be adopted to solve the Toeplitz structure constrained problem at a lower cost than applying the sequential SDP algorithm (Algorithm 1).

The idea of embedding a Toeplitz matrix as the upper-left part of a larger circulant matrix has been discussed in [5], [7], [28]. It was proved in [6] that any positive definite Toeplitz matrix $\mathbf{R}$ of size $K \times K$ can be embedded in a positive definite circulant matrix $\mathbf{C}$ of larger size $L \times L$ parametrized by its first row of the form

$$[r_0, r_1, \ldots, r_{K-1}, *, \ldots, *, r_{K-1}, \ldots, r_1],$$

where $*$ denotes some real number. $\mathbf{R}$ then can be written as

$$\mathbf{R} = \left[\mathbf{I}_K\ \mathbf{0}\right]\mathbf{C}\left[\mathbf{I}_K\ \mathbf{0}\right]^T. \tag{36}$$

Clearly, for any fixed L, if $\mathbf{C}$ is symmetric positive semidefinite, so is $\mathbf{R}$. However, the statement is false the other way around. In other words, the set

$$T_K^L \triangleq \left\{\mathbf{R}|\mathbf{R} = \left[\mathbf{I}_K\ \mathbf{0}\right]\mathbf{C}\left[\mathbf{I}_K\ \mathbf{0}\right]^T, \mathbf{C} \in C_L\right\}, \tag{37}$$

where C_L denotes the set of real-valued positive semidefinite circulant matrices of size $L \times L$, is a subset of T_K.

Instead of T_K, we restrict the feasible set to be T_K^L with $L \ge 2K - 1$. Since a circulant matrix can be diagonalized by the Fourier matrix, if $\mathbf{R} \in T_K^L$ then it can be written as

$$\mathbf{R} = \mathbf{A}\operatorname{diag}(p_0, \ldots, p_{L-1})\mathbf{A}^H, \tag{38}$$

where

$$\mathbf{A} = \left[\mathbf{I}_K\ \mathbf{0}\right]\mathbf{F}_L, \tag{39}$$

with $\mathbf{F}_L$ being the normalized Fourier transform matrix of size $L \times L$ and $p_j = p_{L-j}$, $\forall j = 1, \ldots, L-1$.[3]

[2]Following the convention, the indices for the Toeplitz structure start from 0.

[3]The algorithm is developed for estimating a real-valued Toeplitz matrix, it can be adapted to estimating a complex-valued one by removing the constraint $p_j = p_{L-j}$.

Algorithm 3: Robust covariance estimation under the real-valued Toeplitz structure (Circulant Embedding).

1: Set L to be an integer such that $L \geq 2K - 1$.
2: Construct matrix $\mathbf{A} = \left[\mathbf{I}_K \ \mathbf{0}\right] \mathbf{F}_L$
3: Call Algorithm 2 ($\mathbf{p}_t$ is initialized satisfying $p_j = p_{L-j}, \forall j = 1, \ldots, L-1$).

The robust covariance estimation problem over the restricted set of Toeplitz matrices T_K^L then takes the form

$$\begin{aligned} \underset{\mathbf{R}, \mathbf{P} \succeq \mathbf{0}}{\text{minimize}} \quad & \log\det(\mathbf{R}) + \frac{K}{N} \sum_{i=1}^{N} \log\left(\mathbf{x}_i^H \mathbf{R}^{-1} \mathbf{x}_i\right) \\ \text{subject to} \quad & \mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H \\ & p_j = p_{L-j}, \forall j = 1, \ldots, L-1, \end{aligned} \tag{40}$$

which is the same as (28) except that the last equality constraint on the p_j's.

By Proposition 4, the inner minimization problem takes the form

$$\begin{aligned} \underset{\mathbf{p} \geq \mathbf{0}}{\text{minimize}} \quad & \mathbf{w}_t^T \mathbf{p} + \mathbf{d}_t^T \mathbf{p}^{-1} \\ \text{subject to} \quad & p_j = p_{L-j}, \forall j = 1, \ldots, L-1. \end{aligned} \tag{41}$$

Note that by the property of the Fourier transform matrix, we have $\mathbf{a}_j = \overline{\mathbf{a}}_{L-j}, \forall j = 1, \ldots, L-1$, where the upper bar stands for element-wise complex conjugate. As a result, if $(p_j)_t = (p_{L-j})_t$, for $j = 1, \ldots, L-1$, then

$$\begin{aligned} (w_j)_t &= (w_{L-j})_t \\ (d_j)_t &= (d_{L-j})_t, \end{aligned} \tag{42}$$

which implies that the constraint $p_j = p_{L-j}$ will be satisfied automatically.

The algorithm for the Toeplitz structure based on circulant embedding is summarized in Algorithm 3. Notice that Algorithm 3 can be generalized easily to noisy observations by the augmented representation (27).

C. Banded Toeplitz Structure

In addition to imposing the Toeplitz structure on a real-valued covariance matrix, in some applications we can further require that the Toeplitz matrix is k-banded, i.e., $r_j = 0$ if $j > k$. For example, the covariance matrix of a stationary moving average process of order k satisfies the above assumption. One may also consider banding the covariance matrix if it is known in prior that the correlation of x_t and $x_{t-\tau}$ decreases as τ increases.

Based on the circulant embedding technique introduced in the last subsection, the problem can be formulated as

$$\begin{aligned} \underset{\mathbf{R}, \mathbf{P} \succeq \mathbf{0}}{\text{minimize}} \quad & \log\det(\mathbf{R}) + \frac{K}{N} \sum_{i=1}^{N} \log\left(\mathbf{x}_i^H \mathbf{R}^{-1} \mathbf{x}_i\right) \\ \text{subject to} \quad & \mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H \\ & p_j = p_{L-j}, \forall j = 1, \ldots, L-1 \\ & r_j = 0, \forall j = k+1, \ldots, K-1. \end{aligned} \tag{43}$$

By Proposition 4, the inner minimization problem becomes

$$\begin{aligned} \underset{\mathbf{p} \geq \mathbf{0}}{\text{minimize}} \quad & \mathbf{w}_t^T \mathbf{p} + \mathbf{d}_t^T \mathbf{p}^{-1} \\ \text{subject to} \quad & p_j = p_{L-j}, \forall j = 1, \ldots, L-1 \\ & r_j = 0, \forall j = k+1, \ldots, K-1, \end{aligned} \tag{44}$$

which can be rewritten compactly as

$$\begin{aligned} \underset{\mathbf{p} \geq \mathbf{0}}{\text{minimize}} \quad & \mathbf{w}_t^T \mathbf{p} + \mathbf{d}_t^T \mathbf{p}^{-1} \\ \text{subject to} \quad & \left[\mathbf{0}_{(K-k-1) \times k+1} \ \mathbf{I}_{K-k-1}\right] \mathbf{A}\mathbf{p} = \mathbf{0} \\ & p_j = p_{L-j}, \forall j = 1, \ldots, L-1. \end{aligned} \tag{45}$$

Recall that $\mathbf{a}_j = \overline{\mathbf{a}}_{L-j}, \forall j = 1, \ldots, L-1$. For simplicity we assume that L is odd. The constraint $p_j = p_{L-j}$ implies that $p_j \mathbf{a}_j + p_{L-j} \overline{\mathbf{a}}_{L-j} = 2 p_j \mathrm{Re}\{\mathbf{a}_j\}$. Define real-valued quantities

$$\widetilde{\mathbf{A}} = \mathrm{Re}\left\{\left[\mathbf{a}_0, 2\mathbf{a}_1, \ldots, 2\mathbf{a}_{\frac{L-1}{2}}\right]\right\} \tag{46}$$

$$\widetilde{\mathbf{w}} = \left[w_0, 2w_1, \ldots, 2w_{\frac{L-1}{2}}\right] \tag{47}$$

$$\widetilde{\mathbf{d}} = \left[d_0, 2d_1, \ldots, 2d_{\frac{L-1}{2}}\right], \tag{48}$$

we have the equivalent problem

$$\begin{aligned} \underset{\widetilde{\mathbf{p}} \geq \mathbf{0}}{\text{minimize}} \quad & \widetilde{\mathbf{w}}_t^T \widetilde{\mathbf{p}} + \sum_{j=0}^{\frac{L-1}{2}} \tilde{d}_j / \tilde{p}_j \\ \text{subject to} \quad & \widetilde{\mathbf{A}} \widetilde{\mathbf{p}} = \mathbf{0}, \end{aligned} \tag{49}$$

where the variables $\widetilde{\mathbf{p}}$ and $\mathbf{p}$ are related by

$$\widetilde{\mathbf{p}} = \left[p_0, p_1, \ldots, p_{\frac{L-1}{2}}\right]. \tag{50}$$

Compared to (45), the equivalent problem has a lower computational cost as both the number of variables and constraints are reduced. Using the epigraph form, problem (49) can be cast as the following second-order-cone programming (SOCP)

$$\begin{aligned} \underset{\widetilde{\mathbf{p}}, \mathbf{t}}{\text{minimize}} \quad & \widetilde{\mathbf{w}}_t^T \widetilde{\mathbf{p}} + \sum_{j=0}^{\frac{L-1}{2}} d_j t_j \\ \text{subject to} \quad & \widetilde{\mathbf{A}} \widetilde{\mathbf{p}} = \mathbf{0}, \\ & \left\| \begin{bmatrix} 2 \\ \tilde{p}_j - t_j \end{bmatrix} \right\| \leq \tilde{p}_j + t_j, \forall j. \end{aligned} \tag{51}$$

The algorithm for the banded Toeplitz structure is summarized in Algorithm 4.

Algorithm 4: Robust covariance estimation under the real-valued Banded Toeplitz structure (Circulant Embedding).

Set L to be an integer such that $L \geq 2K-1$.
Construct matrix $\mathbf{A} = \begin{bmatrix} \mathbf{I}_K & \mathbf{0} \end{bmatrix} \mathbf{F}_L$ and $\widetilde{\mathbf{A}}$ with (46).
Set $t=0$, initialize $\mathbf{p}_t$ to be any positive vector.
repeat
  $\widetilde{\mathbf{R}}_t = \mathbf{A}\mathbf{P}_t\mathbf{A}^H, \mathbf{R}_t = \widetilde{\mathbf{R}}_t / \mathrm{Tr}\left(\widetilde{\mathbf{R}}_t\right)$.
  Compute $\mathbf{M}_t, \mathbf{w}_t, \mathbf{d}_t$ with (31).
  Compute $\widetilde{\mathbf{w}}$ and $\widetilde{\mathbf{d}}$ with (47) and (48), and update $\widetilde{\mathbf{p}}$ as the minimizer of (51).
  Compute $\mathbf{p}$ with (50), $\mathbf{p}_t \leftarrow \mathbf{p}$
  $t \leftarrow t+1$
until some convergence criterion is met

D. Convergence Analysis

We consider Algorithm 2, and the argument for Algorithms 3, and 4 would be similar.

As Proposition 4 is established under the condition $\mathbf{P}_t \succ \mathbf{0}$, the general convergence result of the MM algorithm cannot be applied here since the surrogate function $g(\mathbf{P}|\mathbf{P}_t)$ is required to upperbound $L(\mathbf{P}), \forall \mathbf{P}, \mathbf{P}_t \succeq \mathbf{0}$. Therefore, we consider the following ϵ-approximation of problem (28):

$$\begin{aligned} \underset{\mathbf{R}, \mathbf{p} \geq \mathbf{0}}{\text{minimize}} \quad & \log\det\left(\mathbf{R} + \epsilon\mathbf{A}\mathbf{A}^H\right)\} \\ & + \frac{K}{N}\sum_{i=1}^{N} \log\left(\mathbf{x}_i^H\left(\mathbf{R} + \epsilon\mathbf{A}\mathbf{A}^H\right)^{-1}\mathbf{x}_i\right) \\ \text{subject to} \quad & \mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H \end{aligned} \tag{52}$$

with $\varepsilon > 0$, where the upperbound derived in Proposition 4 now can be applied to $\widetilde{\mathbf{P}} \triangleq \mathbf{P} + \epsilon\mathbf{I} \succ \mathbf{0}$. Algorithm 2 can be easily modified to solve problem (52), and under Assumption 1, the limit point of the sequence $\{\mathbf{p}_t^{\epsilon}\}$ generated by Algorithm 2 converges to the set of stationary points of (52).

That is, if $(\mathbf{p}^{\epsilon})^{\star}$ is a limit point of $\{\mathbf{p}_t^{\epsilon}\}$, then

$$\nabla L^{\epsilon}\left((\mathbf{p}^{\epsilon})^{\star}\right)^T \mathbf{d} \geq \mathbf{0} \tag{53}$$

for any feasible direction $\mathbf{d}$, where $\nabla L^{\epsilon}\left((\mathbf{p}^{\epsilon})^{\star}\right)$ is the gradient of the objective function $L^{\epsilon}(\mathbf{p})$ at $(\mathbf{p}^{\epsilon})^{\star}$.

Proposition 5: Under Assumption 1, let ϵ_k be a positive sequence with $\lim_{k\to+\infty}\epsilon_k = 0$, then any limit point $\mathbf{p}^{\star}$ of the sequence $\{(\mathbf{p}^{\epsilon_k})^{\star}\}$ is a stationary point of problem (28).

Proof: The conclusion follows from the continuity of $\nabla L^{\epsilon}\left((\mathbf{p}^{\epsilon})^{\star}\right)$ in $(\mathbf{p}^{\epsilon})^{\star}$ and ϵ under Assumption 2. ■

In practice, as ϵ can be chosen as an arbitrarily small number, directly applying Algorithms 2, 3 and 4 or adapting them to solving the ϵ-approximation problem would be virtually the same.

V. Tyler's Estimator With Non-Convex Structure

In the previous sections we have proposed algorithms for Tyler's estimator with a general convex structural constraint and discussed in detail some special cases. For the non-convex structure, the problem is more difficult to handle. In this section, we are going to introduce two popular non-convex structures that are tractable by applying the MM algorithm, namely the spiked covariance structure and the Kronecker structure.

A. The Spiked Covariance Structure

The term "spiked covariance" was introduced in [29] and refers to the covariance matrix model

$$\mathbf{R} = \sum_{j=1}^{L} p_j \mathbf{a}_j \mathbf{a}_j^H + \sigma^2\mathbf{I}, \tag{54}$$

where L is some integer that is less than K, and the $\mathbf{a}_j$'s are unknown orthonormal basis vectors. Note that although (54) and (26) share similar form, they differ from each other essentially since the $\mathbf{a}_j$'s in (26) are known and are not necessarily orthogonal. The model is directly related to principle component analysis, subspace estimation, and also plays an important role in sensor array applications [4], [9]. This model, referred to as factor model, is also very popular in financial time series analysis [30].

The constrained optimization problem is formulated as

$$\begin{aligned} \underset{\mathbf{R}, \mathbf{a}_j, \mathbf{p} \geq \mathbf{0}, \sigma}{\text{minimize}} \quad & \log\det(\mathbf{R}) + \frac{K}{N}\sum_{i=1}^{N}\log\left(\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i\right) \\ \text{subject to} \quad & \mathbf{R} = \sum_{j=1}^{L} p_j\mathbf{a}_j\mathbf{a}_j^H + \sigma^2\mathbf{I}, \\ & \mathbf{A}^H\mathbf{A} = \mathbf{I}, \end{aligned} \tag{55}$$

where $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_L]$.

Applying the upperbound (16) for the second term in the objective function yields the following inner minimization problem:

$$\begin{aligned} \underset{\mathbf{R}, \mathbf{a}_j, \mathbf{p} \geq \mathbf{0}, \sigma}{\text{minimize}} \quad & \log\det(\mathbf{R}) + \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i^H\mathbf{R}^{-1}\mathbf{x}_i}{\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i} \\ \text{subject to} \quad & \mathbf{R} = \sum_{j=1}^{L} p_j\mathbf{a}_j\mathbf{a}_j^H + \sigma^2\mathbf{I}, \\ & \mathbf{A}^H\mathbf{A} = \mathbf{I}. \end{aligned} \tag{56}$$

Although the problem is non-convex, a global minimizer can be found in closed-form as

$$\begin{aligned} (\sigma^{\star})^2 &= \frac{1}{K-L}\sum_{j=L+1}^{K}\lambda_j \\ p_j^{\star} &= \lambda_j - (\sigma^{\star})^2 \\ \mathbf{a}_j^{\star} &= \mathbf{u}_j, \end{aligned} \tag{57}$$

where $\lambda_1 \geq \cdots \geq \lambda_K$ are the sorted eigenvalues of matrix $\frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^H}{\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i}$ and the $\mathbf{u}_j$'s are the associated eigenvectors [31]. The algorithm for the spiked covariance structure is summarized in Algorithm 5.

Algorithm 5: Robust covariance estimation under the spiked covariance structure.

1: Initialize $\mathbf{R}_0$ to be an arbitrary feasible positive definite matrix.
2: **repeat**
3: $\mathbf{M}_t = \frac{K}{N}\sum_{i=1}^{N}\frac{\mathbf{x}_i\mathbf{x}_i^H}{\mathbf{x}_i^H\mathbf{R}_t^{-1}\mathbf{x}_i}$.
4: Eigendecompose $\mathbf{M}_t$ as $\mathbf{M}_t = \sum_{j=1}^{K}\lambda_j\mathbf{u}_j\mathbf{u}_j^H$, where $\lambda_1 \geq \cdots \geq \lambda_K$.
5: Compute $\sigma^\star, p_j^\star, \mathbf{a}_j^\star$ with (57)
6: $\widetilde{\mathbf{R}}_{t+1} = \sum_{j=1}^{L} p_j^\star\mathbf{a}_j^\star\left(\mathbf{a}_j^\star\right)^H + (\sigma^\star)^2\mathbf{I}$.
7: $\mathbf{R}_{t+1} = \widetilde{\mathbf{R}}_{t+1}/\mathrm{Tr}\left(\widetilde{\mathbf{R}}_{t+1}\right)$.
8: $t \leftarrow t+1$.
9: **until** Some convergence criterion is met.

As the feasible set is not convex, the convergence statement of the MM algorithm in [26] needs to be modified as follows.

Proposition 6: Any limit point $\mathbf{R}^\star$ generated by the algorithm satisfies

$$\mathrm{Tr}\left(\nabla L(\mathbf{R}^\star)^H\mathbf{R}\right) \geq 0,\ \forall\mathbf{R} \in \mathcal{T}_{\mathcal{S}}\left(\mathbf{R}^\star\right),$$

where $\mathcal{T}_{\mathcal{S}}\left(\mathbf{R}^\star\right)$ stands for the tangent cone of $\mathcal{S}$ at $\mathbf{R}^\star$.

Proof: The result follows by combining the standard convergence proof of the MM algorithm [26] and the necessity condition of $\mathbf{R}^\star$ being the global minimal of $g\left(\mathbf{R}|\mathbf{R}^\star\right)$ over an arbitrary set $\mathcal{S}$ (see Proposition 4.7.1 in [32]):

$$\mathrm{Tr}\left(\nabla g(\mathbf{R}^\star|\mathbf{R}^\star)^H\mathbf{R}\right) \geq 0,\ \forall\mathbf{R} \in \mathcal{T}_{\mathcal{S}}\left(\mathbf{R}^\star\right). \qquad \blacksquare$$

B. The Kronecker Structure

In this subsection we consider the covariance matrix that can be expressed as the Kronecker product of two matrices, i.e.,

$$\mathbf{R} = \mathbf{A} \otimes \mathbf{B}, \tag{58}$$

where $\mathbf{A} \in \mathbb{S}_+^p$ and $\mathbf{B} \in \mathbb{S}_+^q$.

Substituting $\mathbf{R} = \mathbf{A} \otimes \mathbf{B}$ into the objective function yields the equivalent problem:

$$\begin{aligned}\underset{\mathbf{A}\succeq\mathbf{0},\mathbf{B}\succeq\mathbf{0}}{\text{minimize}}\quad & \frac{pq}{N}\sum_{i=1}^{N}\log\mathrm{Tr}\left(\mathbf{A}^{-1}\mathbf{M}_i^H\mathbf{B}^{-1}\mathbf{M}_i\right)\\ & + q\log\det\left(\mathbf{A}\right) + p\log\det\left(\mathbf{B}\right)\end{aligned} \tag{59}$$

where $\mathbf{M}_i \in \mathbb{C}^{q\times p}$ and $\mathrm{vec}\left(\mathbf{M}_i\right) = \mathbf{x}_i$. Denote the objective function of (59) as $L\left(\mathbf{A},\mathbf{B}\right)$.

Note that although the objective function of the equivalent problem is still non-convex, the constraint set of the equivalent problem (59) becomes the Cartesian product of two convex sets, which is convex.

1) Gauss-Seidel: Since $L\left(\mathbf{R}\right)$ is scale-invariant, we can make the restriction that $\mathrm{Tr}\left(\mathbf{A}\right) = 1$ and $\mathrm{Tr}\left(\mathbf{B}\right) = 1$ and then problem (59) can be solved by updating $\mathbf{A}$ and $\mathbf{B}$ alternately.

Algorithm 6: Robust covariance estimation under the Kronecker structure (Gauss-Seidel).

1: Initialize $\mathbf{A}_0$ and $\mathbf{B}_0$ to be arbitrary positive definite matrices of size $p \times p$ and $q \times q$, respectively.
2: **repeat**
3: Update $\mathbf{A}$ with (62).
4: Update $\mathbf{B}$ with (63).
5: $t \leftarrow t+1$.
6: **until** Some convergence criterion is met.

Specifically, for fixed $\mathbf{B} = \mathbf{B}_t$, we need to solve the following problem:

$$\begin{aligned}\underset{\mathbf{A}\succeq\mathbf{0}}{\text{minimize}}\quad & \log\det\left(\mathbf{A}\right) + \frac{p}{N}\sum_{i=1}^{N}\log\mathrm{Tr}\left(\mathbf{A}^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)\\ \text{subject to}\quad & \mathrm{Tr}\left(\mathbf{A}\right) = 1.\end{aligned} \tag{60}$$

Setting the gradient of the objective function to zero yields the fixed-point equation

$$\mathbf{A} = \frac{p}{N}\sum_{i=1}^{N}\frac{\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i}{\mathrm{Tr}\left(\mathbf{A}^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}. \tag{61}$$

As objective function of (60) is essentially the same as the Tyler's cost function (5), an argument similar to Theorem 2.1 in [15] reveals that the solution to (61) is unique up to a positive scaling factor, and under Assumption 1, the iteration

$$\begin{aligned}\widetilde{\mathbf{A}} &= \frac{p}{N}\sum_{i=1}^{N}\frac{\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i}{\mathrm{Tr}\left(\mathbf{A}_r^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}\\ \mathbf{A}_{r+1} &= \widetilde{\mathbf{A}}/\mathrm{Tr}\left(\widetilde{\mathbf{A}}\right)\end{aligned} \tag{62}$$

converges to the unique global minimum of (60) as $r \to +\infty$.

Assign $\mathbf{A}_{t+1} = \lim_{r\to+\infty}\mathbf{A}_r$, similarly we have the fixed-point iteration for $\mathbf{B}$ as

$$\begin{aligned}\widetilde{\mathbf{B}} &= \frac{q}{N}\sum_{i=1}^{N}\frac{\mathbf{M}_i\mathbf{A}_{t+1}^{-1}\mathbf{M}_i^H}{\mathrm{Tr}\left(\mathbf{A}_{t+1}^{-1}\mathbf{M}_i^H\mathbf{B}_r^{-1}\mathbf{M}_i\right)}\\ \mathbf{B}_{r+1} &= \widetilde{\mathbf{B}}/\mathrm{Tr}\left(\widetilde{\mathbf{B}}\right),\end{aligned} \tag{63}$$

and $\mathbf{B}_{t+1} = \lim_{r\to+\infty}\mathbf{B}_r$.

Proposition 7: Under Assumption 1, every limit point, denoted by $(\mathbf{A}^\star,\mathbf{B}^\star)$, of the sequence $\{(\mathbf{A}_t,\mathbf{B}_t)\}$ generated by Algorithm 6 is a global minimizer of (59).

Proof: The convergence of block coordinate descent algorithm states that the pair $(\mathbf{A}^\star,\mathbf{B}^\star)$ is a stationary point of problem (59) (Proposition 2.7.1 in [33]). Moreover, it has been proven that the Tyler's cost function (5) is geodesically convex on $\mathbb{S}_{++}$ [22]. Lemma 3 in [23] then implies that $L\left(\mathbf{A}\otimes\mathbf{B}\right)$ is also geodesically convex. Finally, Corollary 3.1 in [34] implies that the stationary point $(\mathbf{A}^\star,\mathbf{B}^\star)$ is a global minimum since $L\left(\mathbf{A}\otimes\mathbf{B}\right)$ is continuously differentiable. $\blacksquare$

2) Block Majorization-Minimization: A stationary point of $L(\mathbf{A}, \mathbf{B})$ can also be found by block majorization-minimization algorithm (Block MM).

By Proposition 1, with the value of $\mathbf{B}_t$ fixed to be $\mathbf{B}_t$, a convex upperbound of $L(\mathbf{A}, \mathbf{B})$ on $\mathbb{S}_+^p$ at point $\mathbf{A}_t$ (ignoring a constant term and up to a scale factor of q) can be found as

$$g(\mathbf{A}|\mathbf{A}_t, \mathbf{B}_t) = \mathrm{Tr}\left(\mathbf{A}_t^{-1}\mathbf{A}\right) + \frac{p}{N}\sum_{i=1}^{N}\frac{\mathrm{Tr}\left(\mathbf{A}^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}{\mathrm{Tr}\left(\mathbf{A}_t^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}. \tag{64}$$

Lemma 8: Under Assumption 1, for any $\mathbf{A}_t, \mathbf{B}_t \succ \mathbf{0}$, the matrix

$$\mathbf{M}(\mathbf{A}_t, \mathbf{B}_t) = \frac{p}{N}\sum_{i=1}^{N}\frac{\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i}{\mathrm{Tr}\left(\mathbf{A}_t^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}$$

is nonsingular.

Proof: At $(\mathbf{A}_t, \mathbf{B}_t)$ (ignoring a constant term and up to a scale factor of q) the function $L(\mathbf{A}, \mathbf{B}_t)$ can be upperbounded by

$$\tilde{g}(\mathbf{A}|\mathbf{A}_t, \mathbf{B}_t) = \log\det(\mathbf{A}) + \mathrm{Tr}\left(\mathbf{A}^{-1}\mathbf{M}(\mathbf{A}_t, \mathbf{B}_t)\right). \tag{65}$$

If $\mathbf{M}(\mathbf{A}_t, \mathbf{B}_t)$ is singular, we can eigendecompose $\mathbf{M}(\mathbf{A}_t, \mathbf{B}_t)$ as $\mathbf{M}(\mathbf{A}_t, \mathbf{B}_t) = \mathbf{U}\mathrm{diag}(\lambda_1, \ldots, \lambda_p)\mathbf{U}^H$ with $\lambda_1 = 0$, and set $\mathbf{A}^{-1} = \mathbf{U}\mathrm{diag}(\sigma_1, \ldots, \sigma_p)\mathbf{U}^H$.

Letting $\sigma_1 \to 0$ would result in $\tilde{g}(\mathbf{A}|\mathbf{A}_t, \mathbf{B}_t)$ unbounded below, which implies $L(\mathbf{A}, \mathbf{B}_t)$ is also unbounded below and contradicts Assumption 1. ■

An immediate implication of Lemma 8 is that $g(\mathbf{A}|\mathbf{A}_t, \mathbf{B}_t)$ is strictly convex on $\mathbb{S}_{++}^p$ and has a unique closed-form minimizer given by

$$\mathbf{A}_{t+1} = \mathbf{A}_t^{1/2}\left(\mathbf{A}_t^{-1/2}\mathbf{M}\mathbf{A}_t^{-1/2}\right)^{1/2}\mathbf{A}_t^{1/2}, \tag{66}$$

where

$$\mathbf{M} = \frac{p}{N}\sum_{i=1}^{N}\frac{\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i}{\mathrm{Tr}\left(\mathbf{A}_t^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}.$$

Symmetrically, we have the update for $\mathbf{B}$ given by

$$\mathbf{B}_{t+1} = \mathbf{B}_t^{1/2}\left(\mathbf{B}_t^{-1/2}\mathbf{M}\mathbf{B}_t^{-1/2}\right)^{1/2}\mathbf{B}_t^{1/2}, \tag{67}$$

where

$$\mathbf{M} = \frac{q}{N}\sum_{i=1}^{N}\frac{\mathbf{M}_i\mathbf{A}_{t+1}^{-1}\mathbf{M}_i^H}{\mathrm{Tr}\left(\mathbf{A}_{t+1}^{-1}\mathbf{M}_i^H\mathbf{B}_t^{-1}\mathbf{M}_i\right)}.$$

Proposition 9: Under Assumption 1, every limit point, denoted by $(\mathbf{A}^\star, \mathbf{B}^\star)$, of the pair generated by Algorithm 7 is a global minimizer of problem (59).

Proof: Theorem 2 (a) in [26] implies that $(\mathbf{A}^\star, \mathbf{B}^\star)$ is a stationary point of problem (59). The rest of the proof is the same as that of Proposition 7. ■

Compared to Algorithm 6, which is a double loop algorithm, Algorithm 7 only performs a single loop iteration. However, the updates of Algorithm 6 are simpler to compute than those of Algorithm 7.

Algorithm 7: Robust covariance estimation under the Kronecker structure (Block Majorization-Minimization).

1: Initialize $\mathbf{A}_0$ and $\mathbf{B}_0$ to be arbitrary positive definite matrices of size $p \times p$ and $q \times q$, respectively.
2: **repeat**
3: Update $\mathbf{A}$ with (66) or (68) if $\mathbf{A} \in \mathcal{A}$.
4: Update $\mathbf{B}$ with (67) or (69) if $\mathbf{B} \in \mathcal{B}$.
5: $t \leftarrow t + 1$.
6: **until** Some convergence criterion is met.

Note that with the surrogate function of the form (64), we can easily impose additional convex structures on $\mathbf{A}$ and $\mathbf{B}$, and the update is found by solving the convex problem:

$$\mathbf{A}_{t+1} = \arg\min_{\mathbf{A}\in\mathcal{A}} g(\mathbf{A}|\mathbf{A}_t, \mathbf{B}_t), \tag{68}$$

$$\mathbf{B}_{t+1} = \arg\min_{\mathbf{B}\in\mathcal{B}} g(\mathbf{B}|\mathbf{A}_{t+1}, \mathbf{B}_t), \tag{69}$$

with $\mathcal{A}$ and $\mathcal{B}$ being the convex structural constraint sets.

VI. Numerical Results

In this section, we present numerical results that demonstrate the effect of imposing structure on the covariance estimator on reducing estimation error, and provide a comparison of the proposed estimator with some state-of-the-art estimators. The estimation error is evaluated by the normalized mean-square error, namely

$$\mathrm{NMSE}\left(\widehat{\mathbf{R}}\right) = \frac{E\left\|\widehat{\mathbf{R}} - \mathbf{R}_0\right\|_F^2}{\|\mathbf{R}_0\|_F^2}, \tag{70}$$

where all of the matrices are normalized by their trace. The expected value is approximated by 100 Monte Carlo simulations. In the following, we mainly compare the performance of four estimators, namely, the SCM, unconstrained Tyler's estimator (fixed-point equation of (3)), COCA (solution to (12)), and the proposed structure constrained Tyler's estimator. The samples in all of the simulations of this section, if not otherwise specified, are *i.i.d.* following $\mathbf{x}_i \sim \sqrt{\tau}\mathbf{u}$, where $\tau \sim \chi_1^2$, $\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_0)$, and τ and $\mathbf{u}$ are independent. The dimension K is set to be 15.

All simulations were coded in MATLAB and performed on a PC with a 3.20 GHz i5-3470 CPU and 8 GB RAM. Convergence criteria for all derived algorithms was set to be $|L(\mathbf{R}_{t+1}) - L(\mathbf{R}_t)| / \max(1, |L(\mathbf{R}_t)|) \le 10^{-6}$. For all algorithms that involve a convex programming, namely, COCA (12), Algorithm 1, and Algorithm 4, we used CVX [35], [36] with solver MOSEK. In the following simulations, all of the algorithms (except for COCA) shared the same initial point, which was randomly generated each time the data set changed.

A. Toeplitz Structure

In this simulation, $\mathbf{R}_0$ is set to be a real-valued Toeplitz matrix. The parameter $\mathbf{R}_0$ is set to be $\mathbf{R}(\beta)$, whose ij-th entry is of the form

$$(\mathbf{R}(\beta))_{ij} = \beta^{|i-j|}. \tag{71}$$

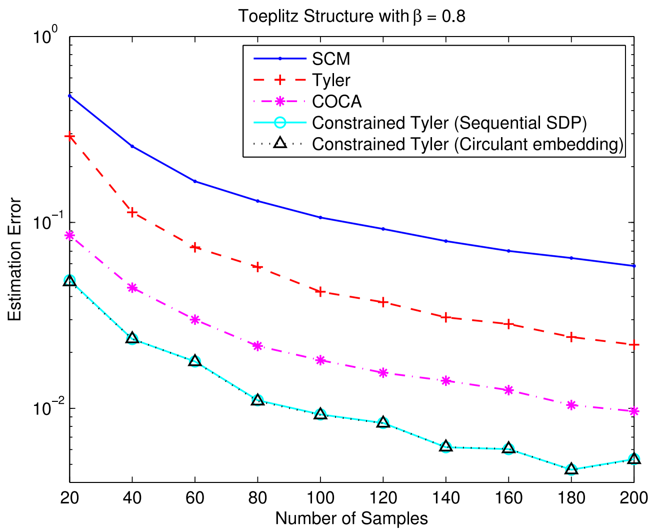

Fig. 1. Estimation error (NMSE) versus the number of samples N of different estimators under the Toeplitz structure of the form (71).

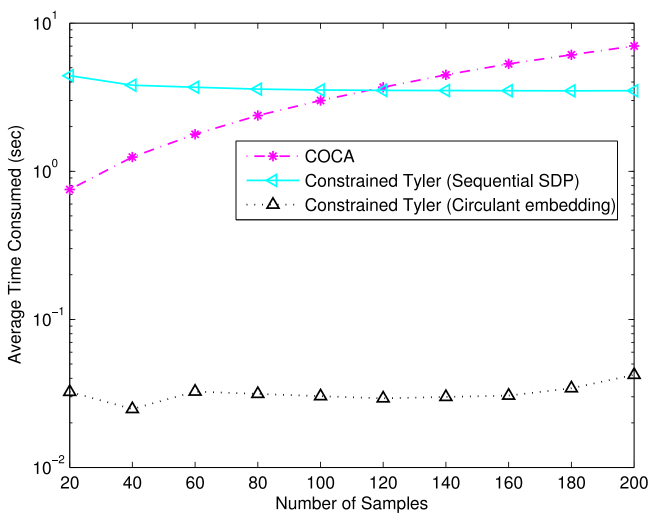

Fig. 2. Average time (in seconds) consumed by COCA and the constrained Tyler's estimator via sequential SDP (Algorithm 1) and circulant embedding (Algorithm 2) as the number of samples N varies from 20 to 200.

Fig. 1 shows the NMSE of the estimators with $\beta = 0.8$. The result indicates that the structure constrained Tyler's estimator achieves the smallest estimation error. In addition, we see that although the circulant embedding algorithm (Algorithm 2) with $L = 2\ K - 1$ approximately solves the Toeplitz structure constrained problem, it achieves virtually the same estimation error as imposing the Toeplitz structure and solving the problem via the sequential SDP algorithm (Algorithm 1). However, the computational cost of circulant embedding is much lower than that of sequential SDP and COCA, as shown in the average time cost plotted in Fig. 2.

B. Banded Toeplitz Structure

Next we investigate the case that $\mathbf{R}_0$ is a k-banded Toeplitz matrix $B_k\,(\mathbf{R})$, where $B_k\,(\mathbf{R})$ defines a matrix with the ij-th entry equals to that of $\mathbf{R}$ if $|i - j| \le k$, and equals zero otherwise.

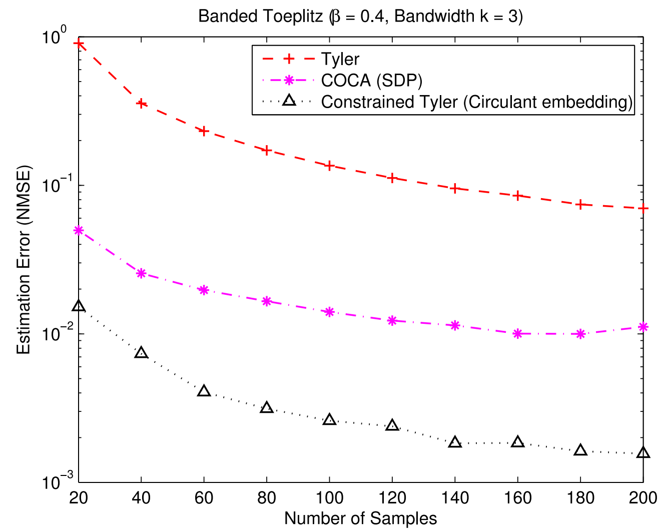

Fig. 3. Estimation error (NMSE) versus the number of samples N of different estimators under the banded Toeplitz structure.

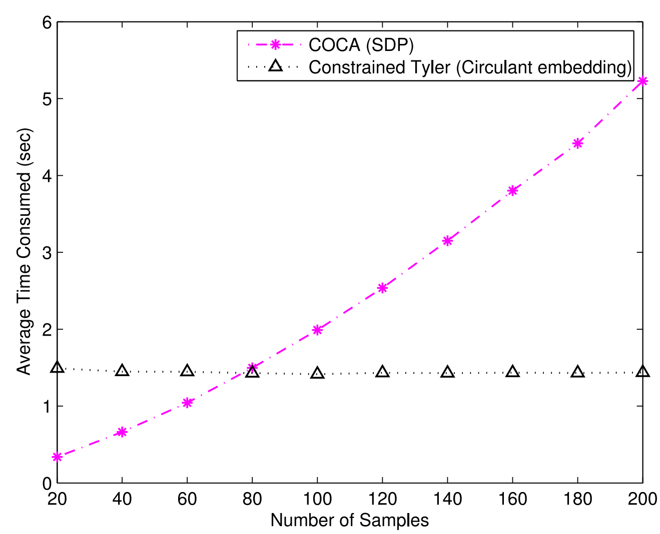

Fig. 4. Average time (in seconds) consumed by COCA and constrained Tyler's estimator as the number of samples N varies from 20 to 200.

We set $\mathbf{R} = \mathbf{R}\,(0.4)$ as defined in (71), and the bandwidth k is chosen to be 3. The NMSE is plotted in Fig. 3, where the constrained Tyler's estimator achieves the smallest estimation error. Fig. 4 plots the average time consumed by COCA and the constrained Tyler's estimator. As the number of semidefinite constraints that COCA has is proportional to the number of samples N, the time consumption is increasing in N, while the time cost by the algorithm for the constrained Tyler's estimator remains roughly the same as N grows. When N is small, the algorithm for COCA runs faster than ours since the scale of the SDP that COCA solves is small. In the regime that N is large, the computational cost of COCA increases, as reflected both in the time and the memory required to run the algorithm.

In the third simulation, we consider $\mathbf{R}_0$ being a non-banded Toeplitz matrix with the property that $(\mathbf{R}_0)_{ij}$ decays rapidly as $|i - j|$ increases. We investigate the cases of $\mathbf{R}_0 = \mathbf{R}\,(0.4)$

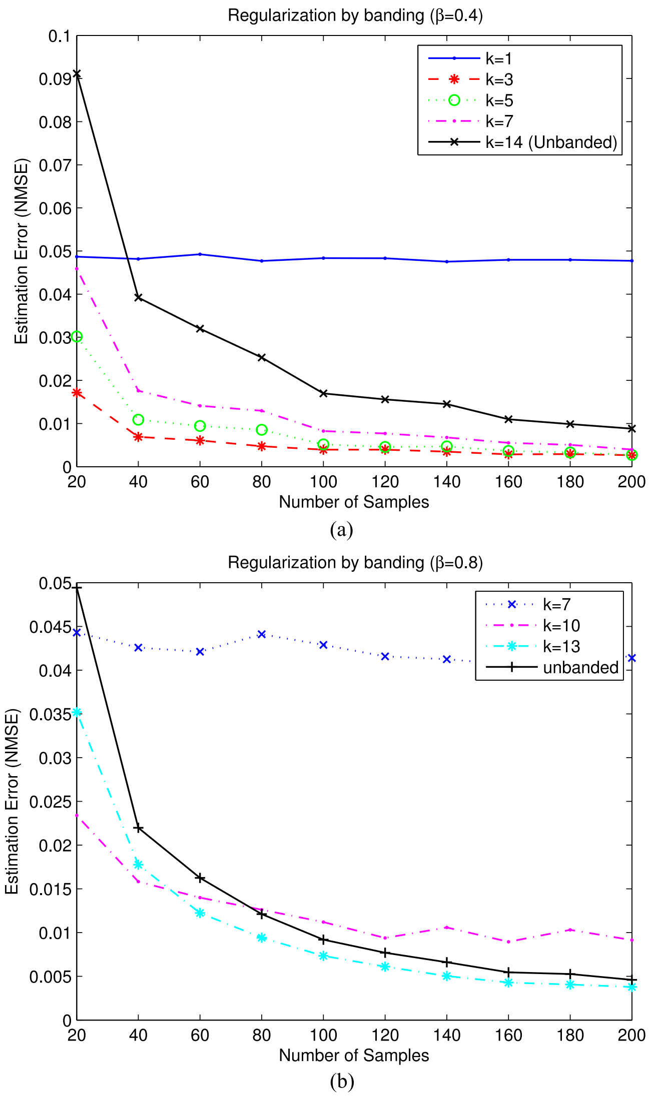

Fig. 5. NMSE of the Tyler's estimator versus the number of samples N under a banded Toeplitz structure of bandwidth k: (a) population parameter $\mathbf{R}_0 = \mathbf{R}(0.4)$ (fast decay), (b) population parameter $\mathbf{R}_0 = \mathbf{R}(0.8)$ (slow decay).

(fast decay) and $\mathbf{R}_0 = \mathbf{R}(0.8)$ (slow decay) and impose a banded Toeplitz structure on the Tyler's estimator with a varying bandwidth k to regularize the estimator. Fig. 5 shows that the smallest error is obtained when $k = 3$ in the $\beta = 0.4$ case, and when $k = 13$ in the $\beta = 0.8$ case. In either case, with the right choice of bandwidth k, the regularized estimator outperforms the unbanded one when the number of samples is relatively small compared to the dimension of the covariance matrix to be estimated.

C. Direction of Arrival Estimation

In this subsection, we examine the robustness of the proposed estimator in the context of the direction of arrival estimation problem with the following signal model:

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta}_s)\,\mathbf{s}(t) + \mathbf{n}(t),$$

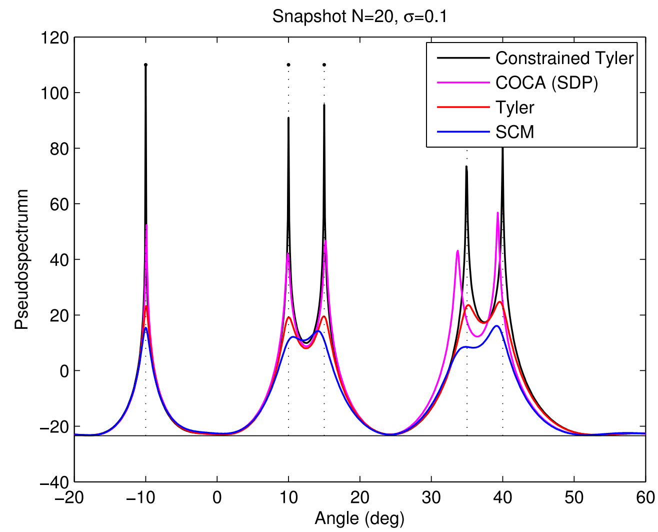

Fig. 6. Arrival angle estimated by MUSIC with different covariance estimators.

where $\boldsymbol{\theta}_s = [\theta_1, \ldots, \theta_m]$ is a vector with elements representing the arriving directions of signal $\mathbf{s}(t)$,

$$\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\boldsymbol{\theta}_m)] \quad (72)$$

is the steering matrix, and $\mathbf{n}(t)$ is zero mean additive noise. We study the simple case of an ideal uniform linear array (ULA) with half-wavelength inter-element spacing, where

$$\mathbf{a}(\theta) = \left[1, e^{-j\pi\sin(\theta)}, \ldots, e^{-j\pi(K-1)\sin(\theta)}\right]^T. \quad (73)$$

Assuming that the signal $\mathbf{s}(t)$ is a wide-sense stationary random process with zero mean, the covariance of $\mathbf{x}(t)$ is

$$\mathbf{R} = \mathbf{A}(\boldsymbol{\theta}_s)\,\mathrm{Cov}(\mathbf{s})\,\mathbf{A}(\boldsymbol{\theta}_s)^H + \mathrm{Cov}(\mathbf{n}).$$

Further assume that the signals arriving from different directions are uncorrelated and that the noise is spatially white, i.e., $\mathrm{Cov}(\mathbf{s}) = \mathrm{diag}(p_1, \ldots, p_m) \triangleq \mathbf{P}_s$ and $\mathrm{Cov}(\mathbf{n}) = \sigma^2\mathbf{I}$, the covariance model simplifies to be

$$\mathbf{R} = \mathbf{A}(\boldsymbol{\theta}_s)\,\mathbf{P}_s\mathbf{A}(\boldsymbol{\theta}_s)^H + \sigma^2\mathbf{I}. \quad (74)$$

In our simulation, $m = 5$ random signals are assumed arriving from directions $-10^\circ, 10^\circ, 15^\circ, 35^\circ, 40^\circ$ with equal power $p = 1$ and the noise power is set to be $\sigma^2 = 0.1$. The received signal is assumed to be elliptically distributed and m is assumed to be known. The number of sensors is $K = 15$.

We first estimate $\mathbf{R}$ and then apply the MUltiple SIgnal Classification (MUSIC) algorithm to estimate the arriving angles. The performance of SCM, Tyler's estimator, COCA and the constrained Tyler's estimator are compared. For the latter two estimators, which require a specification of the structure set $\mathcal{S} = \left\{\mathbf{R} | \mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H\right\}$ parameterized by $\mathbf{P} \succeq \mathbf{0}$, we construct the matrix $\mathbf{A}$ according to (72) and (73) with $\boldsymbol{\theta} = \left[-90^\circ, -85^\circ, \ldots, 80^\circ, 85^\circ\right]$. Fig. 6 shows the estimated arrival direction using different estimators with the number of snapshots $N = 20$, and only the constrained Tyler's estimator correctly recovers all of the arriving angles.

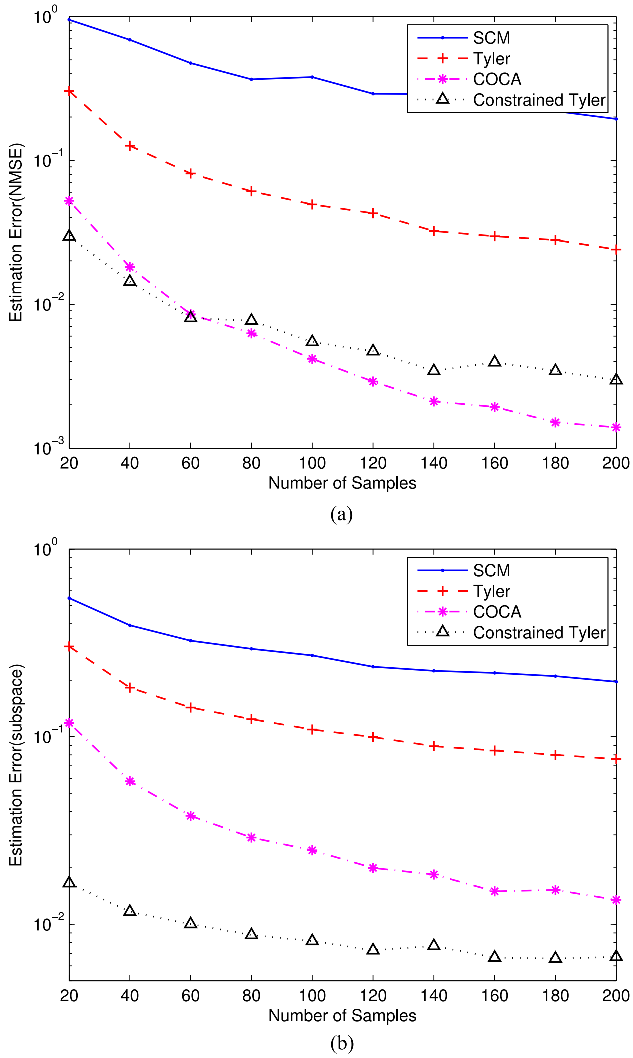

Fig. 7. The estimation error of different estimators versus the number of samples N under the DOA structure: (a) NMSE, (b) estimation error of the noise subspace given by different estimators evaluated by (75).

Fig. 7 shows the performance of different estimators in terms of NMSE and the estimation error of noise subspace evaluated by

$$\left\| \widehat{\mathbf{E}}_c \widehat{\mathbf{E}}_c^H - \mathbf{E}_c \mathbf{E}_c^H \right\|_F, \tag{75}$$

with the number of snapshots N varying from 20 to 200, where $\mathbf{E}_c$ denotes the noise subspace and $\widehat{\mathbf{E}}_c$ denotes its estimate. $\widehat{\mathbf{E}}_c$ is constructed by eigendecomposing $\widehat{\mathbf{R}}$ as $\widehat{\mathbf{R}} = \sum_{j=1}^{K} \lambda_j \mathbf{u}_j \mathbf{u}_j^H, \lambda_1 \geq \cdots \geq \lambda_K$, and $\widehat{\mathbf{E}}_c = \sum_{j=m+1}^{K} \lambda_j \mathbf{u}_j \mathbf{u}_j^H$. Fig. 7(a) reveals that the constrained Tyler's estimator achieves the smallest NMSE when N is small, while COCA performs better when N is large. However, Fig. 7(b) indicates that the constrained Tyler's estimator can estimate the noise subspace more accurately for all values of investigated N, which is beneficial for algorithms that are based on $\widehat{\mathbf{E}}_c$ such as MUSIC.

The average time cost by COCA and the constrained Tyler's estimator is plotted in Fig. 8. It can be seen that the proposed method is much faster than COCA. In addition, unlike COCA,

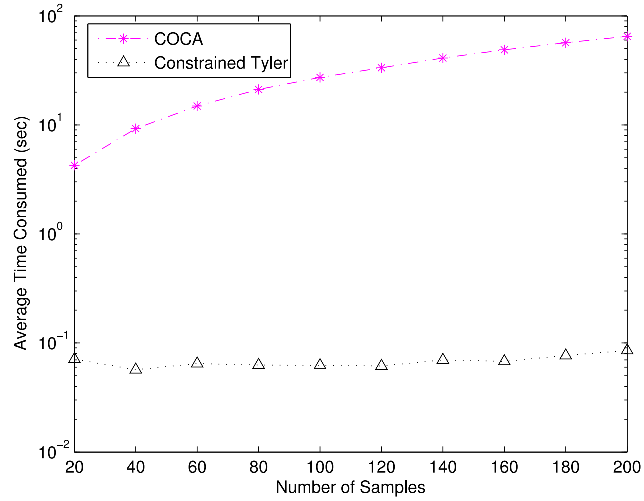

Fig. 8. Average time (in seconds) consumed by COCA and constrained Tyler's estimator as the number of samples N varies from 20 to 200.

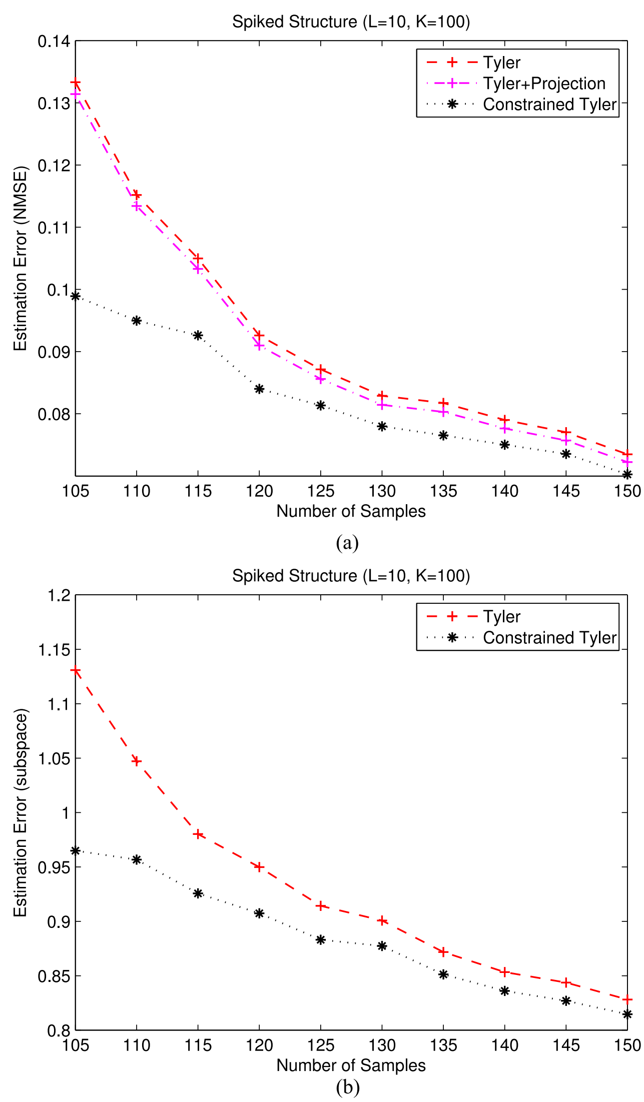

Fig. 9. The estimation error of different estimators versus the number of samples N under the spiked covariance structure: (a) NMSE, (b) estimation error of the noise subspace given by different estimators evaluated by (75).

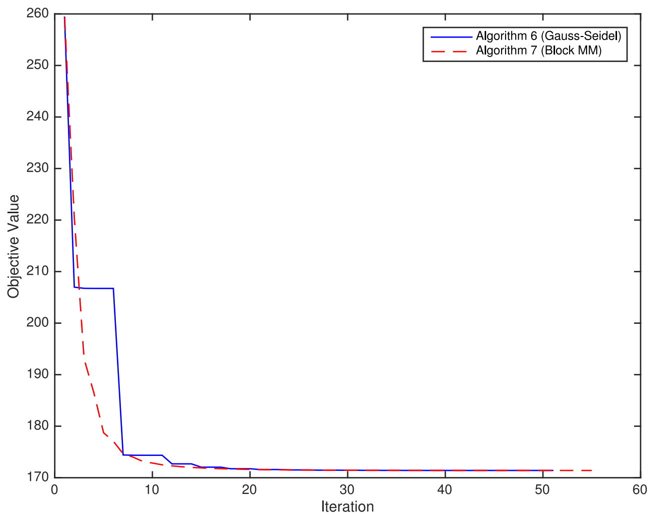

Fig. 10. Convergence comparison of Algorithm 6 and 7 under the Kronecker structure $(N = 4, \mathbf{A}_0 = \mathbf{I}, p = 10, \mathbf{B}_0 = \mathbf{R}(0.8), q = 8)$.

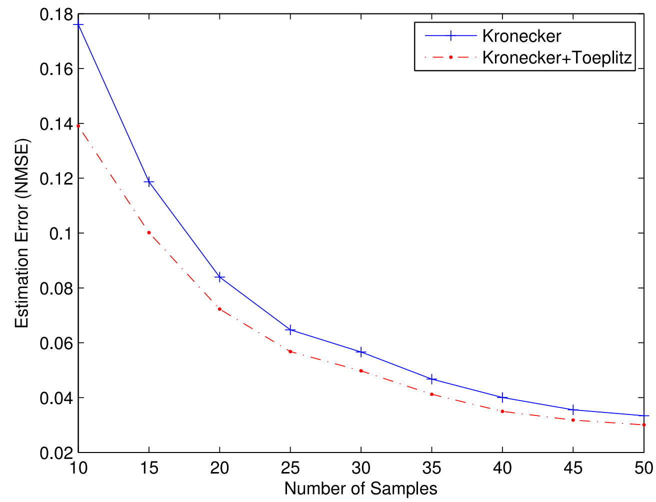

Fig. 11. NMSE of Tyler's estimator with a Kronecker structural constraint compared to that with both a Kronecker and a Toeplitz structural constraint.

the consumed time of our algorithm is not sensitive to the number of samples N.

D. Spiked Covariance Structure

We construct the true covariance $\mathbf{R}_0$ by the following model:

$$\mathbf{R}_0 = \sum_{j=1}^{L} p_j \mathbf{a}_j \mathbf{a}_j^H + \sigma^2 \mathbf{I},$$

where the $\mathbf{a}_j$'s are randomly generated orthonormal basis and the p_j's are randomly generated corresponding eigenvectors uniformly distributed in $[0.01, 1]$. σ^2 is set to be 0.01. The number of spikes $L = 10$ is assumed to be known in prior. The matrix dimension is fixed to be $K = 100$, and the number of samples is varied from $N = 105$ to $N = 150$. As COCA applies only for convex structural set and cannot be used here, we replace it by the projected Tyler's estimator, which is a two step procedure that first obtains the Tyler's estimator and then performs projection according to (57). Fig. 9 shows that imposing the spiked structure helps in reducing the NMSE and subspace estimation error measured by (75).

E. Kronecker Structure

The parameters are set to be $\mathbf{A}_0 = \mathbf{I}, \mathbf{B}_0 = \mathbf{R}(0.8), p = 10, q = 8$, in the simulations. We first plot the convergence curve of Algorithms 6 and 7 with the number of samples $N = 4$ in Fig. 10. The two algorithms converges in roughly the same number of iterations, however, the objective value corresponds to Algorithm 7 (block MM) decreases more smoothly than Algorithm 6 (Gauss-Seidel), as the latter is a double loop algorithm while the former is a single loop algorithm.

Fig. 11 plots the NMSE of Tyler's estimator with a Kronecker constraint on $\mathbf{R}$ and that with both a Kronecker constraint on $\mathbf{R}$ and a Toeplitz constraint on $\mathbf{B}$. We can see that further imposing a Toeplitz structure on $\mathbf{B}$ helps in reducing the estimation error.

VII. Conclusion

In this paper, we have discussed the problem of robustly estimating the covariance matrix with a prior structure information. The problem has been formulated as minimizing the negative log-likelihood function of the angular central Gaussian distribution subject to the prior structural constraint. For the general convex constraint, we have proposed a sequential convex programming algorithm based on the majorization-minimization framework. The algorithm has been particularized with higher computational efficiency for several specific structures that are widely considered in the signal processing community. The spiked covariance model and the Kronecker structure, although belonging to the non-convex constraint, are also discussed and shown to be computationally tractable. The proposed estimator has been shown outperform the state-of-the-art methods in the numerical section.

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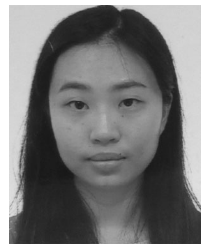

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